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ASYMPTOTIC PROPERTIES OF THE LEAST SQUARES ESTIMATOR IN PERIODIC $ARCH\,(p)$ MODELS

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ABSTRACT. We study the asymptotic properties of the least squares estimator (LSE) for periodic ARCH(p) (PARCH(p)) models constructed through the periodic AR representation (PAR) on the squared observations of the PARCH model. After linearization, we show that the LSE is strongly consistent, asymptotically normal, and we investigate their asymptotic properties in stationary and integrated cases (in a periodic sense). The theory is applied to the study of goodness-of-fit tests and specification tests for some restrictions of PARCH models via Wald statistics.

Keywords. Periodic ARCH models, Periodically correlated processes, LSE, Consistency, Normality asymptotic; Wald test
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1. Introduction

Periodic GARCH models (PGARCH) initiated by Bollerslev and Ghysels [15] have been extensively studied by Aknouche and Bibi (hereafter *AB* [1]) and Bibi and Aknouche [6]. The results obtained by these authors are based on the quasi-maximum likelihood estimator (QMLE) when the strong consistency and the asymptotic normality (CAN) are examined of PGARCH models and of periodic ARMA(PARMA) models driven by PGARCH error (PARMA-PGARCH). Besides the CAN properties, *AB* [1] have gaven some conditions assuring the stability, stationarity, and the ergodicity (in a periodic sense) of PGARCH and of PARMA-PGARCH models. Even if the obtained estimators are strongly consistent and asymptotically normal only under moment conditions on the conditional density, it could be, but theoretically, the divergence of the true innovation density can greatly increase the variance of the estimates, thus increasing the cost of ignoring the true distribution of innovations. This finding allows us to call on the least squares estimator (LSE) method to remedy such a problem. So, in this paper, we are interested in the CAN properties of LSE for PARCH(p) models. Despite the fact that PARCH models are a particular case of PGARCH one. The approaches and the aims of this paper differ greatly from the previous paper by *AB* [1]. Indeed, our approach is focused on, the one hand,(i): The LSE method is applied on the Autoregressive representation of the PARCH model, is driven by a martingale difference

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errors, and an explicit parameter estimates are obtained beside their CAN properties with rates of convergence. (ii): The parametrization AR with intercept encompasses many commonly used models in the literature. (iii): The CAN properties are essentially involved some conditions on the stochastic eigenvalues of a certain matrix entailed in the multivariate AR representation. A shorten review of the methodology followed in this paper are discussed as follows.

1.1. **Literature review.** The statistical inference for the PARCH process has been considered by several authors and some methods were proposed for estimating the parameters involved in such models and their the CAN properties. Indeed, for instance, among others, [1] have studied the PGARCH (p, q) model where the CAN properties via the Quasi-Maximum likelihood estimation (QMLE) are investigated. The study also considered the periodic PARMA with PGARCH (p, q) as innovations processes.

The CAN properties of LSE of PGARCH (p, q) and PARMA with PGARCH (p, q) errors has been investigated by Bibi and Lescheb [7]. The same authors [8] have extended their results to the conditional LSE approach to PGARCH and PARMA with PGARCH errors. The generalized QME of PGARCH (p, q) model and their CAN properties were studied recently by Aknouche et al. [3]. For certain transformations of PGARCH (p, q), Bibi and Hmadi [10] investigated certain structural probabilistic properties for periodic log GARCH(p, q) models and established its CAN properties. Other transformations have been proposed by Sadoun and Bentarzi [26] for periodic EGARCH models and its CAN properties. A power periodic threshold GARCH model was introduced by Guerbyenne and Kessira [20] in which the CAN properties of QMLE were established. Note here that in all above setting the models are supposed to be stationary (in the periodic sense). The non-stationary case was investigated by Aknouche et al. [2].

1.2. **Contribution.** The parameters which characterize the PARCH model should be estimated via the data generating the processes. As we know that in nonlinear models, the QMLE remains the main issue for estimation. In this paper, a LSE is proposed as an alternative issue for estimating the $PGARC$ model. So, our contribution is motivated by

(1) We briefly describe the linear representation of PARCH by embedding it into a stationary multivariate AR (VAR) version. The VAR representation allows us to give sufficient conditions ensuring the existence of PC solution of PARCH model.
(2) We study the LSE of PARCH model based on a multivariate AR representation with martingale difference errors, and we show that under assumptions similar to those used by Voffal [29], the LSE estimator is asymptotically stable, strongly consistent, asymptotically normal, and we determine its rate of convergence.
(3) The asymptotic properties also encompass the integrated case.
(4) To study the goodness-of-fit tests and specification tests for PARCH model, so a Wald statistic is derived for this purpose.

1.3. **Content.** An overview of the paper is as follows. In the next Section, we first state several properties related to a vectorial representation, and we give conditions leaving the process PARCH to be periodically strictly stationary and ergodic. In Section 3, the formulation of LSE is presented and the necessary conditions of the strong consistency and the asymptotic normality are listed, followed by an illustrative example. Some asymptotic properties when the process in integrate are described. As a consequence, in Section 4, we construct the Wald statistics for testing certain appropriate null hypotheses against an alternative. Section 5 is an appendix in which we give explicit expansion for moments up to eighth order and the proofs of the fundamental theorems. Concluding remarks and suggestions for future research constitute the Section 6.

2. Background: PARCH models

Let us recall that a second order process $(X_t)_{t\in\mathbb{Z}}$ defined on probability space $(\Omega,\mathcal{A},\mathcal{P})$ is called periodic $ARCH(p)$ process ($PARCH(p)$) of known period $s>0$, if it satisfies the following equations for any $n\in\mathbb{Z}:=\{0,\pm1,\pm2,...\}$

$$X_n = e_n\sqrt{h_n} \text{ with } h_n := a_0(s_n)+\sum_{i=1}^{p} a_i(s_n)X_{n-i}^2 \tag{1.1}$$

where $(e_n)_{n\in\mathbb{Z}}$ is a sequence of random variables $i.i.d.\,(0,1)$ defined on the same probability space $(\Omega,\mathcal{A},\mathcal{P})$ and admitting finite moments up to the fourth order. The $(s_n)_n$ is periodic sequence with values in $\mathbb{S}=\{1,...,s\}$ and defined by: $s_n=\sum\limits_{v=1}^{s} v\mathbb{I}_{\Delta(v)}(n)$ where $\mathbb{I}_{\Delta(v)}$ denote the indicator function of the set $\Delta(v)=\{st+v|t\in\mathbb{Z}\}$, $v\in\mathbb{S}$. The period s is taken to be the smallest positive integer satisfying (1.1). When $s=1$, (1.1) corresponds to the standard ARCH model. The parameters $(a_i(s_n),\,0\le i\le p)$ are such that $a_0(s_n)>0$ and $a_i(s_n)\ge0$, $1\le i\le p$ for all $n\in\mathbb{Z}$. By setting $n=st+v$ and putting $X_t(\nu)=X_{st+v}$ (resp. $e_t(\nu)=e_{st+v}$, $h_t(\nu)=h_{st+v}$),we will obtain the following "seasonal" parametrization

$$X_t(\nu)=e_t(\nu)\sqrt{h_t(\nu)} \text{ with } h_t(\nu)=a_0(v)+\sum_{i=1}^{p} a_i(v)X_t^2(\nu-i). \tag{1.2}$$

In (1.2), $X_t(\nu)$ (resp. $e_t(\nu)$, $h_t(\nu)$) denote the process X_t (resp. e_t, h_t) during the $v-$th "season" $\nu\in\mathbb{S}$ of "the year" t. For convenience, the non$-$periodic notation $\{X_t$, e_t and $h_t\}$ will be used interchangeably with the periodic notations $\{X_t(\nu), e_t(\nu)$ and $h_t(\nu)\}$, moreover, when the argument ν is negative, $X_t(\nu)=X_{t-1}(\nu+s)$, etc. Let $\eta_t(\nu)=X_t^2(\nu)-h_t(\nu)=h_t(\nu)\left(e_t^2(\nu)-1\right)$, then we obtain a representation $PAR(p)$ on the square process $(X_t)_t$ with intercept and non-independent innovations i.e.,

$$X_t^2(\nu)=a_0(v)+\sum_{i=1}^{p} a_i(v)X_t^2(\nu-i)+\eta_t(\nu),\ v\in\mathbb{S}. \tag{1.3}$$

When the process $\left(X_t^2\right)_t$ is integrable, the nuisance innovation $(\eta_t)_{t\in\mathbb{Z}}$ may be regarded as a periodic martingales differences adapted to filtration $\Im:=(\Im_t)_t$ where $\Im_t:=\sigma\{e_{t-j},j\ge0\}$ with $cov(\eta_t(\nu),\eta_t(\nu'))=\sigma_\eta^2(\nu)\,\mathbb{I}_{\{\nu=\nu'\}}$. However, thanks to Gladyshev [19], with periodic coefficients, it is possible to embed the representation (1.3) into a multivariate autoregressive (or vector of seasons) with constant parameters. More precisely, $\underline{X}_t^2=\left(X_t^2(1),...,X_t^2(s)\right)'$, is an $VAR(p^*)$ model in the form

$$\left(\mathbf{I}_{(s)}-\mathbf{A}_0\right)\underline{X}_t^2=\underline{a}_0+\sum_{i=1}^{p^*}\mathbf{A}_i\underline{X}_{t-i}^2+\underline{\eta}_t \tag{1.4}$$

where $\underline{\eta}_t=(\eta_t(1),...,\eta_t(s))'$, $\underline{a}_0=(a_0(1),...,a_0(s))'$ and $\mathbf{I}_{(s)}$ is the identity matrix of order $s\times s$. The order of the model (1.4) is given by $p^*=\left[\frac{p}{s}\right]$ where $[x]$ is the integer part of x (it is the smallest integer satisfying $[x]-1<x\le[x]$). The $s\times s$ matrices $(\mathbf{A}_i)_{0\le i\le p^*}$ are calculated as follows (cf. Basawa et al. [4]) $(\mathbf{A}_0)_{i,j}=a_{i-j}(i)\mathbb{I}_{\{i>j\}}(i,j)$, and $(\mathbf{A}_m)_{i,j} = a_{ms+i-j}(i)$ for $1 \leq m \leq p^*$. The process $(X_t)_t$ is said to be periodically second order stationary, if the following hypothesis is verified

$\mathbf{H}$: $\det\left(\mathbf{I}_{(s)} - \sum\limits_{i=0}^{p^*} \mathbf{A}_i z^i\right) \neq 0, \forall z \in \mathbb{C}$: such that $|z| \leq 1$.

Remark 2.1. Note that under $\mathbf{H}$, the process $\left(X_t^2\right)_t$ admits a periodically correlated solution in the sense that $E\left\{X_{t+s}^2\right\} = E\left\{X_t^2\right\}$ and $Cov(X_{t+s}^2, X_{t'+s}^2) = Cov(X_t^2, X_{t'}^2)$ for all $t, t' \in \mathbb{Z}$. Furthermore, the solution process is unique, $\Im_t$-measurable and periodically ergodic (cf., Basawa et al. [4] for a PARMAmodel driven by an *i.i.d.*, errors). In contrast, when there is at least one root of $\det\left(\mathbf{I}_{(s)} - \sum\limits_{i=0}^{p^*} \mathbf{A}_i z^i\right) = 0$, on the unit circle, $\left(X_t^2\right)_t$ is said to be periodically integrated ARCH model noted PIARCH (see Bibi et al. [9] for more details). Additionally, it is worth noting that $(X_t)_t$ are uncorrelated processes unlike $\left(X_t^2\right)_t$.

Remark 2.2. Under the hypothesis $\mathbf{H}$, it is possible to relate $\left(X_t^2\right)_t$ and $(\eta_t)_t$ through the infinite order moving-average (causal) with periodic coefficients (PMA) expansion

$$X_t^2(\nu) = \sum_{i=0}^{\infty} \phi_i(\nu)\, \eta_t(\nu - i) + \widetilde{a}_0(v) \tag{1.5}$$

where $\widetilde{a}_0(v) = \sum\limits_{i=0}^{\infty} \phi_i(\nu)\, a_0(\nu - i)$ in which the "seasonal weight" $\phi_i(\nu)$ satisfy, $\max\limits_{1 \leq \nu \leq s} \sum_{i=0}^{\infty} \phi_i(\nu) < +\infty$, and can be computed as $\phi_i(\nu) = \mathbb{I}_{\{i=0\}} + \sum\limits_{j=1}^{\min(i,p)} a_j(v)\phi_{i-j}(\nu - j)\, \mathbb{I}_{\{i \geq 1\}}$, $\nu \in \mathbb{S}$. The representation PMA constitutes an approximation of the Wold decomposition of periodically stationary processes and plays an important role when dealing with diagnostic checking for periodic residuals. The notation used above and elsewhere interprets $\phi_i(j)$ for each $i \geq 0$, periodically in j with period s.

3. Least squares estimator issues

Through the sequel, let $\underline{\theta} = (\underline{\theta}'(1), ..., \underline{\theta}'(s))' \in \mathbb{R}^{(p+1)s}$, be the unknown vector of parameters involved in model (1.1), $\mathbf{Z}_t' := diag\left\{\underline{Z}_t'(1), ..., \underline{Z}_t'(s)\right\}$, where $\underline{\theta}(v) = (a_0(v), a_1(v), ..., a_p(v))'$ and $\underline{Z}_t'(v) := (1, X_t^2(\nu - 1), ..., X_t^2(\nu - p))$, $v \in \mathbb{S}$. With these notations, the model (1.3) transforms into a linear regression model

$$\underline{X}_t^2 = \mathbf{Z}_t'\underline{\theta} + \underline{\eta}_t, \tag{2.1}$$

In this paper, we are interested in the unconstrained least squares estimators (LSE) of the parameter $\underline{\theta}$ with a view to a realization $\{X_1, X_2, ..., X_{sn}\}$ (or $\left\{\underline{X}_1^2, \underline{X}_2^2, ..., \underline{X}_n^2\right\}$) of the process $(X_t)_t$ defined by (1.2) (or of its vectorial version $\left(\underline{X}_t^2\right)_t$ defined by (1.4)). The parametric space is therefore $\Theta = \{\underline{\theta} \in \mathbb{R}^{(p+1)s}$: $\mathbf{H}$ holds true$\}$ supposed compact. For this purpose, let us define some matrices and vectors, $\mathbf{S}_n = \sum\limits_{t=1}^{n} \mathbf{Z}_t\mathbf{Z}_t' = diag\left\{\mathbf{S}_n(1), ..., \mathbf{S}_n(s)\right\}$, $\underline{S}_n = \sum\limits_{t=1}^{n} \mathbf{Z}_t\underline{X}_t^2 := (\underline{S}_n'(1), ..., \underline{S}_n'(s))$, $\underline{T}_n = \sum\limits_{t=1}^{n} \mathbf{Z}_t\underline{\eta}_t = (\underline{T}_n'(1), ..., \underline{T}_n'(s))'$ where $\mathbf{S}_n(v) = \sum\limits_{t=1}^{n} \underline{Z}_t(v)\, \underline{Z}_t'(v)$, $\underline{S}_n(v) := \sum\limits_{t=1}^{n} \underline{Z}_t(v) X_t^2(v)$, $\underline{T}_n(v) = \sum\limits_{t=1}^{n} \underline{Z}_t(v)\eta_t(\nu)$, and denote by $\lambda_{\max}^{(v)}(n)$ (resp. $\lambda_{\min}^{(v)}(n)$) the largest (resp. the smallest) stochastic eigenvalue of $\mathbf{S}_n(v)$ for all $v \in \mathbb{S}$. The LSE of $\underline{\theta}(\nu)$ is obtained by minimizing the least squares criterion: $S(\underline{\theta}) = \sum_{t=0}^{n-1} \underline{\eta}_t' \underline{\eta}_t$. Hence, the LSE for $\underline{\theta}$ is a solution of the $s(p+1)$ equations obtained by using classical results on matrix differentiation of $S(\underline{\theta})$. So, the minimum of $S(\underline{\theta})$ is reached at

$$\widehat{\underline{\theta}}_n = \mathbf{S}_n^{-1} \underline{S}_n = \underline{\theta} + \mathbf{S}_n^{-1} \underline{T}_n \tag{2.2}$$

in particular $\widehat{\underline{\theta}}_n(\nu) = \mathbf{S}_n^{-1}(\nu) \underline{S}_n(\nu) = \underline{\theta}(\nu) + \mathbf{S}_n^{-1}(\nu) \underline{T}_n(\nu)$ for each $\nu \in \mathbb{S}$. Additionally, the LSE of $\sigma_\eta^2(\nu) := Var\eta_t(\nu)$ is $\widehat{\sigma}_\eta^2(\nu) = \frac{1}{n-p-1} \left\| \underline{X}_t^2(\nu) - \mathbf{Z}_t'(\nu) \underline{\theta}(\nu) \right\|^2$ $= \frac{1}{n-p-1} \sum_{t=1}^{n} \left(X_t^2(\nu) - \widehat{a}_0(v) - \sum_{i=1}^{p} \widehat{a}_i(v) X_t^2(\nu - i) \right)^2$. In the sequel, the asymptotic properties of LSE is investigated for two cases stationary and integrated one as follows (the explosive case is already studied by Aknouche [2]). In the sequel, the symbols '$\overset{d}{\rightsquigarrow}$', '$\xrightarrow{P}$', '$\xrightarrow{a.s}$' and '$\Longrightarrow$' denote respectively for convergence in distribution, in probability, almost surely, and the weak convergence of the associated probability measures.

3.1. **Limiting distributions in stationary case.** In order to establish the asymptotic properties of LSE, let us make the following hypothesis,
$\mathbf{H1}(\varepsilon)$: $\sup_t E\left\{ \left\| \underline{X}_t^2 \right\|^{2+\varepsilon} \right\} < +\infty$, where ε is a positive real.
$\mathbf{H2}(\lambda)$: $\lambda_{\min}^{(v)}(n) \xrightarrow{a.s} \infty$, and $\log\left(\lambda_{\max}^{(v)}(n)\right) = o_p(\lambda_{\min}^{(v)}(n))$, $a.s.$, for all $v \in \mathbb{S}$.
For the asymptotic normality of LSE, additional assumptions are also required. Indeed, define the strong mixing coefficients associated to $\left(\underline{X}_t^2\right)_t$ by $\alpha_X(h) = \sup_{A \in \mathcal{F}_{-\infty}^n, B \in \mathcal{F}_{n+h}^{+\infty}} |P(A \cap B) - P(A) P(B)|$ where $\mathcal{F}_{-\infty}^n = \sigma\left(\underline{X}_u^2, u \le n\right)$ and $\mathcal{F}_{n+h}^{+\infty} = \sigma(\underline{X}_u^2, u \ge n+h)$ and assume that
$\mathbf{H3}(X)$: The process $\left(\underline{X}_t^2\right)_t$ is strong $\alpha-$mixing, i.e., $\mathbf{H1}(\varepsilon)$ holds true and $\sum_{h=0}^{\infty} \{\alpha_\eta(h)\}^{\frac{\varepsilon}{2+\varepsilon}} < +\infty$, for some $\varepsilon > 0$.
Now, a few comments can be madeThe hypothesis $\mathbf{H1}(\varepsilon)$ can be translated by simple conditions on the parameters $(a_i(v))_{0 \le i \le p}$, $\nu \in \mathbb{S}$ (see Milhøj [23]). Moreover, it is not difficult to see that under $\mathbf{H1}(\varepsilon)$, the hypothesis $\mathbf{H3}(\alpha)$: $\sup_t E\left\{ |\eta_t|^\alpha | \Im_{t-1} \right\} < +\infty$, $a.s.$, holds true for some $\alpha \ge 2$. The hypothesis $\mathbf{H2}(\lambda)$ constitutes in some sense the weakest possible conditions for which $\widehat{\underline{\theta}}_n \xrightarrow{a.s} \underline{\theta}$ as $n \to \infty$ (cf. Lai, et al. [21]), Lai, et al. [21] also construct a counterexample in which $P\left(\widehat{\underline{\theta}}_n \to \underline{\theta}\right) = 0$ where $\log\left(\lambda_{\max}^{(v)}(n)\right) / \lambda_{\min}^{(v)}(n)$ converge $a.s.$ to a finite nonzeros limit. . $\mathbf{H3}(X)$ is valid for large classes of processes and constitutes a powerful condition for the asymptotic independence. Note that the asymptotic properties of LSE are closely linked to those of the process $(\underline{T}_n)_{n \ge 1}$ and of the random matrices $\mathbf{S}_n$. When $E\left\{ \underline{\eta}_t \underline{\eta}_t' | \Im_{t-1} \right\} = \Sigma_\eta < \infty$ and $E\{\mathbf{Z}_n \mathbf{Z}_n'\} < \infty$ for all n, the random matrix $\mathbf{S}_n$ (up to a constant matrix) is none other than the conditional covariance of $\underline{T}_n$, i.e., $\sum_{t=1}^{n} E\left\{ \left(\mathbf{Z}_t \underline{\eta}_t\right) \left(\mathbf{Z}_t \underline{\eta}_t\right)' | \Im_{t-1} \right\} = \left(\Sigma_\eta \otimes \mathbf{I}_{(p+1)}\right) \mathbf{S}_n$. The main results of this paper are as follows:

Theorem 3.1. *Let $(X_t)_{t\in\mathbb{Z}}$ be a $PARCH(p)$ process admitting a representation* (1.4). *Then under* **H**, **H1** (0) *and* **H2** (λ), $\widehat{\underline{\theta}}_n$ *of* $\underline{\theta}$ *given by* (2.2) *exists and* $\widehat{\underline{\theta}}_n \xrightarrow{a.s} \underline{\theta}$, *as* $n\to\infty$, *further for all* $v\in\mathbb{S}$,

$$\left\|\widehat{\underline{\theta}}_n(\nu)-\underline{\theta}(\nu)\right\|^2=o_p\left(\frac{\log\left(\lambda_{\max}^{(v)}(n)\right)}{\lambda_{\min}^{(v)}(n)}\right),\ a.s.$$

Proof. The proof of the Theorem appears in subsection 5.2. □

Theorem 3.2. *Beside the assumptions of Theorem 3.1, we assume that* **H3** (X) *holds true and* $E\left\{\eta_t^4\right\}<+\infty$, *then for each* $\nu\in\mathbb{S}$

(1) $n^{-1/2}\underline{T}_n(\nu)\overset{d}{\rightsquigarrow}N(\underline{O},\ (\varkappa_4-1)\Sigma_T(\nu))$ *where*
$\Sigma_T(\nu)=E\left\{h_t^2(\nu)Z_t(\nu)\underline{Z}_t'(\nu)\right\}$ *and* $\varkappa_4=E\left\{e_t^4\right\}$,

(2) $\left(\frac{1}{n}\mathbf{S}_n\right)^{-1}\xrightarrow{P}\Sigma_S^{-1}(\nu)$ *where* $\Sigma_S(\nu)=E\left\{Z_t(\nu)\underline{Z}_t'(\nu)\right\}$

(3) $\sqrt{n}\left(\widehat{\underline{\theta}}_n(\nu)-\underline{\theta}(\nu)\right)\overset{d}{\rightsquigarrow}N(\underline{O},\ (\varkappa_4-1)\Sigma(\nu))$, *where*
$\Sigma(\nu)\ =\Sigma_S^{-1}(\nu)\Sigma_T(\nu)\Sigma_S^{-1}(\nu)$

(4) $\sqrt{n}\left(\widehat{\underline{\theta}}_n-\underline{\theta}\right)\overset{d}{\rightsquigarrow}N(\underline{O},(\varkappa_4-1)\Sigma)$ *where the asymptotic variance-covariance* Σ *is a block matrix with variance matrices are* $\Sigma(\nu)$, $\nu\in\mathbb{S}$ *and covariance matrices are*
$Cov\left(\sqrt{n}\left(\widehat{\underline{\theta}}_n(\nu)-\underline{\theta}(\nu)\right),\sqrt{n}\left(\widehat{\underline{\theta}}_n(\nu')-\underline{\theta}(\nu')\right)\ \right)$, $\nu\neq\nu'\in\mathbb{S}$.

Proof. See subsection 5.2. □

Remark 3.3. In the framework of linear regression models, it is well known that for heteroscedastic observations the LSE might be inefficient and it is outperformed by the weighted LSE ($WLSE$) (See Basawa et al. [4]). In our framework, the $WLSE$ is defined by $\widehat{\underline{\theta}}_n^{(WLSE)}=\left(\sum\limits_{t=1}^{n}\mathbf{Z}_t\widehat{\Omega}_t\mathbf{Z}_t'\right)^{-1}\sum\limits_{t=1}^{n}\mathbf{Z}_t\widehat{\Omega}_t\underline{X}_t^2$ where $\widehat{\Omega}_t$ is a consistent estimator of $\Omega_t:=diag\left\{h_t^{-2}(1),...,h_t^{-2}(s)\right\}$. So, if $\widehat{\underline{\theta}}_n$ is computed in a first step, then $\widehat{\Omega}_t$ can be obtained by replacing $h_t(\nu)$ by $\widehat{a}_0(v)+\sum\limits_{i=1}^{p}\widehat{a}_i(v)X_t^2(\nu-i)$ in Ω_t. Therefore, the two-steps $\widehat{\underline{\theta}}_n^{(WLSE)}$ is consistent and asymptotically normal, i.e., under **H1** (0) and for any $\nu\in\mathbb{S}$,

$\sqrt{n}\left(\widehat{\underline{\theta}}_n^{(WLSE)}(\nu)-\underline{\theta}(\nu)\right)\overset{d}{\rightsquigarrow}N(\underline{O},\ (\varkappa_4-1)\Omega^{-1}(\nu))$ where
$\Omega(\nu):=E\left\{h_t^{-2}(\nu)Z_t(\nu)\underline{Z}_t'(\nu)\right\}$ whenever a slightly stronger moments conditions are verified (see Bose et al. [11])

3.2. **Estimating the asymptotic variance matrix.** For statistical inference problem, the asymptotic variance-covariance matrix Σ has to be estimated. Indeed, the quantity $\varkappa_4$ can be estimated by $\frac{1}{n}\sum_{t=1}^{n}\ e_t^4$ and

3.2.1. *Estimation of the asymptotic matrix* $\Sigma_T(\nu)$. The estimation of the $\Sigma_T(\nu)$ matrix is more complicated. In the literature, two types of estimators are generally employed: heteroskedasticity and autocorrelation consistent (HAC) estimators based on kernel methods (see Basawa et al. [4]) and Francq et al. [17] for a weak PARMAmodels) and the spectral density estimators (see e.g. Berk [5]). Following Boubacar Mainassara et al.[4] , the matrix $\Sigma_T(\nu)$ can be estimated using Berk's [5]) approach. Moreover, an estimator of $\Sigma_T(\nu)$ can be obtained by replacing $h_t(\nu)$ by $\widehat{a}_0(v)+\sum\limits_{i=1}^{p}\widehat{a}_i(v)X_t^2(\nu-i)$ and hence averaging $\frac{1}{n}\sum_{t=1}^{n}\widehat{h}_t(\nu)\,Z_t(\nu)\,Z_t'(\nu)$ to obtain an estimator $\widehat{\Sigma}_T(\nu)$ of $\Sigma_T(\nu)$.

3.2.2. *Estimation of the asymptotic matrix* $\Sigma_S(\nu)$. The matrix $\Sigma_S(\nu)$ can be estimated empirically by the square matrix $\widehat{\Sigma}_S(\nu)$ of order $(p+1)\times(p+1)$ defined by $\frac{1}{n}\mathbf{S}_n(\nu)$ and its convergence to $\Sigma(\nu)$ follows essentially the same arguments as in Boubacar et al. [13].

Example 3.4. Consider the first order 4−periodic $ARCH\,(1)$ model with an $PAR(1)$ representation

$$X_t^2(\nu)=a(v)+b(v)X_t^2(\nu-1)+\eta_t(\nu)\,,\ v\in\mathbb{S} \tag{2.3}$$

and $VAR(1)$ representation

$$\underline{X}_t^2=\Phi_0^{-1}\underline{a}+\Phi_0^{-1}\Phi_1\underline{X}_{t-1}^2+\Phi_0^{-1}\underline{\eta}_t, \tag{2.4}$$

where

$$\Phi_0=\begin{pmatrix}1&0&0&0\\-b(2)&1&0&0\\0&-b(3)&1&0\\0&0&-b(4)&1\end{pmatrix}\quad\text{and}\quad\Phi_1=\begin{pmatrix}0&0&0&b(1)\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}..$$

Noting that Φ_0 is non-singular so, the condition **H** reduces to $b^{(4)}=\prod\limits_{i=1}^{4}b(\nu)<1$ under which the LSE of $\underline{\theta}(\nu)=(a(v),b(v))'$ is given by $\widehat{\underline{\theta}}_n(\nu)=\underline{\theta}(\nu)+\mathbf{S}_n^{-1}(\nu)\,\underline{T}_n(\nu)$ where $\mathbf{S}_n(\nu)=\sum\limits_{t=1}^{n}\underline{Z}_t(\nu)\ \underline{Z}_t'(\nu)$, $\underline{T}_n(\nu)=\sum\limits_{t=1}^{n}\underline{Z}_t(\nu)\,\eta_t(\nu)$ with $\underline{Z}_t(\nu)=\left(1,X_t^2(\nu-1)\right)'$. The conditions ensuring the existence of moments for $\left(\underline{X}_t^2\right)_t$ up to $r-th$ order are given subsection 5.1. Moreover, some tedious computation shows that the hypothesis $\mathbf{H2}\,(\lambda)$ is satisfies and hence the strong consistency of $\widehat{\underline{\theta}}_n(\nu)$ follows. However, the explicit form of the parameters estimates are

$$\begin{cases}\widehat{a}_n(\nu)-a(\nu)=\left(\widehat{Var}(X_t^2(\nu-1))\right)^{-1}\\ \times\left(\left\langle X_t^4(\nu-1)\right\rangle_n\left\langle\eta_t(\nu)\right\rangle_n-\left\langle X_t^2(\nu-1)\right\rangle_n\left\langle X_t^2(\nu-1)\,\eta_t(\nu)\right\rangle_n\right)\\ \qquad\qquad\text{and}\\ \widehat{b}_n(\nu)-b(\nu)=\\ \left(\widehat{Var}(X_t^2(\nu-1))\right)^{-1}\left(\left\langle X_t^2(\nu-1)\,\eta_t(\nu)\right\rangle_n-\left\langle X_t^2(\nu-1)\right\rangle_n\left\langle\eta_t(\nu)\right\rangle_n\right)\end{cases}$$

where $\langle Y_t\rangle_n$ means the empirical mean of the sequence $(Y_t)_t$ and $\widehat{Var}(Y_t)$ its empirical variance. The LSE of $\sigma_\eta^2(\nu):=Var\,(\eta_t(\nu))$ is $\widehat{\sigma}_\eta^2(\nu)=\frac{1}{n-2}\sum_{t=1}^{n}(X_t^2(\nu)-\widehat{a}_n(\nu)-\widehat{b}_n(\nu)\,X_t^2(\nu-1))^2$. Now, suppose that $E\left\{\eta_t^4(\nu)\right\}<+\infty$, then we have $\sqrt{n}\left(\widehat{\underline{\theta}}_n(\nu)-\underline{\theta}(\nu)\right)\overset{d}{\rightsquigarrow}N(\underline{O},(\varkappa_4-1)\,\Sigma(\nu))$ where $\Sigma(\nu)=\Sigma_S^{-1}(\nu)\,\Sigma_T(\nu)\,\Sigma_S^{-1}(\nu)$ with

$$\Sigma_S^{-1}(\nu)=\frac{1}{\mu_{(2)}(\nu-1)-\mu_{(1)}^2(\nu-1)}\begin{pmatrix}\mu_{(2)}(\nu-1)&-\mu_{(1)}(\nu-1)\\-\mu_{(1)}(\nu-1)&1\end{pmatrix}$$

$$\Sigma_T(\nu)=\begin{pmatrix}E\left\{h_t^2(\nu)\right\}&E\left\{h_t^2(\nu)\,X_t^2(\nu-1)\right\}\\E\left\{h_t^2(\nu)\,X_t^2(\nu-1)\right\}&E\left\{h_t^2(\nu)\,X_t^4(\nu-1)\right\}\end{pmatrix},$$

in particular $\sqrt{n}\left(\widehat{a}_n\left(\nu\right)-a\left(\nu\right)\right)\overset{d}{\rightsquigarrow}N(0,(\varkappa_4-1)\,\sigma_a^2\left(\nu\right))$ and
$\sqrt{n}\left(\widehat{b}_n\left(\nu\right)-b\left(\nu\right)\right)\overset{d}{\rightsquigarrow}N(0,(\varkappa_4-1)\,\sigma_b^2\left(\nu\right))$, where

$$\begin{aligned}
\sigma_a^2\left(\nu\right) &= \mu_{(4)}^2\left(\nu-1\right)E\left\{h_t^2\left(\nu\right)\right\}-E\left\{X_t^2\left(\nu-1\right)h_t^2\left(\nu\right)\right\}\mu_{(2)}\left(\nu-1\right)\\
&\quad\times\left(\mu_{(2)}\left(\nu-1\right)-2\mu_{(4)}\left(\nu-1\right)\right)/Var(X_t^2\left(\nu-1\right)),\\
\sigma_b^2\left(\nu\right) &= \left(E\left\{X_t^4\left(\nu-1\right)h_t^2\left(\nu\right)\right\}-2\mu_{(2)}\left(\nu-1\right)E\left\{X_t^2\left(\nu-1\right)h_t^2\left(\nu\right)\right\}\right.\\
&\quad\left.+\,\mu_{(2)}^2\left(\nu-1\right)E\left\{h_t^2\left(\nu\right)\right\}\right)/Var(X_t^2\left(\nu-1\right)),
\end{aligned}$$

with

$$\begin{aligned}
E\left\{h_t^2\left(\nu\right)\right\} &= a^2\left(\nu\right)+2a\left(\nu\right)b\left(\nu\right)\mu_{(2)}\left(\nu-1\right)+b^2\left(\nu\right)\mu_{(4)}\left(\nu-1\right),\\
E\left\{X_t^2\left(\nu-1\right)h_t^2\left(\nu\right)\right\} &= a^2\left(\nu\right)\mu_{(2)}\left(\nu-1\right)+2a\left(\nu\right)b\left(\nu\right)\mu_{(4)}\left(\nu-1\right)\\
&\quad+b^2\left(\nu\right)\mu_{(6)}\left(\nu-1\right)\\
E\left\{X_t^4\left(\nu-1\right)h_t^2\left(\nu\right)\right\} &= a^2\left(\nu\right)\mu_{(4)}\left(\nu-1\right)+2a\left(\nu\right)b\left(\nu\right)\mu_{(6)}\left(\nu-1\right)\\
&\quad+b^2\left(\nu\right)\mu_{(8)}\left(\nu-1\right)
\end{aligned}$$

in which $\mu_{(i)}$ are the moments of $\left(\underline{X}_t^2\right)_t$ defined in subsection 5.1 The unknown variances $\widetilde{\sigma}_a^2\left(\nu\right)=\left(\varkappa_4-1\right)\,\sigma_a^2\left(\nu\right)$ and $\widetilde{\sigma}_b^2\left(\nu\right)=\left(\varkappa_4-1\right)\sigma_b^2\left(\nu\right)$ can be consistency estimated by replacing the parameters by their estimates and the theoretical moments by empirical one to obtain the estimates $\widehat{\widetilde{\sigma}}_a^2\left(\nu\right)$ and $\widehat{\widetilde{\sigma}}_a^2\left(\nu\right)$. These estimates can be used to evaluate the standard errors $SE_a\left(\nu\right)=\widehat{\widetilde{\sigma}}_a^2\left(\nu\right)/\sqrt{n}$ and $SE_b\left(\nu\right)=\widehat{\widetilde{\sigma}}_b^2\left(\nu\right)/\sqrt{n}$ for each $v\in\mathbb{S}$ often required to build the confidence interval for each parameter. In end, the conventional $t-$statistic of estimates $\widehat{\widetilde{\sigma}}_a^2\left(\nu\right)$ and $\widehat{\widetilde{\sigma}}_a^2\left(\nu\right)$ defined by

$$\left\{\begin{array}{l}
t_a\left(\nu\right)=\frac{\left(\widehat{a}_n(v)-a(v)\right)}{\widehat{\sigma}_\eta(\nu)}\left(\sum_{t=1}^{n-1}\left(X_t^2(\nu-1)-\left\langle X_t^2(\nu-1)\right\rangle_n\right)^2/\sum_{t=1}^{n-1}X_t^4(\nu-1)\right)^{1/2}\\
t_b\left(\nu\right)=\left(\widehat{b}_n(v)-b(v)\right)\left(\sum_{t=1}^{n-1}\left(X_t^2\left(\nu-1\right)-\left\langle X_t^2\left(\nu-1\right)\right\rangle_n\right)^2\right)^{1/2}/\widehat{\sigma}_\eta\left(\nu\right)
\end{array}\right.$$

are Gaussian with zero means and finite variances.

Remark 3.5. It is worth noting that the LSE is asymptotically equivalent to the Yule-Walker estimator of PAR (2.3) and to the Wittle estimator studied by Straumann [27].

3.3. **Limiting distributions in integrated case.** As already mentioned in Remark 2.1, that the process $\left(X_t^2\right)_t$ is said to be periodically integrated if the characteristic equation $\det\left(\mathbf{I}_{(s)}-\sum\limits_{i=0}^{p^*}\mathbf{A}_iz^i\right)=0$, has at least a root equal to 1. For simplicity of exposition, we consider the $PARCH(1)$ with $PAR(1)$ and $VAR(1)$ representations given by (2.3) and (2.4) respectively. In this case, it is straightforward to see from the $VAR(1)$ representation (2.4) that $b^{(s)}=\prod_{j=1}^{s}b(j)$ is the single root of $\det\left(\mathbf{I}_{(s)}-A_0-A_1z\right)=0$ and hence if $b^{(s)}=1$, the model (2.3) is periodically integrated of order one. This condition can be obtained straightforwardly from $PARCH(1)$ representations (2.3) . Indeed, recursing (2.3)., $s-1-$time we obtain for any $v\in\mathbb{S}$,

$$X_t^2\left(\nu\right)=\mu(v)+b^{(s)}X_{t-1}^2\left(\nu\right)+V_t\left(\nu\right), \tag{2.5}$$

where $V_t(\nu) = \eta_t(\nu) + \sum_{i=1}^{s-1} b^{(i)} \eta_t(\nu - i)$ and $\mu(\nu) = a(\nu) + \sum_{i=1}^{s-1} b^{(i)}\; a(\nu - i)$. The representation (2.5) is is a $PARMA(1, s-1)$ process driven by a martingale difference with seasonally varying intercept coefficient $\mu(\nu)$. However, the vector of unknown parameters $\underline{\beta}(\nu) = \left(\mu(\nu), b^{(s)}, \nu = 1, ..., s\right)'$ involved in model (2.5) maybe estimated by minimizing the sum of squares errors $S(\underline{\beta}) = \sum_{t=1}^{ns} V_t^2$. So, we define the LSE of $\mu(v)$ and $b^{(s)}$ as

$$\begin{aligned}
\underline{\widehat{\beta}}_n(\nu) - \underline{\beta}(\nu) &= \begin{pmatrix} \widehat{\mu_n}(\nu) - \mu(\nu) \\ \widehat{b}_n^{(s)} - b^{(s)} \end{pmatrix} \\
&= \begin{pmatrix} n & \sum_{t=1}^{n} X_{t-1}^2(\nu) \\ \sum_{t=1}^{n} X_{t-1}^2(\nu) & \sum_{t=1}^{n} X_{t-1}^4(\nu) \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{n} V_t(\nu) \\ \sum_{t=1}^{n} X_{t_{t-1}}^2 V_t(\nu) \end{pmatrix} \\
&= \left(\sum_{t=1}^{n} \underline{X}_t^2(\nu)\, \underline{X}_t^{2\prime}(\nu)\right)^{-1} \underline{T}_n(\nu)
\end{aligned}$$

The proper standardization is to premultiply $\underline{\widehat{\beta}}_n(\nu) - \underline{\beta}(\nu)$ by the diagonal matrix $D_n = diag(\sqrt{n}, n)$, hence

$$D_n\left(\underline{\widehat{\beta}}_n(\nu) - \underline{\beta}(\nu)\right) = \left(D_n^{-1} \sum_{t=1}^{n} \underline{X}_t^2(\nu)\, \underline{X}_t^{2\prime}(\nu)\, D_n^{-1}\right)^{-1} D_n^{-1} \underline{T}_n(\nu)$$

$$= \begin{pmatrix} 1 & n^{-3/2} \sum_{t=1}^{n} X_{t-1}^2(\nu) \\ n^{-3/2} \sum_{t=1}^{n} X_{t-1}^2(\nu) & n^{-2} \sum_{t=1}^{n} X_{t-1}^4(\nu) \end{pmatrix}^{-1} \begin{pmatrix} n^{-1/2} \sum_{t=1}^{n} V_t(\nu) \\ n^{-1} \sum_{t=1}^{n} X_{t_{t-1}}^2 V_t(\nu) \end{pmatrix}$$

The asymptotic properties of $D_n\left(\underline{\widehat{\beta}}_n(\nu) - \underline{\beta}(\nu)\right)$ are relies on the theory of weak convergence of functional of a Wiener process $(W(t))_t$ on $D = D[O, 1]$, the space of all real valued functions on $[0, 1]$ that are right continuous at each point of $[0, 1]$. The main advantage of the approach is that the results hold for a very wide class of error processes in the model (2.5). However, we have as $n \to \infty$, by Donsker's invariant principle whereas, Philips et al. [24] have shown that the following convergence results hold jointly

$$\begin{aligned}
n^{-3/2} \sum_{t=1}^{n} X_{t-1}^2(\nu) &\Longrightarrow \sigma_S(\nu) \int_0^1 W_\nu(r)\, dr, \\
n^{-2} \sum_{t=1}^{n} X_{t-1}^4(\nu) &\Longrightarrow \sigma_S^2(\nu) \int_0^1 W_\nu^2(r)\, dr, \\
n^{-1/2} \sum_{t=1}^{n} V_t(\nu) &\Longrightarrow \sigma_S(\nu)\, W_\nu(1) = N(0, \sigma_S^2(\nu)),, \\
n^{-1} \sum_{t=1}^{n} V_t(\nu)\, X_{t-1}^2(\nu) &\Longrightarrow \tfrac{1}{2}\left(\sigma_S^2(\nu)\, W_\nu^2(1) - \sigma_V^2(\nu)\right)
\end{aligned}$$

where $\sigma_S^2(\nu) = \lim\limits_{n\to\infty} E\left\{\frac{1}{n}\left(\sum_{t=1}^{n} X_{t-1}^2(\nu)\right)^2\right\}$, $\sigma_V^2(\nu) = \lim\limits_{n\to\infty} E\left\{\frac{1}{n} V_n^2(\nu)\right\}$. So, by the continuous mapping theorem

$$\begin{pmatrix} n\left(\widehat{\mu}_n(\nu) - \mu(\nu)\right) \\ \sqrt{n}\left(\widehat{b}_n^{(s)} - b^{(s)}\right) \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & \sigma_S(\nu) \int_0^1 W_\nu(r)\, dr \\ \sigma_S(\nu) \int_0^1 W_\nu(r)\, dr & \sigma_S^2(\nu) \int_0^1 W_\nu^2(r)\, dr \end{pmatrix}^{-1} \begin{pmatrix} \sigma_S(\nu)\, W_\nu(1) \\ \frac{1}{2}\left(\sigma_S^2(\nu)\, W_\nu^2(1) - \sigma_V^2(\nu)\right) \end{pmatrix}$$

$$= \frac{1}{C(\nu)\,\sigma_S^2(\nu)} \begin{pmatrix} \sigma_S^2 \int_0^1 W_\nu^2(r)\,dr & -\sigma_S \int_0^1 W_\nu(r)\,dr \\ -\sigma_S \int_0^1 W_\nu(r)\,dr & 1 \end{pmatrix} \times \begin{pmatrix} \sigma_S(\nu)\, W_\nu(1) \\ \frac{1}{2}\left(\sigma_S^2(\nu)\, W_\nu^2(1) - \sigma_V^2(\nu)\right) \end{pmatrix}$$

where $C(\nu) = \int_0^1 W_\nu^2(r)\,dr - \left(\int_0^1 W_\nu(r)\,dr\right)^2$, It follows by elementary calculations that as $n \to \infty$

$$\begin{cases} (\widehat{\mu}_n(\nu) - \mu(\nu)) \Longrightarrow \frac{\sigma_S(\nu)}{C(\nu)} \left\{ W_\nu(1) \int_0^1 W_\nu^2(r)\,dr - \frac{1}{2}\left(W_\nu^2(1) - \sigma_V^2(\nu)\right) / \right. \\ \left. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma_S^2(\nu) \int_0^1 W_\nu(r)\,dr \right\} \\ \sqrt{n}\left(\widehat{b_n^{(s)}} - 1\right) \Longrightarrow \frac{1}{C(\nu)} \left\{ \frac{1}{2}\left(W_\nu^2(1) - \sigma_V^2(\nu)/\sigma_S^2(\nu)\right) - W_\nu(1) \int_0^1 W_\nu(r)\,dr \right\} \end{cases}$$

We define the $t-$statistic:s

$$\begin{cases} t_\mu(\nu) = (\widehat{\mu}_n(v) - \mu(v)) \left(\sum_{t=1}^n \left(X_{t-1}^2(\nu) - \left\langle X_{t-1}^2(\nu)\right\rangle_n\right)^2 / \sum_{t=1}^n X_{t-1}^4(\nu)\right)^{1/2} / \widehat{\sigma}_V(\nu) \\ t_b = \left(\widehat{b_n^{(s)}} - 1\right) \left(\sum_{t=1}^{n-1} \left(X_{t-1}^2 - \left\langle X_{t-1}^2\right\rangle_n\right)^2\right)^{1/2} / \widehat{\sigma}_V \end{cases} \tag{2.6}$$

where $\langle Y_t\rangle_n$ is the empirical mean of the sequence $(Y_t)_t$ and $\widehat{\sigma}_V(\nu)$ is the standard error associated to the model (2.5). Both t_b and $t_\mu(\nu)$ in (2.6) have been suggested as test statistics for detecting the presence of a unit root in (2.5). The asymptotic behavior of statistics in (2.6) under $b^{(s)} = 1$ are given in the next theorem.

Theorem 3.6. *Let $(X_t)_t$ be a $PARCH(1)$ process with $PAR(1)$ and $VAR(1)$ representations given by (2.3) and (2.4) respectively and let $(W(t))_t$ be a standard Brownian motion. Then, as $n \to +\infty$*

(1) $t_b \Longrightarrow \frac{\sigma_S^2}{2\sigma_\eta^2}\left(W^2(1) - \sigma_\eta^2/\sigma_S^2\right) / \left(\int_0^1 W^2(t)\,dt\right)^{1/2}$

(2) $t_\mu(\nu) \Longrightarrow \frac{\sigma^2}{\sigma_\eta^2}\left(\int W_\nu^2(r)\,dr \int W_\nu^2(r)\,dr\right)^{-1/2} \left(\int_0^1 W_\nu(t)\,dW_\nu^2(t)\,dt\right)^{1/2}$

Proof. See Philips et al. [24] □

Remark 3.7. Regarding the above results of Theorem 3.6, it is worth noting that (i) $\widehat{b_n^{(s)}}$ is super-consistent, that is $\widehat{b_n^{(s)}} \xrightarrow{P} b^{(s)}$ at rate n instead of the usual rate $\sqrt{n}$. (ii) $\widehat{b_n^{(s)}}$ is not asymptotically normally distributed and t_b is not asymptotically standard normal. (iii) The limiting distribution of t_b is called the Dickey-Fuller (DF) distribution and does not have a closed form representation. Consequently, quantiles of the distribution must be computed by numerical approximation or by simulation. (iv) Since the normalized bias $n\left(\widehat{b_n^{(s)}} - 1\right)$ has a well-defined limiting distribution that does not depend on nuisance parameters, it can also be used as a test statistic for the null hypothesis $H0:\ b^{(s)} = 1$.

Remark 3.8. Regarding the asymptotic properties of parameter estimates $\widehat{\mu}_n\,(\nu)$ listed in Theorem 3.6, it is obvious that (i) $\widehat{\mu}_n\,(\nu)$ is super-consistent, that is for each ν $\widehat{\mu}_n\,(\nu) \xrightarrow{P} 0$ at rate $\sqrt{n}$. (ii) Neither $\widehat{\mu}_n\,(\nu)$ nor t_μ are asymptotically standard normal.

4. The statistical Tests

The statistical tests play an important role in seasonal and/or heteroscastic models. As an essential special case, among others, tests of periodicity, heteroscasticity and unit root, etc..., occupy a central place. So, in statistical literature, several methods were proposed related to some estimate procedures (see for instance White [30] and the references therein).

4.1. **Wald's test.** The most popular test related to LSE is the usual Wald statistics that allow us to test a null hypothesis $H0$ against an alternative hypothesis $H1$ in the form

$$H0 : R\underline{\theta} = \underline{\theta}_0 \text{ against } H1 : R\underline{\theta} = \underline{\theta}_1 \text{ with } \underline{\theta}_0 \neq \underline{\theta}_1$$

where R is a given $r \times s\,(p+1)$ matrix of rank $r \leq s\,(p+1)$ and $\underline{\theta}_0$ is a specific $r \times 1$ vector. Under the null $H0$ subject to the conditions of theorem 3.2 , $\sqrt{n}\left(R\widehat{\underline{\theta}}_n - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}(\underline{O}, R\,\Sigma R')$. Since Σ, is not singular, then the asymptotic matrix variance-covariance involved in non-singular, and we have the following result.

Corollary 4.1. *Under the conditions of Theorem 3.2, we have*

$$\widehat{W}\left(\widehat{\underline{\theta}}_n\right) := n\left(R\widehat{\underline{\theta}}_n - \underline{\theta}_0\right)'\left(R\widehat{\Sigma}_n R'\right)^{-1}\left(R\widehat{\underline{\theta}}_n - \underline{\theta}_0\right) \rightsquigarrow \chi^2_{(r)}.$$

where $\widehat{\Sigma}_n$ is some consistent estimator of Σ. On the other hand, under the alternative $H1$ we have as $n \to +\infty$

$$\frac{W\,(\underline{\theta})}{n} \xrightarrow{P} (R\underline{\theta} - \underline{\theta}_0)'\left(R\Sigma R'\right)^{-1}(R\underline{\theta} - \underline{\theta}_0) > 0. \tag{3.1}$$

Proof. Is a consequence of the Theorem 3.2, □

We first note that the statistic $\widehat{W}\left(\widehat{\underline{\theta}}_n\right)$ is the Wald- test of the null $H0$. Indeed, given the size $\alpha \in\,]0,1[$, choose a critical value β such that under the null $H0$, $P\left(\widehat{W}\left(\widehat{\underline{\theta}}_n\right) > \beta\right) \to \alpha$. Then

$$\begin{cases} \text{Accepted } H0\text{: if } \widehat{W}\left(\widehat{\underline{\theta}}_n\right) \leq \beta \\ \text{Rejected } H0 \text{ (in favor of the alternative): if } \widehat{W}\left(\widehat{\underline{\theta}}_n\right) > \delta \text{ with } \delta \geq \beta \end{cases} \tag{3.2}$$

The threshold δ and β can be determined to ensure that the probabilities of different types of error are as desired. This test is consistent due to (3.1). In the case when R is a raw vector, so $\underline{\theta}_0$ is a scalar, we can modify the statistics $\widehat{W}\left(\widehat{\underline{\theta}}_n\right)$ to $\widehat{T}_n = \sqrt{n}\left(R\widehat{\underline{\theta}}_n - \underline{\theta}_0\right)'\left(R\widehat{\Sigma}_n R'\right)^{-1/2}$, so $\widehat{T}_n \overset{D}{\rightsquigarrow} \mathcal{N}(0,1)$, whereas under the alternative we have $\frac{\widehat{T}_n}{\sqrt{n}} \xrightarrow{P} \left(R\widehat{\Sigma}_n R'\right)^{-1/2}\left(R\widehat{\underline{\theta}}_n - \underline{\theta}_0\right) \neq 0$. These results can be used to construct a two-sided or a one-sided test. Notice that the above properties are not true when Σ is singular and/or in an integrated case, the reason is that in this case some coefficients or linear combinations of them are estimated more efficiently with a faster convergence rate then $\sqrt{n}$.

Remark 4.2. Regardless of the sample size n, the probability of detecting the null $H0$ to be false cannot be made $\underline{\theta}_0 \neq \underline{\theta}_1$, To obtain a procedure with guaranteed power for $\underline{\theta}_0 \neq \underline{\theta}_1$, the sample size must be made to depend on $\underline{\theta}$. This can be achieved by a sequential procedure, with the stopping rule depending on an estimate of $\underline{\theta}$, but not with a procedure of fixed sample size. Moreover, we are also concerned to changepoint detected an/or the value of periodic value s so the resort to sequential analysis is however imposed (interested readers are advised see the monograph by Tartakovsky et al. [28]). This important issue is left to future separated paper.

5. Appendix

First supplementary material related to this article can be added here.

5.1. **Moments .** To derive an explicit expansion of the moments of the $PARCH(p)$ model, an appropriate autoregressive with random coefficients's representation on $\left(X_t^2\right)_t$ is requested, i.e.,

$$\underline{X}_t^2(\nu) = A_t(\nu)\,\underline{X}_t^2(\nu-1) + \underline{\xi}_t(\nu) \tag{1}$$

obtained by multiplying both sides of the second equation in (1.2) by e_t^2, where $A_t(\nu)$ is the $p \times p-$ companion random matrices and $\underline{\xi}_t(\nu)$ is an appropriate vector of innovations. Then, iterate (1) $(s-1)-$time to get $\underline{X}_t^2(\nu) = \underline{\widetilde{\xi}}_t(\nu) + A_t^{(s)}\underline{X}_{t-1}^2(\nu)$, where $\underline{\widetilde{\xi}}_t(\nu) = \sum\limits_{k=0}^{s-1}\left\{\prod\limits_{i=0}^{k-1} A_t(\nu-i)\right\}\underline{\xi}_t(\nu-k)$ and $A_t^{(s)} = \prod\limits_{\nu=1}^{s} A_t(\nu-i)$, where as usual, empty products are set equal to one. Therefore, the process $\left(\underline{X}_t^2(\nu)\right)_{t\in\mathbb{Z}}$ is causal, strictly stationary in $\mathbb{L}_1$ if and only if $\lambda_{(1)}^{(s)} = \rho\left(E\left\{A_t^{(s)}\right\}\right) < 1$. Moreover, if $E\left\{e_t^{2m}\right\} < +\infty$, then the process $\left(\underline{X}_t^2\right)_{t\in\mathbb{Z}}$ is causal, belonging in $\mathbb{L}_m$ if and only if $\lambda_{(m)}^{(s)} = \rho\left(E\left\{A_t^{(s)\otimes m}\right\}\right) < 1$.

Example 5.1. Suppose that $p=1$, and consider the representation $X_t^2(\nu) = a(\nu)\,e_t^2(\nu)$ $X_t^2(\nu-1) + \xi_t(\nu)$ associated to an $PARCH(1)$, with $\xi_t(\nu) = a_0(\nu)\,e_t^2(\nu)$. Assume that $\sigma_{(m)} = E\left\{e_t^{2m}\right\} < +\infty$ with $\sigma_{(1)} = 1$ for any positive integer m. However, the condition $\lambda_{(m)}^{(s)} < 1$ reduces to $\lambda_{(m)}^{(s)} = \sigma_{(m)}^s \prod\limits_{v=1}^{s} a^m(\nu) < 1$ Set $\mu_{(m)}(\nu) = E\left\{X_t^{2m}(\nu)\right\}$, then $X_t^{2m}(\nu) = \sum\limits_{j=0}^{m} C_m^j a^j(\nu)(\nu)\,e_t^{2j} X_t^{2j}(\nu-1)\,\xi_t^{m-j}(\nu)$, so, $\mu_{(m)}(\nu) = \sigma_{(m)} \sum\limits_{j=0}^{m-1} C_m^j a^j(\nu)\,a_0^{m-j}(\nu)\,\mu_{(j)}(\nu-1) + \sigma_{(m)} a^m(\nu)\,\mu_{(m)}(\nu-1)$. Recursing the above equation $s-$times and requiring $\mu_{(m)}(0) = \mu_{(m)}(s)$ gives

$$\mu_{(m)}(s) = \sum_{j=1}^{s} \sigma_{(m)}^{s-j}\left\{\prod_{k=j+1}^{s} a(k)\right\} m_{(m)}(j)\,/(1-\sigma_{(m)}^s \prod_{j=1}^{s} a(j)),$$

where $m_{(m)}(\nu) = \sigma_{(m)} \sum\limits_{j=0}^{m-1} C_m^j a^j(\nu)\,a_0^{m-j}(\nu)\,\mu_{(j)}(\nu-1)$. Further manipulation, provide $\mu_{(m)}(\nu) = \left\{\sigma_{(m)}^{\nu} \prod\limits_{j=1}^{\nu} a(j)\right\}\mu_{(m)}(s) + \sum\limits_{j=1}^{\nu} \sigma_{(m)}^{\nu-k}\left\{\prod\limits_{k=j+1}^{\nu} a(k)\right\} m_{(m)}(j)$ for any $\nu \in \{1,...,s-1\}$. In particular,

(1) If $m=1$, then in this case $\lambda_{(1)}^{(s)}<1$ reduces to $\prod\limits_{v=1}^{s} a(\nu)<1$ and

$$\mu_{(1)}(\nu)=\begin{cases}\sum\limits_{j=1}^{s}\left\{\prod\limits_{k=j+1}^{s} a(k)\right\} a_0(j)/(1-\prod\limits_{k=1}^{s} a(k)) \text{ if } \nu=s\\ \sum\limits_{j=1}^{\nu}\left\{\prod\limits_{k=j+1}^{\nu} a(k)\right\} a_0(\nu)+\left\{\prod\limits_{j=1}^{\nu} a(j)\right\}\mu_{(1)}(s) \text{ if } \nu\in\{1,...,s-1\}\end{cases}$$

(2) If $m=2$, in this case $\lambda_{(2)}^{(s)}<1$ reduces to $\sigma_{(2)}^{s}\prod\limits_{v=1}^{s} a^2(\nu)<1$ and

$$\begin{cases}\sum\limits_{j=1}^{s}\sigma_{(2)}^{s-j}\left\{\prod\limits_{k=j+1}^{s} a^2(k)\right\} m_{(2)}(j)/(1-\sigma_{(2)}^{s}\prod\limits_{v=1}^{s} a^2(\nu)) \text{ if } \nu=s\\ \sum\limits_{j=1}^{\nu}\sigma_{(2)}^{\nu-j}\left\{\prod\limits_{k=j+1}^{\nu} a^2(k)\right\} m_{(2)}(j)+\sigma_{(2)}^{\nu}\left\{\prod\limits_{j=1}^{\nu} a^2(j)\right\}\mu_{(2)}(s) \text{ if }\\ \qquad \nu\in\{1,...,s-1\}\end{cases}$$

where $m_{(2)}(\nu)=2a(\nu)\,a_0(\nu)\,\mu_{(1)}(\nu-1)+a_0^2(\nu)$.

(3) Let $\gamma_\nu(h)=Cov(X_t^2(\nu),X_t^2(\nu-h)$ be the season ν covariance function at lag $h\geq 0$, then $\gamma_\nu(h)=\left\{\prod\limits_{j=0}^{h-1} a(j)\right\}\left(\mu_{(2)}(\nu-h)-\mu_{(1)}^2(\nu-h)\right)$.

5.2. **Lemmas and proofs .** Before proceeding, the proofs of our main results, it is important to start with some discussions. The LSE of $\underline{\theta}$ given by $\widehat{\underline{\theta}}_n=\mathbf{S}_n^{-1}\underline{S}_n=\underline{\theta}+\mathbf{S}_n^{-1}\underline{T}_n$, provided that $\mathbf{S}_n$ is non-singular, hence for $\widehat{\underline{\theta}}_n$ to converge as $n\to\infty$, to $\underline{\theta}$ in quadratic mean (and hence in probability), it is sufficient that $\mathbf{S}_n^{-1}\xrightarrow{a.s} O$, as $n\to\infty$. This result has been showed by Lai et al [21] under the hypothesis $\mathbf{H2}(\lambda)$ and $\mathbf{H3}(\alpha)$. Moreover, Lai et al [21] have gave example where $\log\lambda_{\max}^{(v)}(n)/\lambda_{\min}^{(v)}(n)$ converges *a.s.*, to finite nonzero limit but $\widehat{\underline{\theta}}_n(\nu)$ faille to be strongly consistent. The proof of above Theorem needs some intermediate results gathered in the following Lemmas

Lemma 5.2. *Let* $\mathbf{W}$ *be a matrix* $q\times q$ *and* $\underline{\varphi}$ *be a vector of* $\mathbb{R}^q$. *If the matrix* $\mathbf{V}=\mathbf{W}+\underline{\varphi}\underline{\varphi}'$ *is invertible then*

(1) $\underline{\varphi}'\mathbf{V}^{-1}\underline{\varphi}=(\det(\mathbf{V}))^{-1}(\det\mathbf{V}-\det\mathbf{W})$.

(2) *Define* $\mathbf{V}_n=\sum_{k=1}^{n}\underline{\varphi}_k\underline{\varphi}_k'$ *where* $\left(\underline{\varphi}_k\right)_{1\leq k\leq n}$ *is a sequence of vectors in* $\mathbb{R}^q$ *and* $\lambda_{\max}(n)$ *(resp.* $\lambda_{\min}$*) denote the maximum (resp. minimum) eigenvalue of* $\mathbf{V}_n$. *Assume that* $\mathbf{V}_{n_0}$ *is invertible for some positive integer* n_0, *then* $(\lambda_{\max}(n))_n$ *is an increasing sequence and* $\mathbf{V}_n$ *is invertible for all* $n\geq n_0$ *Moreover, if* $\lim\limits_{n\to\infty}\lambda_{\max}(n)\xrightarrow{a.s}\infty$, *then* $\sum_{k=1}^{n}\underline{\varphi}_k\mathbf{V}_k^{-1}\underline{\varphi}_k'=O_p(\log\lambda_{\max}(n))$,
will $\lim\limits_{n\to\infty}\lambda_{\max}(n)<+\infty$, *implies* $\sum_{k=1}^{\infty}\underline{\varphi}_k\mathbf{V}_k^{-1}\underline{\varphi}_k'<+\infty$.

Lemma 5.3. (1) *Let* $(\epsilon_t)_{t\in\mathbb{N}}$ *be a martingale difference sequence with respect to the increasing sequence of* $\sigma-$*field* $(\mathcal{G}_t)_t$ *such that* $\sup\limits_t E\left\{\epsilon_t^2|\mathcal{G}_{t-1}\right\}<+\infty$, *a.s. Let* $(\nu_t)_{t\in\mathbb{N}}$ *be a sequence* $\mathcal{G}_{t-1}-$*measurable random variables. Then*

i.: *On* $\left\{w:\sum_{t=1}^{\infty}\nu_t^2<\infty\right\}$, *the series* $\sum_{t=1}^{+\infty}\epsilon_t\nu_t$ *converges a.s.*

ii.: *On* $\left\{w : \sum_{t=1}^{\infty} \nu_t^2 = \infty\right\}$,
$\left(\left(\sum_{t=1}^{n} \nu_t^2\right)^{1/2} \left(\log\left(\sum_{t=1}^{n} \nu_t^2\right)\right)^{\tau}\right)^{-1} \sum_{t=1}^{n} \epsilon_t \nu_t \xrightarrow{a.s} 0$, *as* $n \to +\infty$, *for evry* $\tau > 1/2$ *and hence* $\sum_{t=1}^{n} \epsilon_t \nu_t = o_p\left(\sum_{t=1}^{n} \nu_t^2\right) + O_p(1)$

iii.: *On* $\{w : \sum |\nu_t| < \infty\}$, *the series* $\sum_{t=1}^{+\infty} \epsilon_t^2 |\nu_t|$ *converges a.s., and on* $\{w : \sum |\nu_t| = \infty\}$, $\left(\sum_{t=1}^{n} |\nu_t|\right)^{-r} \sum_{t=1}^{n} \epsilon_t^2 |\nu_t| \to 0$, *a.s., as* $n \to +\infty$ *for any* $r > 1$.

iv.: *If* $\sup_t E\{|\epsilon_t|^{\alpha} |\mathcal{G}_{t-1}\} < +\infty$, *a.s. for some* $\alpha > 2$, *then the last assertion in (iii) can be enhanced on* $\left\{w : \sup_t |\nu_t| < \infty\right\}$ *to* $\limsup_{n\to\infty} \left(\sum_{t=1}^{n} |\nu_t|\right)^{-1} \sum_{t=1}^{n} \epsilon_t^2 |\nu_t| < +\infty$, *a.s. as* $n \to +\infty$.

Proof.
(1) The first assertion follows from the matrix inversion lemma and upon the observation that $\det \mathbf{W} = \det(\mathbf{V} - \underline{\varphi}\underline{\varphi}') = \det(\mathbf{V}) \det\left(\mathbf{I}_{(q)} - \mathbf{V}^{-1}\underline{\varphi}\underline{\varphi}'\right) = \det(\mathbf{V})(1 - \underline{\varphi}'\mathbf{V}^{-1}\underline{\varphi})$.

(2) To prove 2. Since $\mathbf{V}_n - \mathbf{V}_{n-1} = \underline{\varphi}_n\underline{\varphi}_n'$ is a positive definite matrix, then $\lambda_{\max}(n) \geq \lambda_{\max}(n-1)$, $\lambda_{\min}(n) \geq \lambda_{\min}(n-1)$ and $\mathbf{V}_n$ is invertible for any $n \geq n_0$. By the assertion(1), we have $\sum_{k=n_0}^{n} \underline{\varphi}_k' \mathbf{V}_k^{-1} \underline{\varphi}_k = \sum_{k=n_0}^{n} (\det \mathbf{V}_k - \det \mathbf{V}_{k-1}) / \det(\mathbf{V}_k)$. So, if $\lim_{n\to\infty} \lambda_{\max}(n) \xrightarrow{a.s} +\infty$, then for $n \geq n_0$, $\det(\mathbf{V}_n) \geq \lambda_{\min}^{q-1}(n) \lambda_{\max}(n) \to \infty$ and hence,
$\sum_{k=n_0}^{n} \underline{\varphi}_k' \mathbf{V}_k^{-1} \underline{\varphi}_k = O_p(\log \det \mathbf{V}_n) = O_p(\log \lambda_{\max})$. On the other hand, if $\lim_{n\to\infty} \lambda_{\max}(n) < +\infty$, then it follows that
$\sum_{k=n_0}^{\infty} \underline{\varphi}_k' \mathbf{V}_k^{-1} \underline{\varphi}_k \leq \lambda_{\min}^{q-1}(n) \lambda_{\max}(n) \sum_{k=n_0}^{\infty} (\det \mathbf{V}_k - \det \mathbf{V}_{k-1}) < +\infty$. (see chapter 6 in Caines [16] for further detail,)

(3) To prove the assertion (3), we observe that the sequence $\left\{\sum_{t=1}^{n} \epsilon_t \nu_t, \mathcal{G}_n\right\}_{n\geq 1}$ need not be a martingale because $E\{\epsilon_t \nu_t\}$ may not exist. However, by choosing sufficiently large constants a_n such that $P(|\nu_n| > a_n) \leq n^{-2}$, so $P(\nu_n = \nu_n^*$ for all large $n) = 1$ where $\nu_n^* = \nu_n \mathbb{I}_{\{|\nu_n| \leq a_n\}}$. Moreover, $E\{\epsilon_t \nu_t^*\} < +\infty$ and hence $\left\{\sum_{t=1}^{n} \epsilon_t \nu_t^*, \mathcal{G}_n\right\}_{n\geq 1}$ is a martingale. However, (i) and (ii) follow from the so-called local convergence theorem and the strong law for martingales. The assertion (iii) follows from Freedman [18], *p.* 919, Kronecker lemma and assertion (i). Assertion (iv) follows by Lai, et al. [21].

□

Now we are in a position to apply the results of Lemma 5.2 in our formulation.

Lemma 5.4. *Consider* $(X_t)_{t\in\mathbb{Z}}$ *a* $PARCH(p)$ *model admitting an AR representation* (1.4). *Let* $N = \inf\{n : \mathbf{S}_n(v)$ *be invertible*$\}$ *and suppose that* $N < +\infty$, *a.s. For all* $v \in \mathbb{S}$ *and* $n \geq N+1$, *let the quadratic form* $Q_n(v) = \underline{T}_n'(v)\mathbf{S}_n^{-1}(v)\underline{T}_n(v)$, *then* $\lambda_{\max}^{(v)}(n)$ *is increasing in* n *and under* $\mathbf{H1}(0)$ *we have almost surely,*

(1) $Q_n(v) = O_p(1)$ *on* $\left\{w : \lim_{n\to\infty} \lambda_{\max}^{(v)}(n) < +\infty\right\}$, *i.e.,* $Q_n(v)$ *is bounded a.s. for all* $v \in \mathbb{S}$.

(2) $Q_n(v) = o_p\left(\left(\log \lambda_{\max}^{(v)}(n)\right)^{1+\delta}\right)$ *on* $\left\{w : \lim_{n\to\infty} \lambda_{\max}^{(v)}(n) \xrightarrow{a.s} \infty\right\}$ *for all* $\delta > 0$.

(3) *Under* $\mathbf{H1}(\varepsilon)$ *for some* $\varepsilon > 0$, $Q_n(v) = O_p(\log \lambda_{\max}^{(v)}(n))$ *on*

$\left\{w : \lim_{n\to\infty} \lambda_{\max}^{(v)}(n) \xrightarrow{a.s} \infty\right\}$

Proof. Using the identity for $n \geq N+1$,

$$\begin{aligned}\mathbf{S}_n^{-1}(v)\underline{Z}_n(v) &= \mathbf{S}_{n-1}^{-1}(v)\underline{Z}_n(v) - P_n(v)\left(\underline{Z}_n'(v)\mathbf{S}_{n-1}^{-1}(v)\underline{Z}_n(v)\right)\mathbf{S}_{n-1}^{-1}(v)\underline{Z}_n(v) = \\ & P_n(v)\mathbf{S}_{n-1}^{-1}(v)\underline{Z}_n(v),\end{aligned}$$

where $P_n(v) := \left(1+\underline{Z}_n'(v)\mathbf{S}_{n-1}^{-1}(v)\underline{Z}_n(v)\right)^{-1}$, we obtain the following recurrence. For all $n \geq N+1$

$$\begin{aligned}Q_n(v) &= \underline{T}_{n-1}'(v)\mathbf{S}_n^{-1}(v)\underline{T}_{n-1}(v) + 2\eta_n(v)\underline{Z}_n'(v)\mathbf{S}_n^{-1}(v)\underline{T}_{n-1}(v) + \eta_n^2(v)\underline{Z}_n'(v)\ \mathbf{S}_n^{-1}(v)\underline{Z}_n(v) \\ &= Q_{n-1}(v) - P_n(v)\left(\underline{Z'}_n(v)\mathbf{S}_{n-1}^{-1}(v)\underline{T}_{n-1}(v)\right)^2 + \eta_n^2(v)\underline{Z}_n'(v)\mathbf{S}_n^{-1}(v)\underline{Z}_n(v) \\ &+ 2P_n(v)\underline{Z}_n'(v)\mathbf{S}_{n-1}^{-1}(v)\underline{T}_{n-1}(v)\eta_n(v).\end{aligned}$$

By summing the previous recurrence we obtain that for $n > N$

$$\begin{aligned}& Q_n(v) - Q_N(v) + \sum_{j=N+1}^{n} P_j(v)\left(\underline{Z}_j'(v)\mathbf{S}_{j-1}^{-1}(v)\underline{T}_{j-1}(v)\right)^2 \\ &= 2\sum_{j=N+1}^{n} P_j(v)\underline{Z}_j'(v)\mathbf{S}_{j-1}^{-1}(v)\underline{T}_{j-1}(v)\eta_j(v) + \sum_{j=N+1}^{n}\eta_j^2(v)\underline{Z}_j'(v)\mathbf{S}_j^{-1}(v)\underline{Z}_j(v).\end{aligned}$$

Set $v_k(v) = P_k(v)\underline{Z}_k'(v)\mathbf{S}_{k-1}^{-1}(v)\underline{T}_{k-1}(v)$ if $k \geq N+1$ and 0 otherwise. According to the assertion(ii) of lemma 5.2, we have $\sum_{j=N+1}^{n} P_j(v)\underline{Z}_j'(v)\mathbf{S}_{j-1}^{-1}(v)\underline{T}_{j-1}(v)\eta_j(v) = o_p\left(\sum_{j=N+1}^{n} v_j^2(v)\right) + O_p(1)$. So,

(1) On $\left\{w : \lim_{N\to\infty} \lambda_{\max}^{(v)}(N) < \infty\right\}$ and according to lemma 5.2 , (iii), we have $\sum_{j=N+1}^{\infty}\underline{Z}_j'(v)\mathbf{S}_j^{-1}(v)\underline{Z}_j(v) < +\infty,\ a.s.$, and consequently $\sum_{j=N+1}^{\infty}\eta_j^2(v)\underline{Z}_j'(v)\mathbf{S}_j^{-1}(v)\underline{Z}_j(v) < +\infty$. Thus $Q_n(v) = O_p(1)$ and

$$\sum_{j=N+1}^{\infty} P_j(v)\left(\underline{T}_{j-1}'(v)\mathbf{S}_{j-1}^{-1}(v)\underline{Z}_j(v)\underline{Z}\right)^2 < +\infty, a.s$$

.

(2) On $\left\{w : \lim_{N\to\infty} \lambda_{\max}^{(v)}(N) \xrightarrow{a.s} \infty\right\}$, then according to the assertion (2) of lemma 5.2, we have

$$\begin{aligned}\sum_{j=N+1}^{\infty}\eta_j^2(v)\underline{Z}_j'(v)\mathbf{S}_j^{-1}(v)\underline{Z}_j(v) &= O_p\left(\left(\sum_{j=N+1}^{\infty}\underline{Z}_j'(v)\mathbf{S}_j^{-1}(v)\underline{Z}_j(v)\right)^{1+\delta}\right) \\ &= O_p\left(\left(\log\lambda_{\max}^{(v)}(N)\right)^{1+\delta}\right),\ a.s. \qquad (4.1)\end{aligned}$$

and consequently $Q_n(v) = O_p\left(\left(\log \lambda_{\max}^{(v)}(n)\right)^{1+\delta}\right)$ *a.s.* and

$$\sum_{j=N+1}^{n} P_j(v)\left(T'_{j-1}(v)\mathbf{S}_{j-1}^{-1}(v)\underline{Z}_j(v)\right)^2 = O_p\left(\left(\log \lambda_{\max}^{(v)}(n)\right)^{1+\delta}\right), a.s.$$

(3) Under $\mathbf{H1}(\varepsilon)$, $\varepsilon > 0$, then noting that $\underline{Z}'_t(v)\mathbf{S}_t^{-1}(v)\underline{Z}_t(v) < 1$, *a.s.* using the assertions (1) and (iv) of lemma 5.2 and equation (4.1) with $\delta = 0$, to obtain $Q_n(v) = O_p\left(\log \lambda_{\max}^{(v)}(n)\right)$ on the set $\left\{w : \lim_{n\to\infty} \lambda_{\max}^{(v)}(n) \xrightarrow{a.s} \infty\right\}$.

□

Proof. **3.1.** As under **H** the process $(\underline{T}_n)_{n\geq 1}$ is a martingale difference with respect to filtration $\Im$, and under $\mathbf{H1}(0)$, $\underline{T}_n = o_p(n)$, *a.s.* and $\mathbf{S}_n = O_p(n)$, *a.s.* Additionally, since $\lambda_{\min}^{(v)}(n) \xrightarrow{a.s} \infty$, $\mathbf{S}_n$ is invertible for all n sufficiently large, so the LSE, $\widehat{\underline{\theta}}_n$ of $\underline{\theta}$ given by $\widehat{\underline{\theta}}_n = \mathbf{S}_n^{-1}\underline{S}_n$ exists *a.s.* and satisfying $\widehat{\underline{\theta}}_n - \underline{\theta} = \left(\frac{1}{n}\mathbf{S}\right)_n^{-1}\left(\frac{1}{n}\underline{T}_n\right) \to 0$, *a.s.*, as $n \to +\infty$. This allows us the locally strong consistency of LSE. Moreover, using the spectral norm $\|M\|^2 = \lambda_{\max}(M'M)$, we obtain

$$\begin{aligned}\left\|\widehat{\underline{\theta}}_n(v) - \underline{\theta}(v)\right\|^2 &= \left\|\mathbf{S}_n^{-1}(v)\,\underline{T}_n(v)\right\|^2 \leq \left\|\mathbf{S}_n^{-1/2}(v)\right\|^2 \left\|\mathbf{S}_n^{-1/2}(v)\,\underline{T}_n(v)\right\|^2 \\ &= \left(\lambda_{\min}^{(v)}(n)\right)^{-1}\left[\underline{T}'_n(v)\mathbf{S}_n^{-1}(v)\,\underline{T}_n(v)\right] \\ &= \left(\lambda_{\min}^{(v)}(n)\right)^{-1} Q_n(\nu),\end{aligned}$$

and hence the result follows. □

Proof. **3.2** The proof rests classically on a Taylor series expansion of the score function $S(\underline{\theta})$. (Technical details are similar to those used in Francq et al. [17] and Boubacar et al. [13]). Indeed,

(1) It is easy to check that by replacing each component of $\underline{Z}_n(\nu)$ by their corresponding series (1.5), we get $\eta_t(\nu)\,\underline{Z}_t(\nu) = \eta_t(\nu)\,\underline{\widetilde{\mu}}(\nu) + \underline{\widetilde{U}}_t(\nu)$ where $\underline{\widetilde{\mu}}(\nu) = \left(0, \underline{\widetilde{a}}'_0(v)\right)'$, $\underline{\widetilde{U}}_t(\nu) = \eta_t(\nu)\left(1, \underline{U}'_t(\nu)\right)'$, with $\underline{\widetilde{a}}_0(v) = \left(\underline{a}_0(v-j), j = 1, ..., p\right)'$ and $\underline{U}_t(\nu) = \left(\sum_{i=0}^{\infty}\phi_i(\nu - j)\,\eta_t(\nu - i - j), j = 1, .., p\right)'$. Hence, for any $r \geq 1$, $\underline{U}_t(\nu) = \underline{U}_t^{(r)}(\nu) + \underline{V}_t^{(r)}(\nu)$ where

$$\underline{U}_t^{(r)}(\nu) = \left(\sum_{i=0}^{r}\phi_i(\nu - j)\,\eta_t(\nu - i - j), j = 1, .., p\right)',$$

and

$$\underline{V}_t^{(r)}(\nu) = \left(\sum_{i=r+1}^{\infty}\phi_i(\nu - j)\,\eta_t(\nu - i - j), j = 1, .., p\right)'.$$

Thus, $\underline{\widetilde{U}}_t(\nu) = \underline{\widetilde{U}}_t^{(r)}(\nu) + \underline{\widetilde{V}}_t^{(r)}(\nu)$ where $\underline{\widetilde{U}}_t^{(r)}(\nu) = \eta_t(\nu)\left(1, \underline{U}_t^{(r)\prime}(\nu)\right)'$ and $\underline{\widetilde{V}}_t^{(r)}(\nu) = \eta_t(\nu)\left(0, \underline{V}_t^{(r)\prime}(\nu)\right)'$. The processes $\left(\underline{\widetilde{U}}_t^{(r)}(\nu)\right)_t$ and $\left(\underline{\widetilde{V}}_t^{(r)}(\nu)\right)_t$ are centered periodically correlated. Moreover, for fixed r, $\left(\underline{\widetilde{U}}_t^{(r)}(\nu)\right)_t$ is a martingale difference and strong $\alpha-$mixing with $\left\|\underline{\widetilde{U}}_t^{(r)}(\nu)\right\|_{2+\varepsilon} < +\infty$ for some $\varepsilon > 0$. The theorem of $\alpha-$mixing processes implies that for each $\nu \in \mathbb{S}$, $n^{-1/2}\sum\limits_{t=0}^{n-1}\underline{\widetilde{U}}_t^{(r)}(\nu) \overset{d}{\rightsquigarrow} N(\underline{O},\widetilde{I}^{(r)}(\nu))$ where $\widetilde{I}^{(r)}(\nu) = \lim\limits_{n\to\infty} Var(\frac{1}{\sqrt{n}}\sum\limits_{t=0}^{n-1}\underline{\widetilde{U}}_t^{(r)}(\nu))$. According to Boubacar et al. [13], we have $\lim\limits_{r\to\infty}\sup\limits_{n\to\infty} P(\left\|\frac{1}{\sqrt{n}}\sum\limits_{t=0}^{n-1}\underline{\widetilde{V}}_t^{(r)}(\nu)\right\| > \epsilon) = 0$, for any $\varepsilon > 0$ and $\lim\limits_{r\to\infty} Var(n^{-1/2}\sum\limits_{t=0}^{n-1}\underline{\widetilde{U}}_t^{(r)}(\nu)) = Var(n^{-1/2}\sum\limits_{t=0}^{n-1}\eta_t(\nu)$ $\underline{Z}_t(\nu))$. Thus, $\lim\limits_{r\to\infty}\widetilde{I}^{(r)}(\nu) = (\varkappa_4 - 1)\,\Sigma_T(\nu)$. Therefore, from a standard result, we deduce that

$$\begin{aligned} n^{-1/2}\underline{T}_n(\nu) &= \underline{\widetilde{\mu}}(\nu)n^{-1/2}\sum_{t=0}^{n-1}\eta_t(\nu) \; + n^{-1/2}\sum_{t=0}^{n-1}\underline{\widetilde{U}}_t^{(r)}(\nu) + n^{-1/2}\sum_{t=0}^{n-1}\underline{\widetilde{V}}_t^{(r)}(\nu) \\ &\overset{d}{\rightsquigarrow} N(\underline{O},(\varkappa_4 - 1)\,\Sigma_T(\nu)). \end{aligned}$$

(2) By periodic ergodicity we have $\frac{1}{n}\mathbf{S}_n(\nu)$ converges $a.s.$ to $E\left\{\underline{Z}_t(\nu)\,\underline{Z}_t(\nu)'\right\} = \Sigma_S(\nu)$. Its invertibility has been shown by Francq et al. [17] (see also Boubacar et al. [13]).

(3) Note that we have actuelly proved that $\left\{\frac{1}{n}\underline{S}(v)\right\}^{-1} \xrightarrow{a.s.} \Sigma_S^{-1}(\nu)$ and $n^{-1/2}\ \underline{T}_n(v) \overset{d}{\rightsquigarrow} N(o,\ (\varkappa_4 - 1)\,\Sigma_T(\nu))$. Using the fact that $\underline{\widehat{\theta}}_n(\nu) \ = \underline{\theta}_n(\nu) + \ \mathbf{S}_n^{-1}(\nu)\,\underline{T}_n(\nu)$ we obtain $\sqrt{n}\left(\underline{\widehat{\theta}}_n(\nu) \ - \underline{\theta}_n(\nu)\right) \overset{d}{\rightsquigarrow} N(\underline{O},\Sigma(\nu))$ where
$\Sigma(\nu) = (\varkappa_4 - 1)\,\Sigma_S^{-1}(\nu)\,\Sigma_T(\nu)\,\Sigma_S^{-1}(\nu)$.

(4) Note that the asymptotic joint distribution of $\sqrt{n}(\underline{\widehat{\theta}}_n - \underline{\theta}_n)$ follows essentially the same arguments as described in Theorem.

□

Proof. **of lemma** Let m be an integer for which $my \geq (\delta - \beta)$, then $P\left(\ell_m > (\delta - \beta)\right) \geq P\left(\ell_m > my\right) \geq P\left(\sum\limits_{k=1}^{m} S_k > my\right) = P\left(\sum\limits_{k=1}^{m}\widetilde{S}_k > my\right) = \prod\limits_{k=1}^{m} P\left(\widetilde{S}_k > y\right) = \epsilon^m$, so for all $k \geq 1$, $P\left(N \geq mk\right) = P\left(\beta < S_n < \delta, n = 1,...,mk\right) \leq \left(1 - \epsilon^m\right)^k$. For any n, let k be such that $mk < n \leq (k+1)\,m$. Then $P\left(N > n\right) \leq P\left(N \geq mk\right) \leq \left(1 - \epsilon^m\right)^k \leq \left(1 - \epsilon^m\right)^{n/m-1} = \frac{1}{1-\epsilon^m}\left(1 - \epsilon^m\right)^{n/m} = K\rho^n$ where $K = \frac{1}{1-\epsilon^m}$ and $\rho = \left(1 - \epsilon^m\right)^{1/m}$. □

6. Conclusion and outlook

This paper deals with some theoretical properties of non$-$stationary ARCH models. The nonstationarity of the model is accommodated by allowing the coefficients to switch periodically with given regimes. So, a necessary and sufficient conditions for the existence of the stable solution is given. A simple form of the solution is expressed as a one-sided moving average process obtained from an autoregressive representation on the squared observations. Using these results the paper makes the following contributions: (i): an LSE approach for a PAR representation is heavy investigated for identifying our model. More precisely, we studied the asymptotic behavior of the least squares estimator for a class of $ARCH(p)$ models with periodic coefficients and under relatively weak conditions. This study can be applied to several versions of periodic

$ARCH(p)$ type. (ii): Some asymptotic properties of integrated case is studied. It is worth noting that these results reduce to the well-known stationary results when the coefficients are constants. A Wald statistics was suggested for testing the null AR model against the AR model with measurement error. The test statistic is asymptotically $\mathcal{N}(O,1)$ and positive large values indicate inadequacy of the null AR model. The test was more powerful than usual two-sided tests against the alternatives considered in this paper. It is to be understood, however, that an extensive study on the power of the test is necessary to draw a final conclusion since power depends largely on the alternative models and parameter values chosen.

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