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ENTROPY SOLUTIONS FOR LERAY-LIONS ANISOTROPIC ELLIPTIC PROBLEM VIA THE L^1-VERSION OF MINTY'S LEMMA

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Abstract. We consider a class of nonlinear anisotropic elliptic equations with nonlinearities and measure data, governed by a Leray-Lions type operator associated with a Carathéodory function satisfying the natural growth condition and the coercivity condition, but it only satisfies the large monotonicity condition. We address the problem of the existence of entropy solutions in variable exponent anisotropic Sobolev spaces defined on a bounded domain. To overcome the difficulties arising from the lack of strict monotonicity, our existence theorem is proved by using the L^1-version of Minty's lemma.

Keywords. Entropy solutions, nonlinear elliptic equations, anisotropic Sobolev spaces, measure data.
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1. Introduction

Since the end of the last century, the study of nonlinear elliptic anisotropic problems involving variable exponent has received increasing attention in the field of mathematical research. This is partly due to their frequent appearance in various domains of application, including image processing [1, 14], elasticity [35], electromagnetism [11, 33], porous media [6, 20], nonstandard growth [2, 27, 34], and electrorheological fluids [3, 5, 19, 30, 31]. The natural framework for solving these kinds of problems is that of Sobolev spaces with variable exponents, whose origins date back to the pioneering work of Orlicz [29] in 1931. For example, in [32], Sanchon and Urbano proposed in this framework a notion of entropy solution for the Dirichlet problem of elliptic equations

$$-\operatorname{div} a(x, \nabla u) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L^1(\Omega)$. Azroul et al. in [9] proved an existence result of entropy solutions for the elliptic equations with nonlinearities

$$-\operatorname{div} a(x, u, \nabla u) + g(x, u, \nabla u) = \mu \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

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where μ is a measure in $L^1(\Omega) + W^{-1,p'(.)}(\Omega)$. In [10], Azroul et al. have studied the strongly nonlinear problem

$$-\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f - \operatorname{div} \phi(u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

with $f \in L^1(\Omega)$ and $\phi \in C^0(\mathbb{R}^N)$, they have proved the existence of entropy solutions and some regularity results. In [24], Kozhevnikova established the existence of an entropy solution in anisotropic Sobolev spaces with variable exponents for the following problem

$$-\operatorname{div}(a(x,u,\nabla u) + c(u)) + a_0(x,u,\nabla u) = \mu \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.
In this note, we consider the following more general nonlinear anisotropic elliptic problem

$$\begin{cases} -Au + \lambda(x)|u|^{p_0(x)-2}u = \mu - \operatorname{div}\phi(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω represents a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and Au is a Leray-Lions type operator acting from $W_0^{1,\vec{p}(.)}(\Omega)$ into its dual $W^{-1,\vec{p}'(.)}(\Omega)$ defined by

$$Au = \sum_{i=1}^{N} \partial_{x_i} a_i(x,u,\nabla u). \tag{1.2}$$

Our objective is to establish an existence result of entropy solutions for Problem (1.1) by assuming that the function $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is not strictly monotone with respect to the gradient and without using the technical result of almost everywhere convergence for gradients of suitable approximating solutions. Our strategy is to follow the approach developed by Boccardo et al [13], using as the main tool an L^1-version of Minty's Lemma to overcome the difficulties of the not-strict monotonicity. This approach differs from that used in the above-mentioned works and has been adopted in several other works, such as Boccardo [12], Akdim et al. [4], and Azraibi et al. [8, 7]. The novelty in this work lies in the application of this tool to the anisotropic, variable exponent Problem (1.1).

This paper is organized as follows. In Section 2, we introduce the necessary definitions and propositions regarding generalized Lebesgue-Sobolev spaces. In Section 3, we give some assumptions on the data for which our problem has at least one solution. Furthermore, we present the main result of our notion of solution. Finally, in Section 4, we prove the existence of entropy solutions to Problem (1.1) by utilizing the L^1-version of Minty's Lemma.

2. Functional framework

In what follows, we present various definitions and results related to variable exponent Lebesgue and Sobolev spaces. For more details, we refer to the book [15] and the paper [23]. For each open bounded subset Ω of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, we denote

$$C_+(\overline{\Omega}) = \left\{ p : \overline{\Omega} \longrightarrow \mathbb{R} \text{ measurable function such that } 1 < p^- \leq p^+ < N \right\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) \mid x \in \overline{\Omega}\} \qquad \text{and} \qquad p^+ = \operatorname{ess\,sup}\{p(x) \mid x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx \; < \; \infty \right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(.)} := \inf \left\{ \lambda > 0 : \; \rho_{p(.)}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

where the mapping $\rho_{p(.)} : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ is the convex modular of the $L^{p(\cdot)}(\Omega)$ space defined by

$$\rho_{p(.)}(u) := \int_\Omega |u(x)|^{p(x)} dx$$

and satisfies to the following inequality (see [22, 21]): for any $u \in L^{p(\cdot)}(\Omega)$

$$\min\left\{ |u|_{p(.)}^{p^-}; \; |u|_{p(.)}^{p^+} \right\} \leq \rho_{p(.)}(u) \leq \max\left\{ |u|_{p(.)}^{p^-}; \; |u|_{p(.)}^{p^+} \right\}. \tag{2.1}$$

Proposition 2.1 (see [23])**.** *The space $\big(L^{p(\cdot)}(\Omega), |u|_{p(.)}\big)$ is a separable and reflexive Banach space and its dual space is isomorphic to $L^{p'(x)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$.*

Proposition 2.2 (See [18])**.** *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have the Hölder type inequality*

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(.)} |v|_{p'(.)} \leq 2 |u|_{p(.)} |v|_{p'(.)}. \tag{2.2}$$

We define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows

$$W^{1,p(\cdot)}(\Omega) = \{ u \in L^p(\Omega) : \; \partial_{x_i} u \in L^{p(\cdot)}(\Omega), \;\; i = 1, \ldots, N \}$$

which is a Banach space equipped with the following norm

$$\|u\|_{p(.)} = |u|_{p(.)} + \sum_{i=1}^N |\partial_{x_i} u|_{p(.)}.$$

$W_0^{1,p(\cdot)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Besides, $\big(W^{1,p(\cdot)}(\Omega), \|u\|_{p(.)}\big)$ and $\big(W_0^{1,p(\cdot)}(\Omega), \|u\|_{p(.)}\big)$ are separable and reflexive Banach spaces. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [15, 16, 23].

Now, we introduce the anisotropic variable exponent Sobolev space used in the study of our nonlinear anisotropic elliptic problem. Let $\vec{p}(.) = \big(p_0(.), p_1(.), \ldots, p_N(.)\big)$ such that for any $i = 0, 1, \ldots, N$, $p_i(.) \in C_+(\overline{\Omega})$ with

$$1 < p_i^- := ess \inf_{x \in \Omega} p_i(x) \leq ess \sup_{x \in \Omega} p_i(x) := p_i^+ < \infty; \tag{2.3}$$

and

$$p_0(x) := \max\{p_1(x), \ldots, p_N(x)\} \quad \text{for all } x \in \Omega.$$

We denote by $p_i'(.)$ the conjugate exponent of $p_i(.)$ defined as $1/p_i(x) + 1/p_i'(x) = 1$. Let us introduce

$$p_m(x) := \min\{p_1(x), \ldots, p_N(x)\} \;\; \text{for all } x \in \Omega \quad \text{and} \quad p_m^- := \min\{p_1^-, \ldots, p_N^-\}.$$

We define the anisotropic variable exponent Sobolev space by

$$W^{1,\vec{p}(.)}(\Omega) := \left\{ u \in L^{p_0}(\Omega) \ : \ \partial_{x_i} u \in L^{p_i(.)}(\Omega), \quad i = 1, \dots, N \right\}.$$

This is a separable and reflexive Banach space (see [28]) equipped with the following norm

$$||u||_{\vec{p}(.)} = |u|_{p_0(.)} + \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}. \tag{2.4}$$

The dual of $W^{1,\vec{p}(\cdot)}(\Omega)$ is denoted by $W^{-1,\vec{p}'(\cdot)}(\Omega)$, where $\vec{p}'(\cdot) = \big(p'_0(\cdot), p'_1(\cdot), \dots, p'_N(\cdot)\big)$ with

$$1 < p'_i(\cdot) = p_i(\cdot)/\big(p_i(\cdot) - 1\big), \ \forall \, i = 0, 1, \dots, N.$$

We also define $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.4). According to [16, 17], $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

Lemma 2.3. *We have the following continuous and compact embedding*

- *if* $p_m < N$ *then* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $q \in [p_m, p_m^*[$ *where* $p_m^* = Np_m/(N - p_m)$,
- *if* $p_m = N$ *then* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $q \in [p_m, +\infty[$,
- *if* $p_m > N$ *then* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

Proof. In view of the classical embedding theorems of the Sobolev spaces and the continuous embedding $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,p_m(\cdot)}(\Omega)$, we get the previous embedding compact. □

Remark 2.4. Since we suppose that $p_m < N$, we have the embedding $W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,p_m(\cdot)}(\Omega)$ is compact.

Lemma 2.5. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ $(N \geq 1)$. *If* $\omega \in \big(W_0^{1,\vec{p}(\cdot)}(\Omega)\big)^N$, *then*

$$\int_\Omega div(\omega) dx = 0. \tag{2.5}$$

Proof. Fix a vector $\omega = (\omega^1, \dots, \omega^N) \in \big(W_0^{1,\vec{p}(\cdot)}(\Omega)\big)^N$. Since $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$, then each term ω^i can be approximated by a suitable sequence $\omega_n^i \in \mathcal{D}(\Omega)$ such that ω_n^i converges to ω^i in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Moreover, the Green formula gives

$$\int_\Omega \partial_{x_i} \omega_n^i dx = \int_{\partial\Omega} \omega_n^i \eta_i ds = 0, \tag{2.6}$$

due to the fact that $\omega_n^i \in \mathcal{C}_0^\infty(\Omega)$. On the other hand, $\partial_{x_i}\omega_n^i \longrightarrow \partial_{x_i}\omega^i$ in $L^{p_i(\cdot)}(\Omega)$. Thus $\partial_{x_i}\omega_n^i \to \partial_{x_i}\omega^i$ in $L^1(\Omega)$, which gives in view of (2.6) that $\displaystyle\int_\Omega \mathrm{div}(\omega) dx = 0$. □

3. Assumptions and main results

It is assumed that for any $i = 1, \dots, N$, the function $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory map with the following properties:

$$\text{there exists } \beta > 0 \text{ such that } |a_i(x, s, \xi)| \leq \beta\big[j_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}\big], \tag{3.1}$$

$$\text{there exists } \alpha > 0 \text{ such that } a_i(x,s,\xi).\xi_i \geq \alpha|\xi_i|^{p_i(x)} \quad \forall \xi \in \mathbb{R}^N \tag{3.2}$$

and

$$\big[a_i(x,s,\xi) - a_i(x,s,\eta)\big](\xi_i - \eta_i) \geq 0 \quad \text{if } \xi_i \neq \eta_i, \tag{3.3}$$

where j_i is some positive function in $L^{p_i'(.)}(\Omega)$.
μ is a measure datum assumed to be in $L^1(\Omega) + W^{-1,\vec{p}'(.)}(\Omega)$ such as

$$\mu = f - \operatorname{div}(F) \quad \text{with } f \in L^1(\Omega) \text{ and } F = (F_1,\dots,F_N) \in \big(L^{p_i'(.)}(\Omega)\big)^N. \tag{3.4}$$

It is assumed that the function ϕ satisfies the following condition

$$\phi = (\phi_1,\dots,\phi_N) \in C^0(\mathbb{R},\mathbb{R}^N) \tag{3.5}$$

and

$$\lambda \in L^\infty(\Omega) \text{ and there exists } \lambda_0 > 0 \text{ such that } \quad \lambda(x) \geq \lambda_0 \text{ for all } x \in \Omega. \tag{3.6}$$

We introduce some technical sets used in the definition of our notion of solution:

$$\mathcal{T}_0^{1,\vec{p}(.)}(\Omega) := \Big\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable such that } T_k(u) \in W_0^{1,\vec{p}(.)}(\Omega),\ \forall k > 0 \Big\},$$

where, T_k is the truncation function defined for all $k > 0$ by

$$T_k(s) = \max\{-k; \min\{k; s\}\}.$$

It is clear that

$$\lim_{k\to\infty} T_k(s) = s \text{ and } |T_k(s)| = \min\{|s|; k\}.$$

Let χ_B be the indicator function of a set B. For $u \in \mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$ and any $k > 0$ we have

$$\nabla T_k(u) = \chi_{\{\Omega:|u|<k\}} \nabla u \in L^{\vec{p}(.)}(\Omega). \tag{3.7}$$

Definition 3.1. A function $u \in \mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$ is an entropy solution of problem (1.1) if

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u)\partial_{x_i} T_k(u-\varphi)dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\varphi)dx \\ \displaystyle\leq \sum_{i=1}^N \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\varphi)dx + \int_\Omega fT_k(u-\varphi)dx + \sum_{i=1}^N \int_\Omega F_i\partial_{x_i} T_k(u-\varphi)dx \end{cases} \tag{3.8}$$

for any $\varphi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$ and for every $k \geq 0$.

The main result of the present work is the following theorem.

Theorem 3.2. *Let the assumptions (3.1)–(3.5) be satisfied. Then there exists an entropy solution of the problem (1.1).*

Before going further, we state the following technical and useful lemma which is an L^1 version of Minty's Lemma.

Lemma 3.3. *Let $u \in \mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$. Then the following statements are equivalent:*

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla \varphi)\partial_{x_i} T_k(u-\varphi)dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\varphi)dx \\ \displaystyle\leq \sum_{i=1}^N \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\varphi)dx + \int_\Omega fT_k(u-\varphi)dx + \sum_{i=1}^N \int_\Omega F_i\partial_{x_i} T_k(u-\varphi)dx \end{cases} \tag{3.9}$$

for any $\varphi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$ *and for every* $k \geq 0$;

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega a_i\big(x,u,\nabla u\big)\partial_{x_i} T_k(u-\varphi)dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\varphi)dx \\ = \sum_{i=1}^{N} \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\varphi)dx + \int_\Omega fT_k(u-\varphi)dx + \sum_{i=1}^{N}\int_\Omega F_i \partial_{x_i} T_k(u-\varphi)dx \end{cases} \tag{3.10}$$

for any $\varphi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$ *and for every* $k \geq 0$.

Proof. By adding and subtracting the following term

$$\sum_{i=1}^{N} \int_\Omega a_i\big(x,u,\nabla \varphi\big)\partial_{x_i} T_k(u-\varphi)dx$$

to the left-hand side of (3.10) and then using assumption (3.3), it is obvious that (3.10) $\Longrightarrow$ (3.9). So, it remains to show that (3.9) $\Longrightarrow$ (3.10). Let m and k be positive real numbers, $\theta \in (-1,1)$ and $\psi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. Replacing φ by $T_m\big(u-\theta T_k(u-\psi\big)$ in the inequality (3.9), we obtain

$$\begin{aligned} I(m,k)& \\ &= \sum_{i=1}^{N}\int_\Omega a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &\leq -\int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &\qquad + \sum_{i=1}^{N}\int_\Omega \phi_i(u)\partial_{x_i}T_k(u-T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &\qquad + \int_\Omega fT_k(u-T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &\qquad + \sum_{i=1}^{N}\int_\Omega F_i\partial_{x_i}T_k(u-T_m\big(u-\theta T_k(u-\psi)\big)dx = J(m,k). \end{aligned} \tag{3.11}$$

In order to obtain useful estimates, we introduce the following technical sets

$$\Omega_{0,mk} := \big\{x\in\Omega : |u-T_m\big(u-\theta T_k(u-\psi)\big| \leq k\big\}$$
$$\Omega_{1,mk} := \big\{x\in\Omega : |u-T_m\big(u-\theta T_k(u-\psi)\big| \leq m\big\}.$$

Thus, the left-hand side of Inequality (3.11) can be split as follow

$$\begin{aligned} &I(m,k) \\ &= \sum_{i=1}^{N}\int_{\Omega_{0,mk}\cap\Omega_{1,mk}} a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &+ \sum_{i=1}^{N}\int_{\Omega_{0,mk}\cap\Omega_{1,mk}^c} a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &+ \sum_{i=1}^{N}\int_{\Omega_{0,mk}^c} a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx. \end{aligned}$$

Evidently, $T_k(u - T_m\big(u - \theta T_k(u-\psi)\big) = 0$ on $\Omega^c_{0,mk}$, which implies that

$$\sum_{i=1}^N \int_{\Omega^c_{0,mk}} a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx = 0. \tag{3.12}$$

In view of (3.2) and the continuity of the functions $a_i(x,s,.)$ with respect to ξ_i, we have $a_i(x,s,0) = 0$ for $i = 1,\ldots,N$. Since $\nabla T_m\big(u - \theta T_k(u-\psi)\big) = 0$ on $\Omega^c_{1,mk}$, it follows that

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_{0,mk}\cap\Omega^c_{1,mk}} & a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx \\ &= \sum_{i=1}^N \int_{\Omega_{0,mk}\cap\Omega^c_{1,mk}} a_i\big(x,u,0\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx = 0. \end{aligned}\tag{3.13}$$

From estimates (3.12) and (3.13), we obtain

$$\begin{aligned} & I(m,k) \\ &= \sum_{i=1}^N \int_{\Omega_{0,mk}\cap\Omega_{1,mk}} a_i\big(x,u,\nabla T_m\big(u-\theta T_k(u-\psi)\big)\big)\partial_{x_i} T_k(u - T_m\big(u-\theta T_k(u-\psi)\big)dx. \end{aligned}$$

Sending m to ∞, we have $\Omega_{0,mk}, \Omega_{1,mk} \to \Omega$ which implies that $\Omega_{0,mk} \cap \Omega_{1,mk} \to \Omega$. Then, applying the Lebesgue's dominated convergence theorem, it follows that

$$\lim_{m\to\infty} I(m,k) = \theta \sum_{i=1}^N \int_\Omega a_i\big(x,u,\nabla(u-\theta T_k(u-\psi)\big)\big)\partial_{x_i} T_k(u-\psi)dx. \tag{3.14}$$

On the other hand, we have

$$\begin{aligned}\lim_{m\to\infty} J(m,k) = \theta\Big[&- \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\psi))dx + \int_\Omega fT_k(u-\psi))dx \\ &+ \sum_{i=1}^N \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\psi))dx + \sum_{i=1}^N \int_\Omega F_i \partial_{x_i} T_k(u-\psi))dx\Big]. \end{aligned}\tag{3.15}$$

In view of (3.14) and (3.15), passing to the limit in (3.11), we get

$$\begin{aligned} &\theta \sum_{i=1}^N \int_\Omega a_i\big(x,u,\nabla(u-\theta T_k(u-\psi)\big)\big)\partial_{x_i} T_k(u-\psi)dx \\ &\le \theta\Big[- \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\psi))dx + \sum_{i=1}^N \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\psi))dx \\ &\qquad + \int_\Omega fT_k(u-\psi))dx + \sum_{i=1}^N \int_\Omega F_i \partial_{x_i} T_k(u-\psi))dx\Big] \end{aligned}\tag{3.16}$$

for every $\psi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$. Choosing $\theta > 0$, dividing (3.16) by θ and then letting θ to zero, we have

$$\sum_{i=1}^{N} \int_\Omega a_i\big(x, u, \nabla u\big)\partial_{x_i} T_k(u-\psi)dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\psi))dx \tag{3.17}$$
$$\leq \sum_{i=1}^{N} \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\psi))dx + \int_\Omega fT_k(u-\psi))dx + \sum_{i=1}^{N} \int_\Omega F_i \partial_{x_i} T_k(u-\psi))dx.$$

For $\theta < 0$, dividing (3.16) by θ and sending θ to zero, we have

$$\sum_{i=1}^{N} \int_\Omega a_i\big(x, u, \nabla u\big)\partial_{x_i} T_k(u-\psi)dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\psi))dx \tag{3.18}$$
$$\geq \sum_{i=1}^{N} \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\psi))dx + \int_\Omega fT_k(u-\psi))dx + \sum_{i=1}^{N} \int_\Omega F_i \partial_{x_i} T_k(u-\psi))dx.$$

Combining (3.17) and (3.18), we derive the following equality

$$\sum_{i=1}^{N} \int_\Omega a_i\big(x, u, \nabla u\big)\partial_{x_i} T_k(u-\psi)dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uT_k(u-\psi))dx \tag{3.19}$$
$$= \sum_{i=1}^{N} \int_\Omega \phi_i(u)\partial_{x_i} T_k(u-\psi))dx + \int_\Omega fT_k(u-\psi))dx + \sum_{i=1}^{N} \int_\Omega F_i \partial_{x_i} T_k(u-\psi))dx.$$

Hence, the proof of Lemma 3.3 is complete. $\square$

4. Proof of theorem

Let us consider the approximate problems:

$$\begin{cases} -A_n u_n + \lambda(x)|u_n|^{p_0(x)-2}u_n = \mu_n - \mathrm{div}\phi^n(u) & \text{in } \Omega \\ \\ u_n \in W_0^{1,\vec{p}(.)}(\Omega) & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$A_n(u_n) := \sum_{i=1}^{N} \partial_{x_i} a_i(x, T_n(u_n), \nabla u_n)$$
$$\phi^n(s) := \phi\big(T_n(s)\big) \quad \text{and} \quad \mu_n := T_n(f) - \mathrm{div}F.$$

Note that:

$$f_n = T_n(f) \in L^\infty(\Omega), \quad f_n \underset{n\to+\infty}{\longrightarrow} f \ \text{ in } \ L^1(\Omega) \quad \text{and} \quad ||f_n||_{L^1} \leq ||f||_{L^1}.$$

Thanks to [21] (see [25] and [26]), there exists at least one weak solution $u_n \in W_0^{1,\vec{p}(.)}(\Omega)$ to the Cauchy boundary value Problem (4.1). That is, for any test function $v \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$, the following integral identity holds true:

$$\sum_{i=1}^{N} \int_\Omega a_i\big(x, T_n(u_n), \nabla u_n\big)\partial_{x_i} v dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uvdx$$
$$= \sum_{i=1}^{N} \int_\Omega \phi_i(u_n)\partial_{x_i} v dx + \int_\Omega fvdx + \sum_{i=1}^{N} \int_\Omega F_i \partial_{x_i} vdx. \tag{4.2}$$

Step 1: A priori estimates. Choosing $T_k(u_n)$ as test function in (4.2) and using Hypothesis (3.2), we have

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_\Omega|\partial_{x_i}T_k(u_n)|^{p_i(x)}dx+\int_\Omega\lambda(x)|u_n|^{p_0(x)-2}u_nT_k(u_n)dx\\ \leq&\sum_{i=1}^{N}\int_\Omega\phi_i^n(u_n)\partial_{x_i}T_k(u_n)dx+\int_\Omega f_nT_k(u_n)dx+\sum_{i=1}^{N}\int_\Omega F_i\partial_{x_i}T_k(u_n)dx.\end{aligned}\tag{4.3}$$

We show that the first term on the right-hand side in (4.3) is zero. To do this, we set

$$\Phi_i^n(s)=\int_0^s\phi_i^n(t)\chi_{\{|t|\leq k\}}dt.$$

Then by using Lemma 2.5, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega\phi_i^n(u_n)\partial_{x_i}T_k(u_n)dx\\ &\qquad=\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\phi_i^n(u_n)\partial_{x_i}u_ndx=\sum_{i=1}^{N}\int_\Omega\partial_{x_i}\Phi_i^n(u_n)dx=0.\end{aligned}\tag{4.4}$$

Combining estimate (4.4), the boundedness of f_n and Young's Inequality, we get

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_\Omega|\partial_{x_i}T_k(u_n)|^{p_i(x)}dx+\lambda_0\int_\Omega|T_k(u_n|^{p_0(x)}dx\\ \leq&\, Ck+C(\alpha)\sum_{i=1}^{N}\int_\Omega|F_i|^{p_i'(x)}dx+\frac{\alpha}{2}\sum_{i=1}^{N}\int_\Omega|\partial_{x_i}T_k(u_n)|^{p_i(x)}dx.\end{aligned}\tag{4.5}$$

In addition, from Inequality (4.5) it follows that

$$\min\{\frac{\alpha}{2}+\lambda_0\}\sum_{i=0}^{N}\int_\Omega|\partial_{x_i}T_k(u_n)|^{p_i(x)}dx\leq C(k+1),$$

otherwise

$$\sum_{i=0}^{N}\int_\Omega|\partial_{x_i}T_k(u_n)|^{p_i(x)}dx\leq C(k+1),$$

where C is a constant that does not depend on n. It follows that

$$\sum_{i=0}^{N}|\partial_{x_i}T_k(u_n)|_{p_i(.)}^{p_m^-}\leq\sum_{i=0}^{N}\int_\Omega|\partial_{x_i}T_k(u_n)|^{p_i(x)}dx+N\leq C(k+1)+N,\ \forall k>0.$$

Moreover, by the convexity of the function $t\mapsto t^{p_m^-}$ on $\mathbb{R}^+$ with $p_m^->1$ we obtain

$$\frac{1}{N^{p_m^--1}}\Big(\sum_{i=0}^{N}|\partial_{x_i}T_k(u_n)|_{p_i(.)}\Big)^{p_m^-}\leq\sum_{i=0}^{N}|\partial_{x_i}T_k(u_n)|_{p_i(.)}^{p_m^-}\leq C(k+1)+N,\ \forall k>0.$$

It is evident that

$$\sum_{i=0}^{N}|\partial_{x_i}T_k(u_n)|_{p_i(.)}\leq C(k+1)^{1/p_m^-},$$

from which, we deduce the following estimate

$$\|T_k(u_n)\|_{\vec{p}(.)} = |T_k(u_n)|_{p_0(.)} + \sum_{i=1}^{N} |\partial_{x_i} T_k(u_n)|_{p_i(.)} \leq C(k+1)^{1/p_m^-} \ \forall k > 0, \tag{4.6}$$

where C denotes a positive constant that does not depend on k or n.

Step2: Convergence in measure. Let k be large enough. Combining the Hölder type inequality and (4.6), we find that

$$\begin{aligned} k \ \text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \leq \int_{\Omega} |T_k(u_n)|\, dx \\ &\leq 2 \times |1|_{p_0'(\cdot)} |T_k(u_n)|_{p_0(.)} \\ &\leq 2(meas(\Omega)+1)^{1/(p_0')^-} ||T_k(u_n)||_{\vec{p}(.)} \\ &\leq C(k+1)^{1/p_m^-}, \end{aligned} \tag{4.7}$$

which yields

$$\text{meas}(\{|u_n| > k\}) \leq C\left(\frac{1}{k^{-1+p_m^-}} + \frac{1}{k^{p_m^-}}\right)^{1/p_m^-} \longrightarrow 0 \quad \text{as } k \to \infty. \tag{4.8}$$

In addition, for any $\nu > 0$, we have

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \nu\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\}. \end{aligned} \tag{4.9}$$

Let $\varepsilon > 0$, using (4.8), we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.10}$$

Moreover, since $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, then there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{weakly in } W_0^{1,\vec{p}}(\Omega) \quad \text{as } n \to \infty, \forall k > 0.$$

Next, thanks to the compact embedding $W_0^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p_m^-}(\Omega)$, we obtain

$$T_k(u_n) \to \eta_k \quad \text{strongly in } L^{p_m^-}(\Omega) \quad \text{and a.e. in } \Omega \ \forall k > 0.$$

Hence, it can be assumed that $(T_k(u_n))_n$ is a Cauchy sequence in measure. Thus, for all $k > 0$ and $\nu, \varepsilon > 0$, there exists $n_0 = n_0(k, \nu, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\} \leq \frac{\varepsilon}{3} \quad \text{for all } m, n \geq n_0(k, \nu, \varepsilon). \tag{4.11}$$

Finally, from (4.9), (4.10) and (4.11) we obtain that

$$\forall \nu, \varepsilon > 0, \quad \exists n_0 = n_0(\nu, \varepsilon) \text{ such that } \quad \text{meas}\{|u_n - u_m| > \nu\} \leq \varepsilon \quad \forall m, n \geq n_0.$$

From this, we deduce that $(u_n)_n$ is a Cauchy sequence in measure in Ω, and therefore admits a subsequence converging almost everywhere to a measurable function u. As a result, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,\vec{p}}(\Omega) \text{ and a.e. in } \Omega, \ \forall k > 0, \tag{4.12}$$

and using the Lebesgue's dominated convergence Theorem, we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } L^{p_m^-}(\Omega) \quad \forall k > 0. \tag{4.13}$$

Step 3: Intermediate inequality. Let $\varphi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. By choosing $T_k(u_n - \varphi)$ as test function in (4.2), we obtain

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u_n), \nabla u_n\big)\partial_{x_i} T_k(u_n - \varphi)dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)-2}u_n T_k(u_n - \varphi)dx \\ = \sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_{x_i} T_k(u_n - \varphi)dx + \int_\Omega f_n T_k(u_n - \varphi)dx \\ + \sum_{i=1}^N \int_\Omega F_i \partial_{x_i} T_k(u_n - \varphi)dx. \end{aligned} \tag{4.14}$$

Next, adding and subtracting the term

$$\sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u_n), \nabla \varphi\big)\partial_{x_i} T_k(u_n - \varphi)dx,$$

to the left-hand side of (4.14) and using assumption (3.3), we derive the following inequality

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u_n), \nabla \varphi\big)\partial_{x_i} T_k(u_n - \varphi)dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)-2}u_n T_k(u_n - \varphi)dx \\ \leq \sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_{x_i} T_k(u_n - \varphi)dx \\ + \int_\Omega f_n T_k(u_n - \varphi)dx + \sum_{i=1}^N \int_\Omega F_i \partial_{x_i} T_k(u_n - \varphi)dx. \end{aligned} \tag{4.15}$$

Step 4: Passing to the limit. Let $M = k + ||\varphi||_\infty$. From (4.12) and (4.13), we have

$$T_M(u_n) \rightharpoonup T_M(u) \quad \text{weakly in } W_0^{1,\vec{p}(.)}(\Omega),$$
$$T_M(u_n) \longrightarrow T_M(u) \quad \text{strongly in } L^{\bar{p_m}}(\Omega) \quad \text{and a.e. in} \quad \Omega$$

and

$$\partial_{x_i} T_k(u_n - \varphi) \longrightarrow \partial_{x_i} T_k(u - \varphi) \text{ strongly in } L^{p_i(\cdot)}(\Omega). \tag{4.16}$$

On one hand, thanks to Hypothesis (3.1), we deduce

$$|a_i(x, T_M(u_n), \nabla\varphi)| \leq \beta\big[j_i(x) + |T_M(u_n)|^{p_i(x)-1} + |\partial_{x_i}\varphi|^{p_i(x)-1}\big],$$

hence

$$|a_i(x, T_M(u_n), \nabla\varphi)| \longrightarrow |a_i(x, T_M(u), \nabla\varphi)| \quad \text{a.e. in } \Omega$$

and

$$\begin{aligned}\beta\big[j_i(x) + |T_M(u_n)|^{p_i(x)-1} + |\partial_{x_i}\varphi|^{p_i(x)-1}\big] \\ \longrightarrow \beta\big[j_i(x) + |T_M(u)|^{p_i(x)-1} + |\partial_{x_i}\varphi|^{p_i(x)-1}\big] \quad \text{a.e. in} \quad \Omega.\end{aligned}$$

Thus, according to Vitali's theorem, it follows that

$$a_i(x, T_M(u_n), \nabla\varphi) \longrightarrow a_i(x, T_M(u), \nabla\varphi) \quad \text{strongly in } L^{\bar{p_m}}(\Omega) \text{ as } n \to \infty. \tag{4.17}$$

According to (4.16) and (4.17), we can get

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_M(u_n), \nabla\varphi)\partial_{x_i}T_k(u_n-\varphi)dx\\ &\longrightarrow \int_{\Omega} a_i(x, u, \nabla\varphi)\partial_{x_i}T_k(u-\varphi)dx \quad \text{as } n\to\infty.\end{aligned} \tag{4.18}$$

On the other hand, in view of Convergences (4.12) and (4.13) we have

$$\lambda(x)|u_n|^{p_0(.)-2}u_nT_k(u_n-\varphi) \longrightarrow \lambda(x)|u|^{p_0(.)-2}uT_k(u-\varphi) \quad \text{a.e. in } \Omega$$

and

$$f_nT_k(u_n-\varphi) \longrightarrow fT_k(u-\varphi) \quad \text{a.e. in } \Omega.$$

Next, we make use of Vitali's theorem to derive

$$\int_{\Omega}\lambda(x)|u_n|^{p_0(.)-2}u_nT_k(u_n-\varphi)dx \longrightarrow \int_{\Omega}\lambda(x)|u|^{p_0(.)-2}uT_k(u-\varphi)dx \tag{4.19}$$

and

$$\int_{\Omega} f_nT_k(u_n-\varphi)dx \longrightarrow \int_{\Omega} fT_k(u-\varphi)dx. \tag{4.20}$$

Similarly, we have

$$\int_{\Omega}\phi_i^n(u_n)\partial_{x_i}T_k(u_n-\varphi)dx \longrightarrow \int_{\Omega}\phi_i(u)\partial_{x_i}T_k(u-\varphi). \tag{4.21}$$

and

$$\int_{\Omega} F_i\partial_{x_i}T_k(u_n-\varphi)dx \longrightarrow \int_{\Omega} F_i\partial_{x_i}T_k(u-\varphi). \tag{4.22}$$

In view of (4.18), (4.19), (4.20), (4.21) and (4.22) we pass to the limit in (4.15), so that $\forall\varphi\in W_0^{1,\vec{p}(.)}(\Omega)\cap L^\infty(\Omega)$, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla\varphi)\partial_{x_i}T_k(u-\varphi)dx + \int_{\Omega}\lambda(x)|u|^{p_0(x)-2}uT_k(u-\varphi)dx\\ &\le \sum_{i=1}^{N}\int_{\Omega}\phi_i(u)\partial_{x_i}T_k(u-\varphi)dx + \int_{\Omega} fT_k(u-\varphi)dx + \sum_{i=1}^{N}\int_{\Omega} F_i\partial_{x_i}T_k(u-\varphi)dx.\end{aligned}$$

According to Lemma 3.3, u is an entropy solution (with equality sign) of Problem (1.1) in the sense of the Definition 3.1.

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References

1. Aboulaich, R., Meskine, D. and Souissi, A. (2008). New diffusion models in image processing. Comput. Math. Appl. 56, 874-882. https://doi.org/10.1016/j.camwa.2008.01.017
2. Acerbi, E., Mingione, G. (2001). Regularity results for a class of functionals with non-standard growth. Arch. Ration. Mech. Anal. 156, 121-140. https://doi.org/10.1007/s002050100117

3. Acerbi, E. and Mingione, G. (2002). Regularity results for stationary electro-rheological fluids. Archive for rational mechanics and analysis, 164 (3), 213-259. https://doi.org/10.1007/s00205-002-0208-7
4. Akdime, Y., Azroul, E. and Rhoudaf, M. (2008). Existence of T-solution for degenerated problem via Minty's lemma. Acta. Math. Sin.-English Ser. 24, 431-438. https://doi.org/10.1007/s10114-007-0970-4
5. Antontsev, S. and Rodrigues, J.F. (2007). On stationary thermorheological viscous flows. Univ.Ferrara Sez. VII Sci. Mat. 52, 19-36. https://doi.org/10.1007/s11565-006-0002-9
6. Antontsev, S. and Shmarev, S. (2005). A model porous medium equation with variable exponent of nonlinearity: Existence, uniqueness and localization properties of solutions. Nonlinear Anal. 60, 515-545. https://doi.org/10.1016/j.na.2004.09.026
7. Azraibi, O., Bouzelmate, A., EL HAJI, B. and Elkhannouss, Y. (2024). Anisotropic elliptic Problem involving the L^1-version of Minty's lemma. Palestine Journal of Mathematics. Vol 13(2), 338-348.
8. Azraibi, O., Derham, A. and El Haji, B. (2025). On some anisotropic elliptic problem with measure data Int. J. Nonlinear Anal. Appl. 16, 5, 1-11. DOI:10.22075/ijnaa.2024.33483.4992
9. Azroul, E., Benboubker, M. B. and Rhoudaf, M. (2012). Entropy solution for some $p(x)$-Quasilinear problems with right-hand side measure, Afr. diaspora J. Math 13, no. 2, 23-44. DOI:10.5269/bspm.v13i2.129
10. Azroul, E., Hjaij, H. and Touzani, A. (2013). Existence and regularity of entropy solutions for strongly nonlinear $p(x)$-elliptic equations. Electronic J. Diff. Equ., **68**, 1-27. ttp://ejde.math.txstate.edu
11. Bahrouni A, Repovš D. (2018). Existence and nonexistence of solutions for p(x)-curl systems arising in electromagnetism. Complex Var Elliptic Equ. 63(2): 292-301. https://doi.org/10.1080/17476933.2017.1304390
12. Boccardo, L. .(2002) A remark on some nonlinear elliptic problems. Electronic J. Diff. Equ. Conference, 08, 47-52. http://eudml.org/doc/123393
13. Boccardo, L and Orsina, L. (2000). Existence results for dirichlet problems in L^1 via minty's lemma. Applicable Analysis: An International Journal, 76:(3-4), 309-317. https://doi.org/10.1080/00036810008840887
14. Chen, Y., Slevine, S. and Rao, M. (2006). Variable exponent,linear growth functionals in image restoration. SIAM J. Appl. Math, 66, 1383-1406. https://doi.org/10.1137/050624522
15. Diening, L., Harjulehto, P., Hasto, P. and Ruzicka, M. (2011). Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Heidelberg. https://doi.org/10.1007/978-3-642-18363-8
16. Fan, X. (2011). Anisotropic variable exponent Sobolev spaces and $\vec{p}(x)-$Laplacian equations. Complex. Var. Elliptic. Eq. 46, 623-642. https://doi.org/10.1080/17476931003728412
17. Fan, X. (2005). Solution for $p(x)$-Laplacian Dirichlet problems with singular coefficient. J. Math. Anal. Appl. **312**, 464-477. DOI:10.1016/j.jmaa.2005.03.057
18. Fan, X. and Zhao, D. (2001). On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. 263, 424-446. DOI:10.21136/CMJ.1991.102493
19. Halsey TC. (1992) Electrorheological fluids. Sciences. 258(5083), 761-766. DOI:10.1126/science.258.5083.761
20. Henriques, E., Urbano, J.M. (2006). Intrinsic scaling for PDEs with an exponential nonlinearity. Indiana Univ. Math. J. 55, 1701-1722. 10.1512/iumj.2006.55.2715
21. Houede, D.A., Ibrango, I. and Ouédraogo, A. (2024). Entropy solutions for some elliptic anisotropic problems involving variable exponent with Fourier boundary conditions and measure data. J. Elliptic Parabol. Equ. 10, No. 1, 237–277. https://doi.org/10.1007/s41808-023-00259-z

22. Houede, D.A., Sakandé, I. and Ouédraogo, A. (2025). Existence of renormalized solutions for Fourier boundary-value problems involving variable exponent and measure data. Annals of Mathematics and Computer Science Vol 29, 28-71. https://doi.org/10.56947/amcs.v29.585
23. Kovacik. O. and Rakosnik. J. (1991). On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslovak Math. J. **41**, 592-618. DOI:10.21136/CMJ.1991.102493
24. Kozhevnikova, L. M. (2019): On solutions of anisotropic elliptic equations with variable exponent and measure data, Complex Variables and Elliptic Equations. https://doi.org/10.1080/17476933.2019.1579206
25. Leray, J. and Lions, L. (1965). Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, Bull. Soc. Math. France 93, 97-107. DOI:10.24033/bsmf.1617
26. Leray, J. and Lions, L. (1968). Quelques Méthodes de Résolution des Problèmes aux Limites Non linéaires. Dunod, Paris
27. Marcellini, P. (1989). Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. Arch. Ration. Mech. Anal. 105, 267-284. https://doi.org/10.1007/BF00251503
28. Mihailescu, M. and Pucci, P. and Radulescu, V. (2008). Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. J. Math. Anal. Appl., **340**, 687-698. DOI:10.1016/j.jmaa.2007.09.015
29. Orlicz, W. (1931). Über konjugierte Exponentenfolgen." Studia Mathematica 3.(1) 200-211. http://eudml.org/doc/217564
30. Rajagopal, K.R. and Rùzicka, M. (2001). Mathematicalmodeling of electrorheologicalmaterials. Continuummechanics and thermodynamics. 13(1), 59-78. https://doi.org/10.1007/s001610100034
31. Rùzicka, M. (2000). Electrorheological fuids: modeling and mathematical theory. Springer Science & Business Media. https://doi.org/10.1007/BFb0104029
32. Sanchon, M. and Urbano, J.M. (2009). Entropy solutions for the $p(x)-$Laplace equation. Trans. Amer. Math. Soc. **361**(12), pp. 6387-6405. 10.1090/S0002-9947-09-04399-2
33. Xiang, M., Wang, F. and Zhang, B. (2017). Existence and multiplicity of solutions for $p(x)-$curl systems arising in electromagnetism. J.Math. Anal. Appl., 448(2), 1600-1617. https://doi.org/10.1007/s00009-019-1312-3
34. Zhikov V.V. (2011). On variational problems and nonlinear elliptic equationswith nonstandard growth conditions. J Math Sci. 173(5), 463-570. https://doi.org/10.3846/mma.2020.11031
35. Zhikov, V.V. (1987). Averaging of functionals of the calculus of variations and elasticity theory. Mathematics of the USSR-Izvestiya. 29(1), 33-66. 10.1070/IM1987v029n01ABEH000958

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