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SOME NECESSARY AND SUFFICIENT CONDITIONS FOR DIOPHANTINE GRAPHS

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ABSTRACT. A graph G of order n is called Diophantine if there exists a labeling function f of vertices such that $\gcd(f(u), f(v))$ divides n for every pair adjacent vertices u, v in G. This paper defines, studies and generalizes maximal Diophantine graphs D_n, determining their independence number, number of full-degree vertices, and clique number. These parameters establish necessary conditions for the existence of Diophantine labelings.

Keywords. Diophantine graph, labeling isomorphism, γ-labeled graph.
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1. INTRODUCTION AND PRELIMINARIES

Assuming that a graph $G = (V, E)$ is a finite simple undirected graph with $|V|$ vertices and $|E|$ edges, where $V = V(G)$ is the vertex set, $E = E(G)$ is the edge set, $|V|$ is called the order of the graph G and $|E|$ is called the size of the graph G. In general, $|X|$ denotes the cardinality of a set X. A set of vertices S of a graph G is said to be an independent set or a free set if for all $u, v \in S$, u, v are nonadjacent in G. The independence number, denoted by $\alpha(G)$, is the maximum order of an independent set of vertices of a graph G. The operation of adding an edge $e = uv$ to a graph G joining the vertices u, v yields a new graph with the same vertex set $V(G)$ and edge set $E(G) \cup \{uv\}$, which is denoted $G + \{uv\}$. The operation of deleting an edge $e = uv$ from a graph G removes only that edge, the resulting graph is denoted $G - \{uv\}$. A spanning subgraph of a graph G is a subgraph of G obtained by deleting edges only, adding edges to a graph G yields a spanning supergraph of G. The join of two graphs G and H is denoted by $G + H$, it has the following vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. $K_n, \overline{K_n}$ and C_n denote the complete graph, the null graph and the cycle graph of order n respectively. We follow terminology and notations in graph theory as in A. Bickle [2], J. L. Gross; J. Yellen; P. Zhang [4], F. Harary [5] and K. H. Rosen [10].

The concept of prime labeling was introduced by R. Entringer and was discussed in a paper by A. Tout [13]. A graph G is called a prime graph if there exists a bijective map $f : V \to \{1, 2, \ldots, n\}$ such that for all $uv \in E$, $(f(u), f(v)) = 1$. Some authors investigated algorithms for prime labeling in [3] and necessary and sufficient conditions

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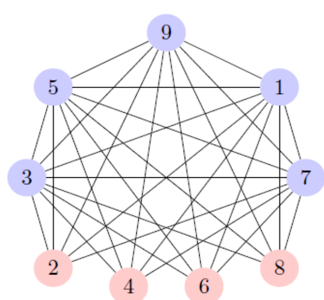
are studied in [11], [12]. The notion of linear Diophantine labeling is an extension of that of prime labeling. In this paper, we give a brief summary of some definitions and some results pertaining to Diophantine graphs. A generalization encompassing prime graphs, Diophantine graphs and another type of graph labeling is introduced and discussed. In maximal Diophantine graphs, an arithmetic function is established to calculate the number of vertices with full degree and the order of the maximal clique or the maximal complete subgraph, the independence number is computed and necessary and sufficient conditions are provided with these bounds. Moreover, an explicit formula for a vertex with minimum degree and minimum label is proved. Furthermore, a new perspective on degree sequences for establishing necessary conditions is presented. Relevant definitions and notations from number theory are mentioned. We follow the basic definitions and notations of number theory as in T. M. Apostol [1].

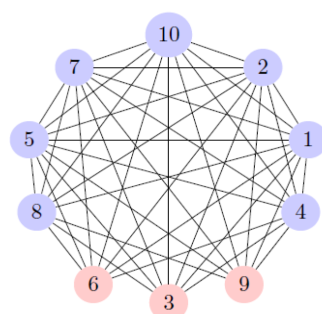
Definition 1.1. [7] Let G be a graph with n vertices. The graph G is called a linear Diophantine graph (or simply, a Diophantine graph) if there exists a bijective map $f : V \to \{1, 2, \dots, n\}$ such that for all $uv \in E$, $(f(u), f(v)) \mid n$. Such a map f is called a Diophantine labeling of G. A maximal Diophantine graph with n vertices, denoted by (D_n, f), is a Diophantine graph such that adding any new edge yields a non-Diophantine graph. If there is no ambiguity, we drop f from (D_n, f) and write it simply D_n.

A formula that computes the number of edges of D_n can be found in [7]. In what follows, we represent (D_n, f) as the maximal Diophantine graph of order n with Diophantine labeling f. Some maximal Diophantine graphs are given in the next example.

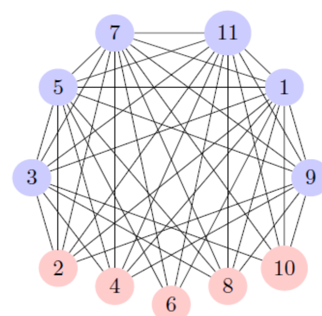
Example 1.2. The following three graphs are examples of maximal Diophantine graphs.



(a) Graph D_9



(b) Graph D_{10}



(c) Graph D_{11}

FIGURE 1. Some maximal Diophantine graphs D_9, D_{10} and D_{11} respectively

The p-adic valuation of $n \in \mathbb{Z}^+$ is defined by $v_p(n) := t$ such that $p^t \mid n$ and $p^{t+1} \nmid n$, where $p \in \mathbb{P}$, $t \in \mathbb{N} := \mathbb{Z}^+ \cup \{0\}$, $\mathbb{Z}^+$ is the set of positive integers and $\mathbb{P}$ is the set of prime numbers. In the rest of this paper, the following arithmetic functions π, ω and τ will be used, (see [1]): Let $n \in \mathbb{Z}^+$.

$$\pi(n) := \big|\{p \in \mathbb{P} : 2 \le p \le n\}\big|,$$
$$\omega(n) := \big|\{p \in \mathbb{P} : p \mid n,\ 2 \le p \le n\}\big|,$$
$$\tau(n) := \big|\{d \in \mathbb{Z}^+ : d \mid n\}\big|.$$

Definition 1.3. [7] A number $p^{\acute{v}_p(n)}$ is called a critical prime power number with respect to a prime number p and a positive integer n, where $\acute{v}_p(n) := v_p(n)+1$ is called the successor of the p-adic valuation.

Lemma 1.4. [7] *For every $u, v \in V(D_n)$, $uv \notin E(D_n)$ if and only if there exists $p \in \mathbb{P}$ such that*

$$f(u), f(v) \in M_{p^{\acute{v}_p(n)}} := \left\{ kp^{\acute{v}_p(n)} : \ k = 1, 2, \dots, \left\lfloor \frac{n}{p^{\acute{v}_p(n)}} \right\rfloor \right\}.$$

Theorem 1.5. [7] *For every $u \in V(D_n)$, $\deg(u) = n-1$ if and only if either $f(u) \mid n$ or $\frac{n}{2} < f(u) = p^{\acute{v}_p(n)} < n$, where $p \in \mathbb{P}$.*

The reduced label $f^*(u)$ of a vertex u in a labeled graph G with n vertices is defined as $f^*(u) := \frac{f(u)}{(f(u),n)}$. Define the set

$$F := \{f(u) : u \in V(D_n), \ \deg(u) = n-1\},$$

where D_n is the maximal Diophantine graph of order n with a Diophantine labeling f.

Lemma 1.6. *If $p^{\acute{v}_p(n)}$ is the critical prime power number with respect to a prime number p and a positive integer n, then the set theoretic complement F^c is given by*

$$F^c = \bigcup_{p^{\acute{v}_p(n)} < \frac{n}{2}} M_{p^{\acute{v}_p(n)}},$$

where $F^c = \{1, 2, \dots, n\} - F$.

Proof. Let $f(u) \in F^c$. Then $f(u) \notin F$. Therefore, $\deg(u) < n-1$. Thus, there exists $w \in V(D_n)$ such that $uw \notin E(D_n)$. Consequently, using Lemma 1.4, there exists $p_0 \in \mathbb{P}$ such that $p_0^{\acute{v}_{p_0}(n)} \mid f(u), f(w)$. Hence,

$$f(u) \in \bigcup_{p^{\acute{v}_p(n)} < \frac{n}{2}} M_{p^{\acute{v}_p(n)}}.$$

On the other hand, let $f(u) \in \bigcup_{p^{\acute{v}_p(n)} < \frac{n}{2}} M_{p^{\acute{v}_p(n)}}$. Then there exists $p_0 \in \mathbb{P}$ such that $p_0^{\acute{v}_{p_0}(n)} \mid f(u)$ and $p_0^{\acute{v}_{p_0}(n)} < \frac{n}{2}$. Therefore, there exists $w \in V(D_n)$ such that $w \neq u$ and $p_0^{\acute{v}_{p_0}(n)} \mid f(w)$. Then, using Lemma 1.4, $uw \notin E(D_n)$. Therefore, $\deg(u) < n-1$. Consequently $f(u) \notin F$. Thus, $f(u) \in F^c$. Hence, the proof follows. $\square$

Lemma 1.7. [7] *Let $u, v \in V(D_n)$. If $f(u) \mid f(v)$, then $\deg(u) \geq \deg(v)$.*

Corollary 1.8. [7] *Let $u, v \in V(D_n)$ such that $f^*(u) > 1$, $f^*(v) > 1$. If $f^*(u), f^*(v)$ have the same prime factors, then $\deg(u) = \deg(v)$.*

Corollary 1.9. [7] *Let $u, v \in V(D_n)$ such that $f(v) = tf(u)$ for some $t \geq 1$. If $t \mid n$ and $(t, f(u)) = 1$, then $\deg(u) = \deg(v)$.*

Theorem 1.10. [7] *Let $u, v \in V(D_n)$ such that $f(u) \mid f(v)$, $f(v)$ is not a prime power number and $f^*(u) > 1$. If $\deg(u) = \deg(v)$, then $f^*(u), f^*(v)$ have the same prime factors.*

This manuscript is structured as follows. Section 2 provides some results of γ-labelings. Section 3 is partitioned into three subsections, each presents some results related to maximal Diophantine graphs. Subsection 3.1 discusses some basic bounds and necessary and sufficient conditions for maximal Diophantine graphs. Subsection

3.2 and 3.3 provided some necessary conditions and explore properties of the minimum degree and the degree sequence in maximal Diophantine graphs. Section 4 includes some examples of non-Diophantine graphs to explain the relation among these necessary conditions.

2. γ-Labelings of Graphs

The following definition is a generalization of Definition 1.1.

Definition 2.1. Let G be a graph with n vertices. The graph G is called an γ-labeled graph if there exists a bijective map $f : V \to \{x_1, x_2, \dots, x_n\}$ such that $f(u), f(v)$ satisfy some conditions, where $\{x_1, x_2, \dots, x_n\}$ is any set of n elements. Such a map f is called an γ-labeling. A maximal γ-labeled graph with n vertices, denoted by (Γ_n, f), is a γ-labeled graph in which for all $uv \notin E(\Gamma_n)$, $\Gamma_n + \{uv\}$ is not a γ-labeled graph.

The reader should not be confused the notion of γ-labeling as provided in Definition 2.1 with the concept of α-valuation that presented in the seminal work of A. Rosa [9].

Definition 2.2. [6] Let $(G_1, f_1), (G_2, f_2)$ be two labeled graphs, where the two maps $f_1 : V(G_1) \to \{x_1, x_2, \dots, x_n\}$ and $f_2 : V(G_2) \to \{x_1, x_2, \dots, x_n\}$ are bijective. The labeled graphs $(G_1, f_1), (G_2, f_2)$ are said to be labeling isomorphic, denoted by $(G_1, f_1) \cong_l (G_2, f_2)$, if there exists a bijective map $\varphi : V(G_1) \to V(G_2)$ such that for all $u, v \in V(G_1)$, $uv \in E(G_1)$ if and only if $\varphi(u)\varphi(v) \in E(G_2)$ and $f_1(u) = \big(f_2 \circ \varphi\big)(u)$.

Theorem 2.3. *A maximal γ-labeled graph Γ_n is unique up to labeling isomorphism.*

Proof. Suppose (Γ_n, f_1) and $(\acute{\Gamma}_n, f_2)$ are two maximal γ-labeled graphs with order n, where $f_1 : V(\Gamma_n) \to \{x_1, x_2, \dots, x_n\}$ and $f_2 : V(\acute{\Gamma}_n) \to \{x_1, x_2, \dots, x_n\}$ are γ-labelings of Γ_n and $\acute{\Gamma}_n$ satisfying certain conditions, say condition C. Define a map

$$\varphi : V(\Gamma_n) \to V(\acute{\Gamma}_n) \quad \text{by} \quad \varphi(u) = f_2^{-1}(f_1(u)).$$

Therefore, φ is one to one (for, let $u, v \in V(\Gamma_n)$, $\varphi(u) = \varphi(v)$. Then we obtain $f_2^{-1}(f_1(u)) = f_2^{-1}(f_1(v))$; accordingly, $f_1(u) = f_1(v)$. Consequently, $u = v$), φ is onto (since φ is one to one and $|V(\Gamma_n)| = |V(\acute{\Gamma}_n)| = n$), φ is preserving the adjacency and non-adjacency of Γ_n and $\acute{\Gamma}_n$ (for the reason that let $u, v \in V(\Gamma_n)$ such that $uv \in E(\Gamma_n)$. Then we have the two labels $f_1(u), f_1(v)$ satisfy C. Since $f_1(u) = f_2(\varphi(u))$ and $f_1(v) = f_2(\varphi(v))$ (see Figure 2), we get $f_2(\varphi(u)), f_2(\varphi(v))$ satisfy C. Consequently, $\varphi(u)\varphi(v) \in E(\acute{\Gamma}_n)$ and the converse is similar) and let $u \in V(\Gamma_n)$, $\varphi(u) = f_2^{-1}(f_1(u))$. Therefore, $f_1(u) = f_2(\varphi(u)) = (f_2 \circ \varphi)(u)$. Hence, the two graphs (Γ_n, f_1) and $(\acute{\Gamma}_n, f_2)$ are labeling isomorphic. $\square$

$$\begin{tikzcd}
V(\Gamma_n) \arrow[rr, "\varphi"] \arrow[dr, "f_1"'] & & V(\acute{\Gamma}_n) \arrow[dl, "f_2"] \\
& \{x_1, x_2, \dots, x_n\} \arrow[u, phantom, "\equiv"] &
\end{tikzcd}$$

Figure 2. $(\Gamma_n, f_1) \cong_l (\acute{\Gamma}_n, f_2)$

Corollary 2.4. *The graphs D_n is unique up to labeling isomorphism.*

Theorem 2.5. *Suppose G is a graph of order n and Γ_n is the maximal γ-labeled graph of order n. G is an γ-labeled graph if and only if G is labeling isomorphic to a spanning subgraph of Γ_n.*

Proof. Suppose Γ_n is the maximal γ-labeled graph with order n and a graph G is a γ-labeled graph with order n. Then there exists $f : V(G) \to \{x_1, x_2, \dots, x_n\}$ is a bijective map such that $f(u), f(v)$ satisfy certain conditions, say condition C and define

$$T := \{uv : uv \notin E(G) \text{ and } f(u), f(v) \text{ satisfy } C\}.$$

Consequently, the spanning supergraph $G + T$ of G is a γ-labeled graph of order n and the set $E(G) \cup T$ is set of all edges such that $f(u), f(v)$ satisfy C. Let $\acute{u}\acute{v} \notin E(G) \cup T$. Then we have that the two labels $f(\acute{u}), f(\acute{v})$ do not satisfy C. Therefore, the spanning supergraph $G + (T \cup \{\acute{u}\acute{v}\})$ of G is not a γ-labeled graph with a γ-labeling satisfying C. Consequently, $G + T$ is the maximal γ-labeled graph of order n. Thus, using Theorem 2.3, we have that $G + T$ is labeling isomorphic to Γ_n. Hence, the graph G is labeling isomorphic to a spanning subgraph of the maximal γ-labeled graph Γ_n.

Conversely, suppose Γ_n is the maximal γ-labeled graph with order n and a graph G is labeling isomorphic to a spanning subgraph of the maximal γ-labeled graph Γ_n. Let T be the set of deleted edges of Γ_n such that the graph G is labeling isomorphic to $\Gamma_n - T$. Then we have

$$|V(G)| = |V(\Gamma_n - T)| = |V(\Gamma_n)| \quad \text{and} \quad V(\Gamma_n) = V(\Gamma_n - T).$$

Therefore, using the same γ-labeling of Γ_n, we have $\Gamma_n - T$ is a γ-labeled graph. Since the graph G is labeling isomorphic to $\Gamma_n - T$, hence the graph G is a γ-labeled graph. $\square$

Corollary 2.6. *A graph G of order n is Diophantine if and only if G is labeling isomorphic to a spanning subgraph of D_n.*

3. Basic Bounds of the Maximal Diophantine Graphs D_n

3.1. **Some Necessary and Sufficient Conditions for D_n.** Through this paper, let $F(G)$ denote the number of full degree vertices of a graph G. The next two theorems present two different methods that compute the quantity $F(D_n)$.

Theorem 3.1. *If $p_i^{\acute{v}_{p_i}(n)} < \frac{n}{2}$, $i = 1, 2, \dots, r$, then the number of full degree vertices in D_n is given by*

$$F(D_n) = n - \sum_{1 \le i \le r} \left\lfloor \frac{n}{p_i^{\acute{v}_{p_i}(n)}} \right\rfloor + \sum_{1 \le i < j \le r} \left\lfloor \frac{n}{p_i^{\acute{v}_{p_i}(n)} p_j^{\acute{v}_{p_j}(n)}} \right\rfloor - \cdots + (-1)^r \left\lfloor \frac{n}{\prod\limits_{1 \le i \le r} p_i^{\acute{v}_{p_i}(n)}} \right\rfloor,$$

where $p_1, p_2, \dots, p_r$ are distinct prime numbers.

Proof. Let $p_i^{\acute{v}_{p_i}(n)} < \frac{n}{2}$, $i = 1, 2, \dots, r$ and the set

$$F := \{f(u) : u \in V(D_n),\ \deg(u) = n - 1\}.$$

Then

$$F(D_n) = |F| = n - |F^c|.$$

Consequently, using Lemma 1.6,

$$F(D_n) = n - \left| \bigcup_{i=1}^{r} M_{p_i^{\acute{v}_{p_i}(n)}} \right|,$$

Therefore, using the inclusion-exclusion principle (see [10]),

$$F(D_n) = n - \sum_{1\le i\le r} \left\lfloor \frac{n}{p_i^{\acute{v}_{p_i}(n)}} \right\rfloor + \sum_{1\le i<j\le r} \left\lfloor \frac{n}{p_i^{\acute{v}_{p_i}(n)} p_j^{\acute{v}_{p_j}(n)}} \right\rfloor - \cdots + (-1)^r \left\lfloor \frac{n}{\prod_{1\le i\le r} p_i^{\acute{v}_{p_i}(n)}} \right\rfloor .$$

□

For a very large $n \in \mathbb{Z}^+$, the above formula does not provide efficient upper and lower bounds for the quantity $F(D_n)$. There is an alternative approach to determine the quantity $F(D_n)$ by using the following arithmetic function

$$\xi_x(n) := \left| \left\{ p^{\acute{v}_p(n)} : p \mid n,\ x < p^{\acute{v}_p(n)} < n,\ p \in \mathbb{P} \right\} \right| ,$$

where $n \in \mathbb{Z}^+$ and a positive real number $x < n$. This function is utilized for computing not only the number of vertices with full degree in D_n but also the order of the maximal clique of D_n as follows in Theorems 3.2, 3.3. Obviously, for every $n \in \mathbb{Z}^+$, $\xi_1(n) \le \omega(n)$ and for every $p \in \mathbb{P}$, $k \in \mathbb{Z}^+$ and a positive real number $x < n$, $\xi_x\left(p^k\right) = 0$.

Theorem 3.2. *The number of vertices with full degree in D_n is given by*

$$F(D_n) = \tau(n) + \pi(n-1) - \pi\left(\frac{n}{2}\right) + \xi_{\frac{n}{2}}(n).$$

In particular, if $n = p^k$, where $p \in \mathbb{P}$ and $k \ge 1$, then

$$F(D_n) = \pi(n-1) - \pi\left(\frac{n}{2}\right) + k + 1.$$

Proof. Let D_n be the maximal Diophantine graph with order n. Define the following three sets

$$\begin{aligned} S_1 &:= \{d \in \mathbb{Z}^+ : d \mid n\}, \\ S_2 &:= \left\{p \in \mathbb{P} : \frac{n}{2} < p < n\right\}, \\ S_3 &:= \left\{p^{\acute{v}_p(n)} : p \mid n,\ \frac{n}{2} < p^{\acute{v}_p(n)} < n,\ p \in \mathbb{P}\right\}. \end{aligned}$$

Consequently, using Theorem 1.5, one can see that $S_1 \cup S_2 \cup S_3$ is the set of labels of the full degree vertices in D_n. Clearly, S_1, S_2 and S_2 are mutually disjoint sets and

$$|S_1| = \tau(n), \quad |S_2| = \pi(n-1) - \pi\left(\frac{n}{2}\right) \quad \text{and} \quad |S_3| = \xi_{\frac{n}{2}}(n).$$

Hence,

$$F(D_n) = \tau(n) + \pi(n-1) - \pi\left(\frac{n}{2}\right) + \xi_{\frac{n}{2}}(n).$$

In case of $n = p^k$, where $p \in \mathbb{P}$ and $k \ge 1$, we have $\tau(n) = k+1$ and $\xi_{\frac{n}{2}}(n) = 0$. Therefore,

$$F(D_n) = \pi(n-1) - \pi\left(\frac{n}{2}\right) + k + 1.$$

□

The clique number, denoted by $Cl(G)$, is the order of the maximal clique of a graph G. Although $\omega(G)$ is the standard notation of the clique number, we have chosen $Cl(G)$ in this study to prevent confusion with the arithmetic function $\omega(n)$. The following theorem gives the order of the maximal clique in D_n.

Theorem 3.3. *The clique number of D_n is given by*

$$Cl(D_n) = \tau(n) + \pi(n) - \omega(n) + \xi_1(n).$$

In particular, if $n = p^k$, where $p \in \mathbb{P}$ and $k \geq 1$, then $Cl(D_n) = \pi(n) + k$.

Proof. Let D_n be the maximal Diophantine graph with order n. Define the following three sets

$$\begin{aligned} S_1 &:= \{d \in \mathbb{Z}^+ : d \mid n\}, \\ S_2 &:= \{p \in \mathbb{P} : p \nmid n,\ 1 < p < n\}, \\ S_3 &:= \left\{p^{\acute{v}_p(n)} : p \mid n,\ 1 < p^{\acute{v}_p(n)} < n,\ p \in \mathbb{P}\right\}. \end{aligned}$$

Therefore, any two vertices in $V(D_n)$ that is labeled by integers from the set $S_1 \cup S_2 \cup S_3$ are adjacent, since for any two distinct labels ℓ_1, ℓ_2, we have

$$\begin{cases} (\ell_1, \ell_2) = 1, & \text{if } \ell_1, \ell_2 \in S_2 \cup S_3. \\ (\ell_1, \ell_2) \mid n, & \text{if } \ell_1, \ell_2 \in S_1 \cup S_2. \\ (\ell_1, \ell_2) \mid n, & \text{if } \ell_1, \ell_2 \in S_1 \cup S_3. \end{cases}$$

Consequently, one can see that $S_1 \cup S_2 \cup S_3$ is the set of labels of vertices that are in the maximal clique of D_n. Suppose contrarily that $u \in V(D_n)$ is a vertex of the maximal clique in D_n such that $f(u) \notin S_1 \cup S_2 \cup S_3$. Then $f(u) \nmid n$. Therefore, there exists a prime number p_0 such that $p_0^{\acute{v}_{p_0}(n)} \mid f(u)$; otherwise, for every prime number p, $p^{\acute{v}_p(n)} \nmid f(u)$, so $v_p(f(u)) < \acute{v}_p(n) = v_p(n) + 1$. Then $v_p(f(u)) \leq v_p(n)$ which is a contradiction of $f(u) \nmid n$. Let $\ell = p_0^{\acute{v}_{p_0}(n)}$ be a certain label. Then we have $\ell \in S_2 \cup S_3$, $\ell \mid f(u)$ and $\ell \neq f(u)$. So, $(f(u), \ell) = \ell \nmid n$, which contradicts the completeness to the maximal clique in D_n. Therefore, the set $S_1 \cup S_2 \cup S_3$ has all labels of vertices in the maximal clique of D_n. Obviously, S_1, S_2 and S_3 are mutually disjoint sets and

$$|S_1| = \tau(n), \quad |S_2| = \pi(n) - \omega(n) \quad \text{and} \quad |S_3| = \xi_1(n).$$

Therefore,

$$Cl(D_n) = \tau(n) + \pi(n) - \omega(n) + \xi_1(n).$$

If $n = p^k$, where $p \in \mathbb{P}$ and $k \geq 1$, we have $\tau(n) = k + 1$, $w(n) = 1$ and $\xi_1(n) = 0$. Hence, $Cl(D_n) = \pi(n) + k$. $\square$

The chromatic number, denoted by $\chi(G)$, is the minimum number of colours needed to colour the vertices of the graph G. So, no two adjacent vertices receive the same colour. Therefore, using Lemma 1.4 and Theorem 3.3, The chromatic number of D_n is given by

$$\chi(D_n) = \tau(n) + \pi(n) - \omega(n) + \xi_1(n).$$

Theorem 3.4. *The independence number of D_n is given by*

$$\alpha(D_n) = \max_{2 \leq p \leq n} \left\lfloor \frac{n}{p^{\acute{v}_p(n)}} \right\rfloor,$$

where $p \in \mathbb{P}$. In particular, if n is odd, then $\alpha(D_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $f(u), f(v)$ be two labels, where $u, v \in V(D_n)$. Then, using Lemma 1.4, $uv \notin E(D_n)$ if and only if there exists $p \in \mathbb{P}$ such that $f(u), f(v) \in M_{p^{\acute{v}_p(n)}}$. Thus, the set of vertices of D_n with labels in $M_{p^{\acute{v}_p(n)}}$ is an independent set. Hence,

$$\alpha(D_n) = \max_{2\leq p\leq n} \left| M_{p^{\acute{v}_p(n)}} \right| = \max_{2\leq p\leq n} \left\lfloor \frac{n}{p^{\acute{v}_p(n)}} \right\rfloor .$$

If n is an odd number, then the set of nonadjacent vertices in D_n with labels in $M_2 = \left\{2, 4, 6, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor\right\}$ is a maximal independent set. Hence, $\alpha(D_n) = \left\lfloor \frac{n}{2} \right\rfloor$. □

Lemma 3.5. *For every a prime number $p \leq \frac{n}{2}$, $p^{\acute{v}_p(n)} > \frac{n}{2}$ if and only if D_n is a complete graph.*

Proof. Assume $p \leq \frac{n}{2}$ is a prime number such that $p^{\acute{v}_p(n)} > \frac{n}{2}$. Suppose contrarily that the maximal Diophantine graph D_n is not a complete graph. Therefore there exist two vertices $u, v \in V(D_n)$ such that $uv \notin E(D_n)$. Then, using lemma 1.4, there exists $p_0 \in \mathbb{P}$ such that $f(u), f(v) \in M_{p_0^{\acute{v}_{p_0}(n)}}$. Let $f(u) = tp_0^{\acute{v}_{p_0}(n)}$ and $f(v) = sp_0^{\acute{v}_{p_0}(n)}$ for some $t, s \geq 1$ and $t < s$. Then, $p_0^{\acute{v}_{p_0}(n)} < \frac{n}{s} \leq \frac{n}{2}$, this contradicts the assumption. Hence, D_n is a complete graph.

Conversely, let D_n be a complete graph and consider contrarily that there exists a prime number $p \leq \frac{n}{2}$ such that $p^{\acute{v}_p(n)} < \frac{n}{2}$; otherwise, if $p^{\acute{v}_p(n)} = \frac{n}{2}$, then $p^{\acute{v}_p(n)} \mid n$ that is a contradiction. Let $f(u) = p^{\acute{v}_p(n)}$ and $f(v) = 2p^{\acute{v}_p(n)}$, where $u, v \in V(D_n)$. Then we have

$$(f(u), f(v)) = \left(p^{\acute{v}_p(n)}, 2p^{\acute{v}_p(n)}\right) = p^{\acute{v}_p(n)} \nmid n.$$

Therefore, $F(D_n) < n$. Consequently, D_n is not a complete graph, this contradicts the hypothesis. □

Theorem 3.6. *(**sufficient condition**) Let G be a graph with order n. If*

$$\alpha(G) \geq n - F(D_n),$$

then G is a Diophantine graph.

Proof. Let G be a graph with n vertices such that $\alpha(G) \geq n - F(D_n)$. Suppose S is a subgraph of G with number of vertices less than or equal to $F(D_n)$ vertices. Then, using Theorem 1.5, the number that is either a divisor of n or is of the form $p^{\acute{v}_p(n)}$, where $\frac{n}{2} < p^{\acute{v}_p(n)} < n$ can be used as labels in the vertices of the subgraph S of G. Therefore, the other labels can be assigned in the remaining independent vertices of order $\alpha(G)$ from the graph G. Hence, G is a Diophantine graph. □

Example 3.7. The graph $G = K_3 + \overline{K_4}$ satisfies the sufficient condition in Theorem 3.6 as $\alpha(G) \geq 7 - F(D_7)$; accordingly, the graph G is a Diophantine graph as seen in Figure 3 part (a).

This sufficient condition is not a necessary condition. For instance, D_7, as seen in Figure 3 part (b), is a Diophantine graph. However, D_7 does not meet the sufficient condition, since $\alpha(D_7) < 7 - F(D_7)$.

3.2. **The Minimum Degree Vertex with Minimum Label.** Let $\delta(G)$ denote the minimum degree of the vertices in a graph G and

$$V_\delta(G) := \{u \in V(G) : \deg(u) = \delta(G)\}.$$

(a) The graph $G = K_3 + \overline{K_4}$

(b) The maximal Diophantine graph D_7

FIGURE 3. Two Diophantine graphs

So, we denote the following $f(u_0) := \min\{f(u) : u \in V_\delta(D_n)\}$, in which the vertex u_0 is the vertex in $V_\delta(D_n)$ with minimum label and $f(u_0)$ is the smallest label of a vertex in $V_\delta(D_n)$.

Remark 3.8. Let u_0 be the vertex in $V_\delta(D_n)$ with minimum label.

1. $f(u_0) = 1$ if and only if D_n is a complete graph.
2. For every $n > 3$, $\delta(D_n) \geq 3$.
3. If D_n is not a complete graph, then $F(D_n) \leq \delta(D_n)$.

Theorem 3.9. *Let u_0 be the vertex in $V_\delta(D_n)$ with minimum label and the finite sequence*

$$p_1^{\acute{v}_{p_1}(n)} < p_2^{\acute{v}_{p_2}(n)} < \cdots < p_s^{\acute{v}_{p_s}(n)} < \frac{n}{2}, \tag{3.1}$$

where p_i, $i = 1, 2, \ldots, s$ are distinct prime numbers, be found. Then there exists $r \leq s$ such that

$$f(u_0) = \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} \quad \text{if and only if} \quad \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} < n < \prod_{i=1}^{r+1} p_i^{\acute{v}_{p_i}(n)}.$$

Proof. Let u_0 be the vertex in $V_\delta(D_n)$ with minimum label and the finite sequence be given in equation (3.1). Let $f(u_0) = \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)}$ for some $r \leq s$. Then $\prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} < n$. Suppose contrarily that $p_{r+1}^{\acute{v}_{p_{r+1}}(n)} f(u_0) < n$. Then there exists a vertex $v \in V(D_n)$ such that $f(v) = p_{r+1}^{\acute{v}_{p_{r+1}}(n)} f(u_0)$. Therefore, using Lemma 1.7, $\deg(v) \leq \deg(u_o)$. Since the degree of u_0 is minimum, we get

$$\deg(v) = \deg(u_0).$$

However, since $f^*(u_0) = \frac{f(u_o)}{(f(u_o),n)} = \prod_{i=1}^{r} p_i$ and $f^*(v) = \frac{f(v)}{(f(v),n)} = \prod_{i=1}^{r+1} p_i$, therefore the reduced labels $f^*(u_0)$ and $f^*(v)$ do not have the same prime factors, which contradicts Theorem 1.10. Hence, $\prod_{i=1}^{r+1} p_i^{\acute{v}_{p_i}(n)} > n$.

Conversely, let $\prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} < n$ and $\prod_{i=1}^{r+1} p_i^{\acute{v}_{p_i}(n)} > n$ for some $r < s$. Since u_0 is the vertex in $V_\delta(D_n)$ with minimum label, therefore $f(u_0) = \min\{f(u) : u \in V_\delta(D_n)\}$. Then we have

$$f(u_0) = \prod_{i=1}^{t} q_i^{\alpha_i} \prod_{j=1}^{t_1} h_j^{\beta_j}. \tag{3.2}$$

where q_i, h_j are distinct prime numbers and α_i, β_j are two non-negative integers such that for every $i = 1, 2, \dots t$, $j = 1, 2, \dots t_1$

$$\alpha_i \geq \acute{v}_{q_i}(n) \quad \text{and} \quad 0 \leq \beta_j < \acute{v}_{h_j}(n). \tag{3.3}$$

Clearly, the two factors of equation (3.2) are relatively prime and $\prod_{j=1}^{t_1} h_j^{\beta_j} \mid n$. Otherwise, if $\prod_{j=1}^{t_1} h_j^{\beta_j} \nmid n$, then there exists a prime number $h_{j_0} \mid f(u_0)$ such that $h_{j_0}^{\beta_{j_0}} \nmid n$. Therefore, $\beta_{j_0} \geq \acute{v}_{h_{j_0}}(n)$, which contradicts equation (3.3). Let $u \in V(D_n)$ such that $f(u) = \prod_{i=1}^{t} q_i^{\alpha_i}$. Then, using equation (3.2),

$$f(u_0) = f(u) \prod_{j=1}^{t_1} h_j^{\beta_j}. \tag{3.4}$$

Therefore, using Corollary 1.9 and equation (3.4),

$$\deg(u) = \deg(u_0) \quad \text{and} \quad f(u) \leq f(u_o),$$

which contradicts the hypothesise of the minimal label of u_0 unless $\beta_j = 0$ in equation (3.4). Thus,

$$f(u_0) = f(u) = \prod_{i=1}^{t} q_i^{\alpha_i}, \tag{3.5}$$

where $\alpha_i \geq \acute{v}_{q_i}(n)$. Then $\alpha_i = \acute{v}_{q_i}(n) + k_i$ for some $k_i \geq 0$, where $i = 1, 2, \dots t$. So, using equation (3.5), we get

$$f(u_0) = \prod_{i=1}^{t} q_i^{\acute{v}_{q_i}(n)+k_i} = \prod_{i=1}^{t} q_i^{\acute{v}_{q_i}(n)} \prod_{i=1}^{t} q_i^{k_i}. \tag{3.6}$$

Let $v \in V(D_n)$ such that $f(v) = \prod_{i=1}^{t} q_i^{\acute{v}_{q_i}(n)}$. Then, using equation (3.6), we obtain

$$f(u_0) = \prod_{i=1}^{t} q_i^{k_i} f(v). \tag{3.7}$$

Consequently, $f^*(u_0) = \prod_{i=1}^{t} q_i^{k_i+1}$ and $f^*(v) = \prod_{i=1}^{t} q_i$. Thus, $f^*(u_0), f^*(v)$ have the same prime factors. Therefore, using corollary 1.8 and equation (3.7),

$$\deg(v) = \deg(u_o) \quad \text{and} \quad f(v) \leq f(u_o).$$

Since $f(u_o)$ is the minimum label, we get $k_j = 0$ in equation (3.7). Therefore,

$$f(u_0) = f(v) = \prod_{i=1}^{t} q_i^{\acute{v}_{q_i}(n)}. \tag{3.8}$$

Given the finite sequence in equation (3.1). Then we have

$$\left| M_{p_1^{\acute{v}_{p_1}(n)}} \right| \geq \left| M_{p_2^{\acute{v}_{p_2}(n)}} \right| \geq \cdots \geq \left| M_{p_s^{\acute{v}_{p_s}(n)}} \right|, \tag{3.9}$$

where p_i, $i = 1, 2, \dots, s$ are distinct prime numbers. Since there exists $r \leq s$ such that $\prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} < n$ and $\prod_{i=1}^{r+1} p_i^{\acute{v}_{p_i}(n)} > n$, therefore we have that there exists a vertex

$w \in V(D_n)$ such that $f(w) = \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)}$. Since u_0 is the vertex in $V_\delta(D_n)$ with minimum label and using equations (3.8) and (3.9), we get

$$f(u_0) \in M_{p_1^{\acute{v}_{p_1}(n)}},\ f(u_0) \in M_{p_2^{\acute{v}_{p_2}(n)}},\ \dots,\ f(u_0) \in M_{p_t^{\acute{v}_{p_t}(n)}},$$

for some $t \le r \le s$. Then $f(u_0) \mid f(w)$. Consequently, using lemma 1.7,

$$\deg(w) \le \deg(u_0)$$

which contradicts the minimum degree of u_0 unless $t = r$. Hence,

$$f(u_0) = f(w) = \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)}.$$

□

One notices that $f(u_0)$ is the largest partial product from the sequence given in equation (3.1) that is less than n as shown in the following equation

$$f(u_0) = \max_{r=1}^{s} \left\{ \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} : \prod_{i=1}^{r} p_i^{\acute{v}_{p_i}(n)} < n \right\}.$$

Moreover, one can see that $\delta(D_n) = \deg(u_0)$ and the degree of every vertex in D_n is found in [7].

Theorem 3.10. *There exists a vertex $w \in V_\delta(D_n)$ such that $\frac{n}{2} < f(w) < n$.*

Proof. let $u \in V_\delta(D_n)$. Then the two cases follow, in the case of $\frac{n}{2} < f(u) < n$, we have nothing to prove. In the other case of $1 < f(u) < \frac{n}{2}$, we get

$$\frac{n}{2} < 2^t f(u) < n$$

for some $t > 0$. Let $f(w) = 2^t f(u)$, where $w \in V(D_n)$, $t > 0$. Then we have $\frac{n}{2} < f(w) < n$. Using Lemma 1.7, we get $\deg(w) \le \deg(u)$. Since the vertex $u \in V_\delta(D_n)$, therefore $\deg(w) = \deg(u) = \delta(D_n)$. Hence, from the two cases, the proof follows. □

3.3. The Degree Sequences of graphs.

Definition 3.11. Let G be a graph of order n. A finite sequence

$$S_G = \{g_i\}_{i=0}^{n-1} = (g_0, g_1, \dots, g_{n-1}),$$

where $g_i := |\{v \in V(G) : \deg(v) = i\}|$, $i = 0, 1, \dots, n-1$ is called the vertex-degree sequence of G.

The reader notices that the standard notion of degree sequence is different in this literature. Moreover, one can obtain the following two equations

$$\sum_{i=0}^{n-1} g_i = |V(G)| \text{ and } \sum_{i=0}^{n-1} i g_i = 2|E(G)|.$$

Example 3.12. $S_{D_7} = (0,0,0,1,2,1,3)$ and $S_{D_{11}} = (0,0,0,0,1,1,3,0,2,1,3)$.

Theorem 3.13. *Let G be a graph of order n with $S_G = \{g_i\}_{i=0}^{n-1}$ and D_n be the maximal Diophantine graph of order n with $S_{D_n} = \{d_i\}_{i=0}^{n-1}$. If the graph G is Diophantine then for each $k \in \{0, 1, \dots, n-1\}$,*

$$\sum_{i=0}^{k} g_i \ge \sum_{i=0}^{k} d_i.$$

Proof. Assume (D_n, f_1) is the maximal Diophantine graph of order n with $S_{D_n} = \{d_i\}_{i=0}^{n-1}$ and (G, f_2) is a Diophantine graph of order n with $S_G = \{g_i\}_{i=0}^{n-1}$, where $f_1 : V(D_n) \to \{1, 2, \ldots, n\}$, $f_2 : V(G) \to \{1, 2, \ldots, n\}$. are Diophantine labelings of D_n, G respectively. Suppose contrarily that there exists $k_0 \in \{0, 1, \ldots, n-1\}$ such that

$$\sum_{i=0}^{k_0} g_i < \sum_{i=0}^{k_0} d_i. \tag{3.10}$$

Using Corollary 2.6, we get that (G, f_2) is labeling isomorphic to a spanning subgraph of (D_n, f_1). Let $(\acute{G}, f_1)$ be a spanning subgraph of (D_n, f_1) such that $(G, f_2) \cong_l (\acute{G}, f_1)$. Then there is a bijective map $\varphi : V(G) \to V(\acute{G})$ such that for all $u, \acute{u} \in V(G)$, $u\acute{u} \in E(G)$ if and only if $\varphi(u)\varphi(\acute{u}) \in E(\acute{G})$ and $f_2(u) = f_1(\varphi(u))$. Then for every $u \in V(G)$,

$$\deg(u) = \deg(\varphi(u)). \tag{3.11}$$

Define a map $\acute{\varphi} : V(G) \to V(D_n)$ such that for every $u \in V(G)$, $\acute{\varphi}(u) := \varphi(u)$. Then the map $\acute{\varphi}$ is bijective and for every $u \in V(G)$, $f_2(u) = f_1(\varphi(u)) = f_1(\acute{\varphi}(u))$. Since $(\acute{G}, f_1)$ is a spanning subgraph of (D_n, f_1) and using equation (3.11), one can see that for every $u \in V(G)$,

$$\deg(u) \leq \deg(\acute{\varphi}(u)). \tag{3.12}$$

Define the following two sets

$$A_{D_n}(k) := \{v \in V(D_n) : \deg(v) \leq k\} \text{ and } A_G(k) := \{u \in V(G) : \deg(u) \leq k\},$$

where $k = 0, 1, \ldots, n-1$. Therefore, using equation (3.10), $|A_G(k_0)| < |A_{D_n}(k_0)|$, for some $k_0 = 0, 1, \ldots, n-1$. Then there exists a vertex $v \in A_{D_n}(k_0) \subseteq V(D_n)$ such that $u = \acute{\varphi}^{-1}(v) \notin A_G(k_0)$. Consequently, we get the following

$$\deg(u) > k_0 \geq \deg(v) = \deg(\acute{\varphi}(u)),$$

which contradicts equation (3.12). Hence, the proof follows. □

Remark 3.14. Let G be a graph of order n with $S_G = \{g_i\}_{i=0}^{n-1}$ and D_n be the maximal Diophantine graph of order n with $S_{D_n} = \{d_i\}_{i=0}^{n-1}$. If $\sum_{i=0}^{k} g_i \geq \sum_{i=0}^{k} d_i$ holds for every $k = 0, 1, \ldots, n-1$, then

$$|E(G)| \leq |E(D_n)|, \quad F(G) \leq F(D_n) \quad \text{and} \quad \delta(G) \leq \delta(D_n).$$

The proof of Remark 3.14 closely resembles the corollaries in [11]. The following table presents the quantities $|E(G)|$, $F(G)$, $Cl(G)$, $\alpha(G)$, $\delta(G)$ and S_G, where $G = D_n$, $n = 4, \ldots, 20$. In what follows, we will refer to these quantities as basic bounds of the graphs G.

4. Necessary Conditions for Diophantine Graphs

According to the basic bounds of D_n , One notes that each of the following six conditions **C1**, …, **C6** listed below constitutes a necessary condition for the existence of a Diophantine labeling for a given graph G of order n:

C1. $|E(G)| \leq |E(D_n)|$. **C2**. $F(G) \leq F(D_n)$. **C3**. $Cl(G) \leq Cl(D_n)$.
C4. $\alpha(G) \geq \alpha(D_n)$. **C5**. $\delta(G) \leq \delta(D_n)$.

C6. For each $k \in \{0, 1, \ldots, n-1\}$ such that $\sum_{i=0}^{k} g_i \geq \sum_{i=0}^{k} d_i$.

The following two examples illustrate some relations among these conditions.

FIGURE 4. Basic bounds of D_n, $n = 4, \ldots, 20$

n	$\|E(D_n)\|$	$F(D_n)$	$Cl(D_n)$	$\alpha(D_n)$	$\delta(D_n)$	$S_{D_n} = (d_i)$
4	6	4	4	1	3	$(0, 0, 0, 4)$
5	9	3	4	2	3	$(0, 0, 0, 2, 3)$
6	15	6	6	1	5	$(0, 0, 0, 0, 0, 6)$
7	17	3	5	3	3	$(0, 0, 0, 1, 2, 1, 3)$
8	27	6	7	2	6	$(0, 0, 0, 0, 0, 0, 2, 6)$
9	30	5	6	4	5	$(0, 0, 0, 0, 0, 4, 0, 0, 5)$
10	41	5	7	3	7	$(0, 0, 0, 0, 0, 0, 0, 3, 2, 5)$
11	41	3	6	5	4	$(0, 0, 0, 0, 1, 1, 3, 0, 2, 1, 3)$
12	65	10	11	2	10	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 10)$
13	57	4	7	6	5	$(0, 0, 0, 0, 0, 2, 1, 3, 0, 2, 0, 1, 4)$
14	81	6	8	4	8	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 3, 2, 2, 6)$
15	83	7	9	7	7	$(0, 0, 0, 0, 0, 0, 0, 1, 6, 0, 0, 0, 0, 1, 7)$
16	106	7	10	5	9	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 4, 0, 2, 2, 7)$
17	95	4	8	8	6	$(0, 0, 0, 0, 0, 0, 2, 1, 1, 4, 1, 0, 2, 0, 1, 1, 4)$
18	143	9	11	4	14	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 3, 2, 9)$
19	119	5	9	9	7	$(0, 0, 0, 0, 0, 0, 0, 3, 1, 1, 4, 1, 0, 2, 0, 0, 1, 1, 5)$
20	173	10	13	6	14	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 4, 10)$

Example 4.1. The basic bounds for the following six graphs G_i, $i = 1, \ldots, 6$ as shown in Figure 5 are given in Table 1 and also the corresponding basic bounds for D_7 and D_{11}. The graph G_1 does not satisfy **C4**, the graph $G_2 = C_4 + \overline{K_3}$ does not satisfy **C5** and **C6** $\left(\text{for } \sum_{i=0}^{3} g_i < \sum_{i=0}^{3} d_i\right)$, the graph G_3 does not satisfy **C3**, the graph $G_4 = K_4 + \overline{K_7}$ does not satisfy **C2** and **C6** $\left(\text{for } \sum_{i=0}^{9} g_i < \sum_{i=0}^{9} d_i\right)$, the graph G_5 does not satisfy **C1** and **C6** $\left(\text{for } \sum_{i=0}^{5} g_i < \sum_{i=0}^{5} d_i\right)$ and the graph G_6 does not satisfy **C6** $\left(\text{for } \sum_{i=0}^{8} g_i < \sum_{i=0}^{8} d_i\right)$. Therefore, the graphs G_i, $i = 1, \ldots, 6$ are not Diophantine. However, the other necessary conditions are satisfied for the graphs G_i, $i = 1, 2, \ldots, 6$.

TABLE 1. Some basic bounds of G_i, $i = 1, \ldots, 6$ and the corresponding bounds of D_7 and D_{11}

Graph G	$\|E(G)\|$	$F(G)$	$Cl(G)$	$\alpha(G)$	$\delta(G)$	S_G
D_7	17	3	5	3	3	$(0, 0, 0, 1, 2, 1, 3)$
G_1	15	0	5	**2**	2	$(0, 0, 1, 0, 2, 4, 0)$
G_2	16	0	3	3	**4**	**(0,0,0,0,3,4,0)**
D_{11}	41	3	6	5	4	$(0, 0, 0, 0, 1, 1, 3, 0, 2, 1, 3)$
G_3	33	3	**7**	5	3	$(0, 0, 0, 4, 0, 0, 4, 0, 0, 0, 3)$
G_4	34	**4**	5	7	4	**(0,0,0,0,7,0,0,0,0,0,4)**
G_5	**42**	2	6	5	4	**(0,0,0,0,1,0,4,0,0,4,2)**
G_6	38	3	6	6	3	**(0,0,0,1,0,5,0,0,0,2,3)**

Notice that, based on Remark 3.14 and Example 4.1, **C6** is stronger than each of the following three conditions **Ci**, i = 1, 2, 5. The two sets of necessary conditions {**Ci** : i = 1, 2, 3, 4, 5} and {**Ci** : i = 3, 4, 6} are mutually independent.

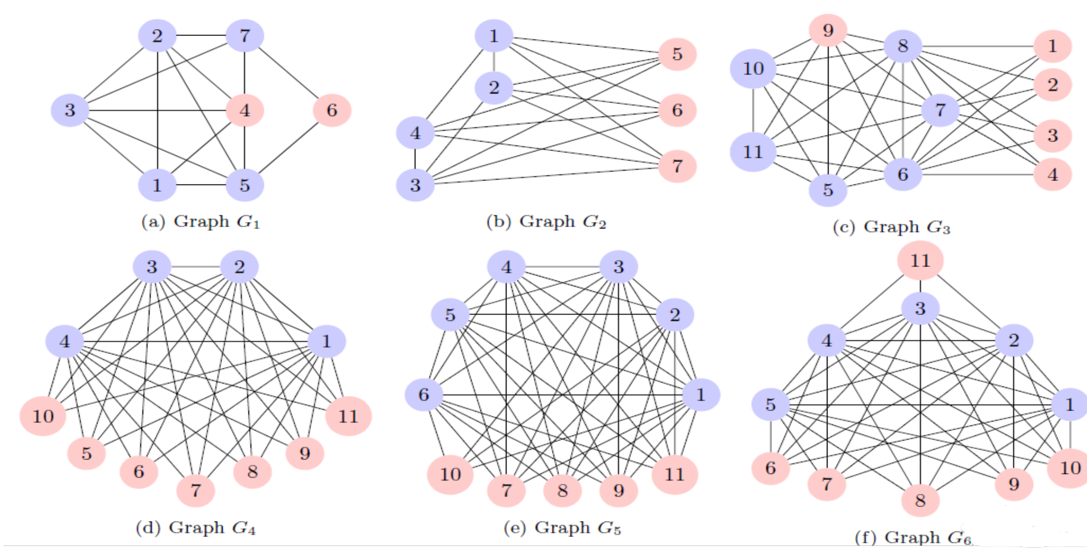


FIGURE 5. The graphs G_1, G_2, G_3, G_4, G_5 and G_6 are non-Diophantine

Example 4.2. The all six conditions are satisfied for the graph G as show in Figure 6. Condition **C1** holds since $|E(G)| = 37 < 41 = |E(D_{11})|$, condition **C2** holds since $F(G) = 3 = F(D_{11})$. condition **C3** holds since $Cl(G) = 6 = Cl(D_{11})$, condition **C4** holds since $\alpha(G) = 6 > 5 = \alpha(D_{11})$, condition **C5** holds since $\delta(G) = 3 < 4 = \delta(D_{11})$ and condition **C6** holds since the sequences $S_G = \{g_i\}_{i=0}^{10} = (0,0,0,1,1,4,0,0,1,1,3)$ and $S_{D_{11}} = \{d_i\}_{i=0}^{10} = (0,0,0,0,1,1,3,0,2,1,3)$ meet the inequality $\sum_{i=0}^{k} g_i \geq \sum_{i=0}^{k} d_i$. for every $k = 0,1,\ldots,10$, Hence, all six conditions **Ci**, i=1, ..., 6 as a whole are not sufficient conditions. However, the graph G is not Diophantine for the multiple of 5 cannot be assign in the independent vertices, although the multiples of 2 and 3 can be labeled in the independent vertices for all possible configurations (see [3]).

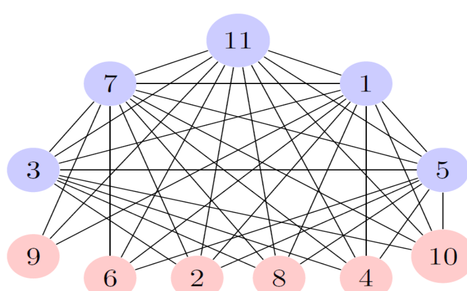


FIGURE 6. G is not Diophantine but hold the six necessary conditions

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