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NUMERICAL INVESTIGATION OF A RIESZ FRACTIONAL DIFFUSION EQUATION WITH DIRICHLET BOUNDARY CONDITIONS VIA THE CRANK-NICOLSON METHOD

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Abstract. This study examines a finite difference method based on the Crank-Nicolson scheme, specifically developed for solving fractional diffusion equations involving the Riesz spatial derivative and the Caputo-Fabrizio time derivative, subject to Dirichlet boundary conditions. The proposed numerical scheme employs a Taylor series expansion for temporal discretization in conjunction with shifted Grünwald-Letnikov operators for spatial approximation. A rigorous technical proof of the scheme's unconditional stability is provided, and its convergence properties are thoroughly analyzed. In addition, the practical effectiveness of the method is demonstrated through a representative numerical example, with corresponding graphical results obtained via MATLAB simulations.

Keywords. Fractional diffusion equation(FDE), Riesz operator, Caputo-Fabrizio operator, error analysis, Stability, finite difference approach, Dirichlet BCs
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1. Introduction

Fractional differential equations (FDEs) form an important and fast-growing area of applied mathematics. They offer useful tools for modeling and studying many natural and engineered systems. Fractional calculus has attracted much attention because it extends classical models to include memory and hereditary effects found in complex systems. FDEs provide a flexible mathematical framework commonly applied in disciplines such as engineering, chemistry, physics, and finance. For more details, readers can consult the references in [12, 10, 16, 23, 19, 3, 18].

The study of FDEs has drawn much attention because many fractional models are hard to solve exactly. In recent years, various numerical methods have been developed to simulate such systems, with the fractional diffusion equation (FDE) being one of the most commonly used models. As mentioned in [4, 28], FDEs are used to describe transport processes in complex systems, often with

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the Caputo–Fabrizio (C-F) time derivative to include memory effects. Moreover, FDEs are widely used in Earth sciences [9] to model anomalous diffusion and scaling behavior in geological processes. They are also useful for studying molecular transport [22] and for describing heat and mass transfer in heterogeneous materials [15].

The finite difference method (FDM) is a widely used technique for approximating solutions to fractional differential equations (FDEs), which are often difficult to solve analytically. Many studies have developed second-order numerical schemes for time-space fractional diffusion equations (TSFDEs), combining approximations for Caputo and Riemann–Liouville derivatives. The stability and convergence of these methods have been thoroughly analyzed and validated numerically. TSFDEs involving the Riesz fractional derivative are particularly important for modeling transport phenomena in various fields. Due to the scarcity of exact solutions, efficient numerical methods are essential. For instance, Crank–Nicolson schemes and second-order spatial approximations have been proposed to address these challenges. Recent advances in finite difference methods have addressed fractional models for diffusion and complex systems. For example, [11] analyzed stability and monotonicity of a fractional Fokker–Planck scheme, while [8] developed methods for time-fractional sub-diffusion equations using the Grünwald–Letnikov formula. More advanced techniques include combining Crank–Nicolson in time with finite elements in space [13], and applying finite differences in time with spectral methods in space [2].

While the Crank–Nicolson scheme and Grünwald–Letnikov (GL) approximations are well-established techniques for fractional diffusion equations [24, 14], their direct application to the Caputo–Fabrizio–Riesz (CF–Riesz) setting presents several nontrivial challenges. First, the Caputo–Fabrizio derivative is defined using a nonsingular exponential kernel, in contrast to the power-law kernels characterizing the classical Caputo or Riemann–Liouville formulations [5]. This kernel structure modifies the temporal memory weighting, producing a finite-memory convolution rather than a weakly singular one. As a result, standard discretizations based on power-law coefficients [22, 7] cannot be directly applied. The temporal approximation must therefore be reformulated to accurately preserve the exponential decay inherent in the CF derivative, ensuring both consistency with the continuous operator and computational efficiency. Second, the Riesz derivative introduces symmetric nonlocality in space, represented through fractional-order differences extending across the domain [23, 15]. When coupled with the CF derivative, this yields a hybrid nonlocal model in which both temporal memory and spatial interactions depend on prior states. Designing a stable and accurate discrete operator under this twofold nonlocality requires careful synchronization between temporal and spatial discretizations. Furthermore, because the CF derivative is nonsingular yet global in time, classical stability results derived for singular-kernel operators do not directly apply [20]. A new stability and convergence analysis is therefore required to demonstrate that the proposed Crank–Nicolson-type scheme remains unconditionally stable, regardless of the chosen time-step size.

In this work, we address these difficulties for a one-dimensional fractional diffusion equation by developing a modified Crank–Nicolson scheme that incorporates the exponential-memory structure of the Caputo–Fabrizio derivative through a Taylor-series-based temporal discretization, while spatial derivatives are handled using shifted Grünwald–Letnikov operators to improve spatial accuracy.

$$ {}_{CF}\mathfrak{D}^{\alpha}_{0,t}\Psi(x,t) = \mathfrak{A}(x,t)\left[\frac{\partial^{\beta}\Psi(x,t)}{\partial|x|^{\beta}}\right] + \mathfrak{F}(x,t), \quad x \in (0,L], \quad t \in (0,T], \tag{1.1} $$

subject to the initial condition

$$ \Psi(x,0) = \phi(x), x \in (0,L], \tag{1.2} $$

the boundary conditions

$$ \Psi(0,t) = 0, \quad \Psi(L,t) = v(t), \quad t \in (0,T], \tag{1.3} $$

where $\alpha \in (0,1)$, $\beta \in (1,2)$, the positive diffusion term $\mathfrak{A}(x,t)$, the source term $\mathfrak{F}(x,t)$, sufficiently smooth functions $\phi(x)$ and $v(t)$. for the spatial direction in (1.1), is the Riesz fractional operator [22],

$$ \frac{\partial^{\beta}}{\partial|x|^{\beta}}\Psi(x,t) = -c_{\beta}\left({}_{0}\mathfrak{D}^{\beta}_{x} + {}_{x}\mathfrak{D}^{\beta}_{L}\right)\Psi(x,t), $$

where the coefficient $c_{\beta} = \frac{1}{2\cos\left(\frac{\beta\pi}{2}\right)}$. The left Riemann-Liouville fractional operator ${}_{a}\mathfrak{D}^{\beta}_{x}$ presented as

$$ {}_{0}\mathfrak{D}^{\beta}_{x}\Psi(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_0^x \frac{\Psi(s,t)}{(x-s)^{\beta-1}}ds. $$

Also, the right Riemann-Liouville fractional operator ${}_{x}\mathfrak{D}^{\beta}_{b}$ presented as

$$ {}_{x}\mathfrak{D}^{\beta}_{L}\Psi(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^L \frac{\Psi(s,t)}{(s-x)^{\beta-1}}ds. $$

From [5], for any $0 < \alpha < 1$, ${}_{CF}\mathfrak{D}^{\alpha}_{0,t}\Psi(x,t)$ is the Caputo-Fabrizio derivative presented as: for $\Psi \in H^1(0,a), b > a$

$$ {}_{CF}\mathfrak{D}^{\alpha}_{0,t}\Psi(x,t) = \frac{M(\alpha)}{1-\eta}\int_0^t \frac{\partial\Psi(x,\xi)}{\partial\xi}\exp\left[-\alpha\frac{t-\xi}{1-\alpha}\right]d\xi, \tag{1.4} $$

where M is a normalization function verified $M(0) = M(1) = 1$ [5].

The study is organized as follows: Section 2 presents a finite difference scheme to approximate the time–space fractional diffusion model defined by equations (1.1)–(1.3). The Caputo-Fabrizio operator is discretized using the Crank-Nicolson method, while the Riemann-Liouville operators are approximated via the Grünwald-Letnikov formula. Stability and error analyses are discussed in Section 3, and numerical experiments evaluating the method's performance are provided in Section 4.

2. Space-time discretization and the FDE scheme

We present several properties and lemmas to approximate the Caputo-Fabrizio operator using the L1 formula for the Caputo-Fabrizio derivative approximation at $t = t_n$. Additionally, we deals with the Riesz operator by examining the right and left Riemann-Liouville fractional operators.

Let $\Delta t = \frac{T}{n}$ the time step , such that $t_{\mathbf{e}} = \mathbf{e}\Delta t$; $\mathbf{e} = 0, 1, \ldots, \mathrm{n}$ be the time-discretization $0 \leq t_{\mathbf{e}} \leq T$. Noted $\Psi(\cdot, t_{\mathbf{e}+1}) = \Psi^{\mathbf{e}+1}, \mathbf{e} = 1, 2, \ldots n$, be the analytical solution of the FDE (1.1), (1.2), and (1.3) at the node point $(x, t_{\mathbf{e}})$. Noting $\tilde{\Psi}^{\mathbf{e}}$ be the numerical approximation to $\Psi(x, t_{\mathbf{e}})$.

Theorem 2.1. [27] *Let $0 < \alpha < 1$, For any $\Psi(t) \in C^2[0, t_{\mathbf{e}}]$, then, we get*

$$\left| {}_{CF}\mathfrak{D}_{0,t}^{\alpha}\Psi(s, t_{\mathbf{e}}) - \frac{\Delta t^{-1}M(\alpha)}{\alpha}\left(\mathfrak{p}_{\mathbf{e},\mathbf{e}}\Psi^{\mathbf{p}} + \sum_{\mathbf{r}=1}^{\mathbf{e}-1}\left(\mathfrak{p}_{\mathbf{r},\mathbf{p}} - \mathfrak{p}_{\mathbf{r}+1,\mathbf{e}}\right)\Psi^{\mathbf{r}} - \mathfrak{p}_{1,\mathbf{e}}\Psi^0\right)\right|$$
$$\leq \frac{(1-\alpha)M(\alpha)}{2\alpha^2}\max_{0\leq t\leq t_{\mathbf{e}}}|\Psi''(t)|\,\Delta t^2. \quad 1 \leq \mathbf{e} \leq n,$$

where
$$\mathfrak{p}_{\mathbf{r},\mathbf{e}} = \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(\mathbf{e}-\mathbf{r})\right] - \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(\mathbf{e}-\mathbf{r}+1)\right].$$

Put $h_x = h = \frac{L}{N}$ the space step, such that $x_{\mathfrak{b}} = \mathfrak{b}h$ for $\mathfrak{b} = 0, 1, \ldots, \mathrm{N}$ be the space discretization. Noted $\Psi(x_{\mathfrak{b}}, t_{\mathbf{e}+1}) = \Psi_{\mathfrak{b}}^{\mathbf{e}+1}, \mathfrak{b} = 1, \ldots N, \mathbf{e} = 1, \ldots n$, be the analytical solution of the FDE (1.1), (1.2), and (1.3) at the node point $(x_{\mathfrak{b}}, t_{\mathbf{e}})$. Noting $\tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}}$ be the numerical approximation to $\Psi(x_{\mathfrak{b}}, t_{\mathbf{e}})$.

Theorem 2.2. [28] *For any $\Psi(x) \in L^1(R)$, ${}_{-\infty}\mathfrak{D}_x^{\beta+2}\Psi(x)$, ${}_x\mathfrak{D}_{\infty}^{\beta+2}\Psi(x)$ and their Fourier transform belong to $L^1(R)$, then the Grünwald-Letnikov operators presented as follow,*

$$ {}_L\mathfrak{D}_{h,\mathbf{s},\mathbf{u}}^{\beta}\Psi(x) = \frac{\lambda_1}{h^\beta}\sum_{\mathbf{r}=0}^{\infty} g_{\mathbf{r}}^{(\beta)}\Psi(x-(\mathbf{r}-\mathbf{s})h) + \frac{\lambda_2}{h^\beta}\sum_{\mathbf{r}=0}^{\infty} g_{\mathbf{r}}^{(\beta)}\Psi(x-(\mathbf{r}-\mathbf{u})h),$$
$$ {}_R\mathfrak{D}_{h,\mathbf{s},\mathbf{u}}^{\beta}\Psi(x) = \frac{\lambda_1}{h^\beta}\sum_{\mathbf{r}=0}^{\infty} g_{\mathbf{r}}^{(\beta)}\Psi(x+(\mathbf{r}-\mathbf{s})h) + \frac{\lambda_2}{h^\beta}\sum_{\mathbf{r}=0}^{\infty} g_{\mathbf{r}}^{(\beta)}\Psi(x+(\mathbf{r}-\mathbf{u})h),$$

where $\mathbf{s}, \mathbf{u}$ are integers and $\mathbf{s} \neq \mathbf{u}$, $\lambda_1 = \frac{\beta-2\mathbf{u}}{2(\mathbf{s}-\mathbf{u})}$, $\lambda_2 = \frac{2\mathbf{s}-\beta}{2(\mathbf{s}-\mathbf{u})}$, and $g_{\mathbf{r}}^{(\beta)} = (-1)^{\mathbf{r}}\begin{pmatrix}\beta\\ \mathbf{r}\end{pmatrix}$ are the binomial coefficients for β order. Then we have

$$ {}_L\mathfrak{D}_{h,\mathbf{s},\mathbf{u}}^{\beta}\Psi(x) = {}_{-\infty}D_x^{\beta}\Psi(x) + O\left(h^2\right), \quad \text{and} \quad {}_R\mathfrak{D}_{h,\mathbf{s},\mathbf{u}}^{\beta}\Psi(x) = {}_xD_{\infty}^{\beta}\Psi(x) + O\left(h^2\right),$$

uniformly for $x \in \mathrm{R}$.

We declare the following properties of the coefficients related to the previous approximations of the fractional derivatives operators as lemmas.

Lemma 2.3. [22, 15] *Let β, β_1 and β_2 be three non-negative real numbers, and $n \in \mathbf{N}$. the coefficients $g_{\mathbf{r}}^{(\beta)}(\mathbf{r} = 0, 1, \cdots)$ verified the following properties:*

(a) $g_0^{(\beta)} = 1, \quad g_{\mathbf{r}}^{(\beta} = \left(1 - \frac{\beta+1}{\mathbf{r}}\right) g_{\mathbf{r}-1}^{(\beta)} \quad for \quad \mathbf{r} \geq 1;$
(b) $\mathrm{g}_1^{(\beta)} < \mathrm{g}_2^{(\beta)} < \ldots < 0, \quad \sum_{\mathbf{r}=0}^{n} \mathrm{g}_{\mathbf{r}}^{(\beta)} > 0 \quad for \quad 0 < \beta < 1;$
(c) $g_2^{(\beta)} > g_3^{(\beta)} > \ldots > 0, \quad \sum_{\mathbf{r}=0}^{n} g_{\mathbf{r}}^{(\beta)} < 0 \quad for \quad 1 < \beta < 2;$
(d) $\sum_{\mathbf{r}=0}^{n} g_{\mathbf{r}}^{(\beta)} = (-1)^n \binom{\beta-1}{n};$
(e) $\sum_{\mathbf{r}=0}^{n} g_{\mathbf{r}}^{(\beta_1)} g_{n-\mathbf{r}}^{(\beta_2)} = g_n^{(\beta_1+\beta_2)}.$

Using Theorem 2.2, in case $(p,q) = (1,0)$, the Riesz fractional operator in (1.1) will be approximated by,

$$\left.\frac{\partial^\beta \Psi(x,t)}{\partial |x|^\beta}\right|_{(x_{\mathfrak{b}}, t_{\mathbf{e}+1})} = \frac{-c_\beta}{h^\beta} \left(\sum_{\mathbf{r}=0}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} \tilde{\Psi}_{\mathfrak{b}-\mathbf{r}+1}^{\mathbf{e}+1} + \sum_{\mathbf{r}=0}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} \tilde{\Psi}_{\mathfrak{b}+\mathbf{r}-1}^{\mathbf{e}+1} \right) + O\left(h^2\right), \tag{2.1}$$

where $w_0^{(\beta)} = \frac{\beta}{2} g_0^{(\beta)}, w_{\mathbf{r}}^{(\beta)} = \frac{\beta}{2} g_{\mathbf{r}}^{(\beta)} + \frac{2-\beta}{2} g_{\mathbf{r}-1}^{(\beta)}, \mathbf{r} = 1, 2, \cdots, M.$

According to Theorems 2.1 and 2.2, we can approach the fractional diffusion equation (1.1), (1.2) and (1.3), by the finite difference equation as follow:

$$\begin{aligned} &\left(1 - 2R_{\mathfrak{b}}^1 w_1^{(\beta)}\right) \tilde{\Psi}_{\mathfrak{b}}^1 - R_{\mathfrak{b}}^1 \left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} \tilde{\Psi}_{\mathfrak{b}-\mathbf{r}+1}^1 + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} \tilde{\Psi}_{\mathfrak{b}+\mathbf{r}-1}^1 \right) \\ &\qquad = \tilde{\Psi}_{\mathfrak{b}}^0 + F_{\mathfrak{b}}^1, \quad for \quad \mathbf{e} = 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\left(1 - 2R_{\mathfrak{b}}^{\mathbf{e}+1} w_1^{(\beta)}\right) \tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}+1} - R_{\mathfrak{b}}^{\mathbf{e}+1} \left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} \tilde{\Psi}_{\mathfrak{b}-\mathbf{r}+1}^{\mathbf{e}+1} + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} \tilde{\Psi}_{\mathfrak{b}+\mathbf{r}-1}^{\mathbf{e}+1} \right) \\ &\qquad = \left(1 - \frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right) \tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}} + \frac{1}{\mathfrak{p}_0} \sum_{\mathbf{r}=1}^{\mathbf{e}-1} \left(\mathfrak{p}_{\mathbf{r}} - \mathfrak{p}_{\mathbf{r}+1}\right) \tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}-\mathbf{r}} + \frac{\mathfrak{p}_{\mathbf{e}}}{\mathfrak{p}_0} \tilde{\Psi}_{\mathfrak{b}}^0 + F_{\mathfrak{b}}^{\mathbf{e}+1}, \\ &\qquad for \quad \mathbf{e} \geq 1, \end{aligned} \tag{2.3}$$

initial condition:

$$\tilde{\Psi}_{\mathfrak{b}}^0 = \phi\left(x_{\mathfrak{b}}\right), \quad \mathfrak{b} = 0, 1, 2, \ldots \tag{2.4}$$

boundary conditions:

$$\tilde{\Psi}_0^{\mathbf{e}} = 0, \quad \tilde{\Psi}_N^{\mathbf{e}} = v(t_{\mathbf{e}}) = v^{\mathbf{e}}, \quad \mathbf{e} = 0, 1, 2, \ldots \tag{2.5}$$

where $\mathfrak{p}_{\mathbf{r}} = \exp\left[-\alpha \frac{\Delta t}{1-\alpha}(\mathbf{r})\right] - \exp\left[-\alpha \frac{\Delta t}{1-\alpha}(\mathbf{r}+1)\right]$, $\mathfrak{F}_{\mathfrak{b}}^{\mathbf{e}} = \mathfrak{F}\left(x_{\mathfrak{b}}, t_{\mathbf{e}}\right)$, $R_{\mathfrak{b}}^{\mathbf{e}+1} = \frac{-\alpha \Delta t c_\beta \mathfrak{A}_{\mathfrak{b}}^{\mathbf{e}+1}}{d_0 h^\beta M(\alpha)} > 0$, and $F_{\mathfrak{b}}^{\mathbf{e}+1} = \frac{\alpha \Delta t}{d_0 M(\alpha)} \mathfrak{F}_i^{k+1}$, $\mathfrak{b}, \mathbf{e} = 0, 1, 2, \ldots$

Denote the column vectors as follows:

$$\tilde{\Psi}^{e} = \left(\tilde{\Psi}_1^{\mathbf{e}}, \tilde{\Psi}_2^{\mathbf{e}}, \cdots, \tilde{\Psi}_{N-1}^{\mathbf{e}}\right)^{\mathrm{T}},$$

$$\tilde{\Psi}^{\mathbf{e}-1} = \left(\tilde{\Psi}_1^{\mathbf{e}-1}, \tilde{\Psi}_2^{\mathbf{e}-1}, \cdots, \tilde{\Psi}_{N-2}^{\mathbf{e}-1}, 0\right)^{\mathrm{T}},$$

$$F^{\mathbf{e}} = \left(\frac{\eta\Delta t}{\mathfrak{p}_0 M(\alpha)}\mathfrak{F}_1^{\mathbf{e}}, \frac{\eta\Delta t}{\mathfrak{p}_0 M(\eta)}\mathfrak{F}_2^{\mathbf{e}}, \cdots, \frac{\eta\Delta t}{\mathfrak{p}_0 M(\eta)}\mathfrak{F}_{N-2}^{\mathbf{e}}, v^{\mathbf{e}}\right)^{\mathrm{T}}.$$

The matrix form of the finite difference scheme (2.2)-(2.5) is as follows:

$$(I-Q)\tilde{\Psi}^1 = \tilde{\Psi}^0 + F^1, \tag{2.6}$$

$$(I-Q)\tilde{\Psi}^{\mathbf{e}+1} = \left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right)\tilde{\Psi}^{\mathbf{e}} + \frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{\mathbf{e}-1}\left(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1}\right)\tilde{\Psi}^{\mathbf{e}-\mathbf{r}} + \frac{\mathfrak{p}_{\mathbf{e}}}{\mathfrak{p}_0}\tilde{\Psi}^0 + F^{\mathbf{e}+1}, \quad for \quad 1 \le \mathbf{e} \le n. \tag{2.7}$$

$Q = (q_{i,\mathbf{r}})$ is a square matrix with $\mathrm{N}-1$ ordered coefficients: $B + B^T = Q$,

$$\text{where} \qquad B = \begin{bmatrix} (\frac{R_1^{k+1}w_1^{(\beta)}}{2}) & -R_1^{k+1}w_0^{(\beta)} & 0 & \cdots & \cdots & \cdots \\ -R_2^{k+1}w_2^{(\beta)} & (\frac{R_2^{k+1}w_1^{(\beta)}}{2}) & -R_2^{k+1}w_0^{(\beta)} & 0 & \cdots & \cdots \\ -R_3^{k+1}w_3^{(\beta)} & -R_3^{k+1}w_2^{(\beta)} & (\frac{R_3^{k+1}w_1^{(\beta)}}{2}) & -R_3^{k+1}w_0^{(\beta)} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots\ddots & \cdots & \cdots \\ \\ -R_{N-1}^{k+1}w_{m-1}^{(\beta)} & -R_{N-1}^{k+1}w_{m-2}^{(\beta)} & -R_{N-1}^{k+1}w_{m-3}^{(\beta)} & \cdots & \cdots & (\frac{R_{N-1}^{k+1}w_1^{(\beta)}}{2}) \end{bmatrix}$$

3. Stability analysis and error estimates

In this section, we discuss the stability and convergence of the numerical formulation method.

Lemma 3.1. [25] *The coefficients* $w_{\mathbf{r}}^{(\beta)}(\mathbf{r}=0,1,\ldots)$ *verified the following properties, for any* $\zeta > 0$

$$(a) w_0^{(\beta)} = \frac{\beta}{2}, \quad w_1^{(\beta)} = \frac{2-\beta-\beta^2}{2}, \quad w_2^{(\beta)} = \frac{\beta\left(\beta^2+\beta-4\right)}{4},$$

$$w_{\mathbf{r}}^{(\beta)} = \frac{\beta}{2}g_{\mathbf{r}}^{(\beta)} + \frac{2-\beta}{2}g_{\mathbf{r}-1}^{(\beta)}, \qquad for\ \mathbf{r} \ge 3;$$

$$(b) 1 > w_0^{(\beta)} > w_3^{(\beta)} > w_4^{(\beta)} > \ldots > 0, \quad \sum_{\mathbf{r}=0}^{n} w_{\mathbf{r}}^{(\beta)} < 0, \quad \forall \quad 1<\beta<2, \quad n \ge 2.$$

$$(c) \quad w_0^{(\beta)} + w_2^{(\beta)} = \frac{-w_1^{(\beta)}}{2} > 0$$

Lemma 3.2. [27] *For* $\mathfrak{p}_{\mathbf{r}} = \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(\mathbf{r})\right] - \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(\mathbf{r}+1)\right]$, *we have* $\mathfrak{p}_{\mathbf{r}} > 0$, *and* $\mathfrak{p}_{\mathbf{r}} > \mathfrak{p}_{\mathbf{r}+1} \quad \mathbf{r} = 0,1,2,\ldots$

3.1. **Stability analysis of the FDE.** denote $\tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}}$ the numerical solution of the finite difference method with the initial value $\tilde{\Psi}_i^0$.

We put $\varepsilon_{\mathfrak{b}}^{\mathbf{e}} = \Psi_{\mathfrak{b}}^{\mathbf{e}} - \tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}}$, $\quad \varepsilon^{\mathbf{e}} = \left(\varepsilon_1^{\mathbf{e}}, \varepsilon_2^{\mathbf{e}}, \ldots, \varepsilon_{N-1}^{\mathbf{e}}\right)^{\mathrm{T}} \quad 1 \le \mathfrak{b} \le N-1, \quad 1 \le \mathbf{e} \le n,$

we have

$$(I-Q)\varepsilon^{1}=\varepsilon^{0}, \tag{3.1}$$

$$(I-Q)\varepsilon^{\mathbf{e}+1}=\left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right)\varepsilon^{\mathbf{e}}+\frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{\mathbf{e}-1}\left(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1}\right)\varepsilon^{\mathbf{e}-\mathbf{r}}+\frac{\mathfrak{p}_{\mathbf{e}}}{\mathfrak{p}_0}\varepsilon^{0}, \quad for \quad k\geq 1. \tag{3.2}$$

Lemma 3.3. *the eigenvalues of the matrix $I-Q$ is more than one.*

Proof. Let $\lambda_{\mathfrak{b}}$, $\mathfrak{b}=1,2,\ldots,N-1$, be the eigenvalues of matrix Q.

To prove that for any $\mathfrak{b}=1,2,\ldots,N-1$, $\lambda_{\mathfrak{b}}<0$. Assume $\exists\ \mathfrak{l}=1,2,\ldots,N-1$, such as $\lambda_l\geq 0$.

We have $R_{\mathfrak{l}}^{\mathbf{e}+1}=\frac{-\alpha\Delta t c_\beta \mathfrak{A}_{\mathfrak{b}}^{\mathbf{e}+1}}{\mathfrak{p}_0 h^\beta M(\alpha)}>0$, and According to Lemma 3.1, $\sum_{\mathbf{e}=0}^{N} w_{\mathbf{e}}^{(\beta)}<0$ for $N\geq 2$, it implies,

$$\lambda_{\mathfrak{l}}>2R_{\mathfrak{l}}^{\mathbf{e}+1}\left(\sum_{\mathbf{e}=0}^{N} w_{\mathbf{e}}^{(\beta)}\right), \tag{3.3}$$

which then gives,

$$\lambda_l-2R_{\mathfrak{l}}^{\mathbf{e}+1}w_1^{(\beta)}>2R_{\mathfrak{l}}^{\mathbf{e}+1}\left(\sum_{\mathbf{e}=0,\mathbf{e}\neq 1}^{N} w_{\mathfrak{e}}^{(\beta)}\right), \tag{3.4}$$

as $\sum_{\mathbf{e}=0,\mathbf{e}\neq 1}^{N} w_{\mathbf{e}}^{(\beta)}>0$ also from Lemma 3.1, and the inequality (3.4) implies

$$\left|\lambda_{\mathfrak{l}}-2R_{\mathfrak{l}}^{\mathbf{e}+1}w_1^{(\beta)}\right|>2R_{\mathfrak{l}}^{\mathbf{e}+1}\left(\sum_{\mathbf{e}=0,\mathbf{e}\neq 1}^{N} w_{\mathbf{e}}^{(\beta)}\right), \tag{3.5}$$

which contradicts Gershgorin's Theorem[16], so for all $\mathfrak{b}=1,2,\ldots,N-1$, $\lambda_{\mathfrak{b}}<0$. The proof is complete.

Theorem 3.4. *The numerical scheme (2.2) - (2.5) is unconditionally stable.*

Proof. Obviously, Q is symmetric, such that $Q=P^{\mathrm{T}}\mathrm{D}P$ where P is an orthogonal matrix and $D=(d_{\mathfrak{b},\mathfrak{r}})$, where is a diagonal (N − 1) ordered square matrix, $d_{\mathfrak{b},\mathfrak{b}}=\lambda_{\mathfrak{b}}$.

The matrix equation (3.1) gives,

$$(I-P^{\mathrm{T}}\mathrm{D}P)\varepsilon^{1}=\varepsilon^{0}, \tag{3.6}$$

multiplying on both sides of the equation (3.6) by P, and noting $P\varepsilon^{\mathfrak{e}}=\xi^{\mathfrak{e}}$, we get

$$(1-\lambda_{\mathfrak{b}})\xi_{\mathfrak{b}}^{1}=\xi_{\mathfrak{b}}^{0},$$

From Lemma 3.3, we have

$$\left|\xi_{\mathfrak{b}}^{1}\right|<\left|\xi_{\mathfrak{b}}^{0}\right|.$$

Now, assume that the following inequality holds for $\mathfrak{e}=0,1,2,\cdots,n-1$,

$$\left|\xi_{\mathfrak{b}}^{\mathfrak{e}}\right|<\left|\xi_{\mathfrak{b}}^{0}\right|. \tag{3.7}$$

Then, by mathematical induction, we can prove $|\xi_{\mathfrak{b}}^{n}| < |\xi_{\mathfrak{b}}^{0}|$, in fact from equation (3.2),

$$(1-\lambda_{\mathfrak{b}})\,|\xi_{\mathfrak{b}}^{n}| = \left|\left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right)\xi_{\mathfrak{b}}^{n-1} + \frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{n-2}(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1})\,\xi_{\mathfrak{b}}^{n-1-\mathbf{r}} + \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_0}\xi_{\mathfrak{b}}^{0}\right|,$$

Using Lemma 3.2, and the inequality (3.7),

$$(1-\lambda_{\mathfrak{b}})\,|\xi_{\mathfrak{b}}^{n}| < \left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right)\left|\xi_{\mathfrak{b}}^{0}\right| + \frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{n-2}(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1})\left|\xi_{\mathfrak{b}}^{0}\right| + \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_0}\left|\xi_{\mathfrak{b}}^{0}\right|,$$

then we have

$$(1-\lambda_{\mathfrak{b}})\,|\xi_{\mathfrak{b}}^{n}| < \left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0} + \frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{n-2}(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1}) + \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_0}\right)\left|\xi_{\mathfrak{b}}^{0}\right|,$$

which means

$$|\xi_{\mathfrak{b}}^{n}| < \frac{1}{(1-\lambda_{\mathfrak{b}})}\left|\xi_{\mathfrak{b}}^{0}\right| < \left|\xi_{\mathfrak{b}}^{0}\right|,$$

then we can conclude that with the 2-norm, $\|\xi^{n}\|_2 < \|\xi^{0}\|_2$, Since P is unitary, it follows that

$$\|\varepsilon^{n}\|_2 < \left\|\varepsilon^{0}\right\|_2.$$

Finally, the scheme (2.2) - (2.5) is unconditionally stable (all norms in a finite-dimensional space are equivalent), the proof is complete.

3.2. **Error estimate analysis of the FDE.** Let $r_{\mathfrak{b}}^{\mathfrak{e}}$ be the local truncation error for $1 \leq \mathbf{b} \leq N-1$. This conclusion stems from Theorem (2.1) and (2.1).

$$\|r_{\mathfrak{b}}^{\mathbf{e}}\|_\infty \leq C\left(\Delta t^2 + h^2\right), \quad 1 \leq \mathbf{e} \leq n. \tag{3.8}$$

and

$$r_{\mathfrak{b}}^{1} = \left(1-2R_{\mathfrak{b}}^{1}w_1^{(\beta)}\right)e_{\mathfrak{b}}^{1} - R_{\mathfrak{b}}^{1}\left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}e_{\mathfrak{b}-\mathbf{r}+1}^{1} + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}e_{\mathfrak{b}+\mathbf{r}-1}^{1}\right), for \quad \mathbf{e}=0, \tag{3.9}$$

$$\begin{aligned} r_{\mathfrak{b}}^{\mathbf{e}+1} = {} & \left(1-2R_{\mathfrak{b}}^{\mathbf{e}+1}w_1^{(\beta)}\right)e_{\mathfrak{b}}^{\mathbf{e}+1} - R_{\mathfrak{b}}^{\mathbf{e}+1}\left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}e_{\mathfrak{b}-\mathbf{r}+1}^{\mathbf{e}+1} + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}e_{\mathfrak{b}+\mathbf{r}-1}^{\mathbf{e}+1}\right) \\ & - \left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right)e_{\mathfrak{b}}^{\mathbf{e}} - \frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{\mathbf{e}-1}(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1})\,e_{\mathfrak{b}}^{\mathbf{e}-\mathbf{r}}, \quad for \quad \mathbf{e} \geq 1, \end{aligned} \tag{3.10}$$

where

$$e_{\mathfrak{b}}^{\mathbf{e}} = \Psi_{\mathfrak{b}}^{\mathbf{e}} - \tilde{\Psi}_{\mathfrak{b}}^{\mathbf{e}}, \quad 1 \leq \mathfrak{b} \leq N-1, \quad \mathbf{e} = 1\cdots n,$$

and

$$e^{0} = 0, \quad e^{\mathbf{e}} = \left(e_1^{\mathbf{e}}, e_2^{\mathbf{e}}, \ldots, e_{N-1}^{\mathfrak{e}}\right)^{\mathrm{T}}, \quad \|e^{\mathbf{e}}\|_\infty = \max_{1\leq\mathfrak{b}\leq N-1}|e_{\mathfrak{b}}^{\mathbf{e}}|.$$

Theorem 3.5. *The numerical scheme (2.2) - (2.5) is unconditionally convergent, where the accuracy order is* $O\left(\Delta t^2+h^2\right)$.

Proof. To prove the theorem, it is enough to find a constant $\tilde{C}$ where

$$\|e^n\|_\infty \leq \tilde{C}\left(\Delta t^2+h^2\right).$$

Suppose $\|e^1\|_\infty = |e_l^1| = \max_{1\leq \mathfrak{b}\leq N-1}|e_{\mathfrak{b}}^1|$.To prove the above inequality, we use mathematical induction. For $\mathfrak{e}=0$, according to the lemma (3.1) and from equation (3.9), we have

$$\begin{aligned}
\left\|e^1\right\|_\infty = \left|e_l^1\right| &\leq \left(1-R_{\mathfrak{b}}^1\Big(\sum_{\mathbf{r}=0}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}+\sum_{\mathbf{r}=0}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\Big)\right)\left|e_l^1\right| \\
&\leq \left|e_l^1\right| - R_{\mathfrak{b}}^1\left(\sum_{\mathbf{r}=0} w_{\mathbf{r}}^{(\beta)}\left|e_{l-\mathbf{r}+1}^1\right| + \sum_{\mathbf{r}=0}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\left|e_{l+\mathbf{r}-1}^1\right|\right) \\
&= \left(1-2R_{\mathfrak{b}}^1 w_1^{(\beta)}\right)\left|e_l^1\right| - R_{\mathfrak{b}}^1\left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\left|e_{l-\mathbf{r}+1}^1\right| + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\left|e_{l+\mathbf{r}-1}^1\right|\right) \\
&\leq \left|\left(1-2R_{\mathfrak{b}}^1 w_1^{(\beta)}\right)e_l^1 - R_{\mathfrak{b}}^1\left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} e_{l-\mathbf{r}+1}^1 + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} e_{l+\mathbf{r}-1}^1\right)\right| \\
&= \left|r_{\mathfrak{b}}^1\right|.
\end{aligned}$$

Then, (3.8) and (3.9) implies that

$$\left\|e^1\right\|_\infty \leq C\left(\Delta t^2+h^2\right). \tag{3.11}$$

Now assume that (induction assumption)

$$\left\|e^{k+1}\right\|_\infty \leq C\left(\Delta t^2+h^2\right), for \quad 0\leq k\leq n-2, \tag{3.12}$$

Supposing $\|e^n\|_\infty = |e_l^n| = \max_{1\leq\mathfrak{b}\leq N}|e_{\mathfrak{b}}^n|$. According to the lemma (3.1) and from equation (3.10), For $\mathfrak{e}=n-1$ we have

$$\begin{aligned}
&\left(1-R_{\mathfrak{b}}^n\Big(\sum_{\mathbf{r}=0}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}+\sum_{\mathbf{r}=0}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\Big)\right)\left|e_l^n\right| \\
&\leq \left|e_l^n\right| - R_{\mathfrak{b}}^n\left(\sum_{\mathbf{r}=0} w_{\mathbf{r}}^{(\beta)}\left|e_{l-\mathbf{r}+1}^n\right| + \sum_{\mathbf{r}=0}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\left|e_{l+\mathbf{r}-1}^n\right|\right) \\
&= \left(1-2R_{\mathfrak{b}}^n w_1^{(\beta)}\right)\left|e_l^n\right| - R_{\mathfrak{b}}^1\left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\left|e_{l-\mathbf{r}+1}^n\right| + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)}\left|e_{l+\mathbf{r}-1}^n\right|\right) \\
&\leq \left|\left(1-2R_{\mathfrak{b}}^n w_1^{(\beta)}\right)e_l^n - R_{\mathfrak{b}}^n\left(\sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} e_{l-\mathbf{r}+1}^n + \sum_{\mathbf{r}=0,\mathbf{r}\neq 1}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} e_{l+\mathbf{r}-1}^n\right)\right| \\
&= \left|\left(1-\frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right)e_l^{n-1} + \frac{1}{\mathfrak{p}_0}\sum_{\mathbf{r}=1}^{n-2}\left(\mathfrak{p}_{\mathbf{r}}-\mathfrak{p}_{\mathbf{r}+1}\right)e_l^{n-1-\mathbf{r}} + r_l^n\right|.
\end{aligned}$$

Then, inequality (3.8) and (induction assumption) (3.12) gives

$$\left(1 - R_{\mathfrak{b}}^n(\sum_{\mathbf{r}=0}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} + \sum_{\mathbf{r}=0}^{N-\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)})\right) |e_l^n|$$
$$\leq \left|\left(1 - \frac{\mathfrak{p}_1}{\mathfrak{p}_0}\right) + \frac{1}{\mathfrak{p}_0}\left(\mathfrak{p}_1 - \mathfrak{p}_{n-1}\right)\right| C\left(\Delta t^2 + h^2\right) + C\left(\Delta t^2 + h^2\right)$$
$$\leq \left(2 - \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_0}\right) C\left(\Delta t^2 + h^2\right).$$

Using Lemma 3.1, we conclude that

$$\left(1 - R_{\mathfrak{b}}^n(\sum_{\mathbf{r}=0}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} + \sum_{\mathfrak{r}=0}^{N-\mathfrak{b}+1} w_{\mathfrak{r}}^{(\beta)})\right) \|e^n\|_\infty \leq C\left(2 - \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_0}\right)\left(\Delta t^2 + h^2\right). \tag{3.13}$$

From Lemma 3.2, we obtain

$$\frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_0} = \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(n-1)\right] \approx 1 - \alpha\frac{\Delta t}{1-\alpha}(n-1) > 1 - \alpha\frac{\Delta t}{1-\alpha}(n)$$

. Finally, we obtain

$$\|e^n\|_\infty \leq C(1 + \frac{\alpha}{1-\alpha}T)(\Delta t^2 + h^2) \leq \tilde{C}(\Delta t^2 + h^2).$$

where $\quad \tilde{C} = \frac{1+(\alpha/(1-\alpha))T}{1-R_{\mathfrak{b}}^n(\sum_{\mathbf{r}=0}^{\mathfrak{b}+1} w_{\mathbf{r}}^{(\beta)} + \sum_{\mathfrak{r}=0}^{N-\mathfrak{b}+1} w_{\mathfrak{r}}^{(\beta)})}.$

Then, the proof is complete.

4. Numerical results

In this section, we test the finite-difference scheme (2.2)–(2.5) with numerical examples. Using MATLAB R2017a, we check the solution, L^2 error, and convergence order for space-time fractional diffusion equations with a Caputo–Fabrizio time derivative and a Riesz space derivative. The L^2 error, $|e(h, \Delta t)|$, is used to measure accuracy, and the convergence order is calculated by,

$$L^2\_\text{convergence order} = \log_2\left(\frac{\|e(2h, 2\Delta t)\|}{\|e(h, \Delta t)\|}\right), \quad \text{where} \quad \|e(h, \Delta t)\| = \sqrt{h\sum_{i=1}^{N-1}\left|U_i^n - \bar{U}_i^n\right|^2}.$$

Example 4.1. Take the FDE (1.1)-(1.3) problem, $L = 1$, with $\mathfrak{F}$ presented as follow

$$\mathfrak{F}(x,t) = -\frac{x^2(x-1)^2}{1-\alpha}\frac{\sigma\pi}{\sigma^2+\pi^2}\left(\sin(\pi t) - \pi\frac{\cos(\pi t)}{\sigma} + \exp(-\sigma t)\frac{\pi}{\sigma}\right) + \cos(\pi t)\sec\left(\frac{\pi\beta}{2}\right)$$
$$\times ((x^{2-\zeta} + (1-x)^{2-\zeta})/\Gamma(3-\beta) - 6(x^{3-\beta} + (1-x)^{3-\beta})/\Gamma(4-\beta)$$
$$+ 12(x^{4-\beta} + (1-x)^{4-\beta})/\Gamma(5-\beta)), \qquad \text{where} \quad \sigma = \alpha/(1-\alpha),$$

$\phi(x) = x^2(x-1)^2$, $\mathfrak{A} = 1$, and $v(t) = 0$.

TABLE 1. Example 4.1: The L^2_- errors, L^2_- convergence order and CPU times at T=1.

α	β	$\Delta t = h$	L^2_- error	L^2_- c.order	CPU time(s)
0.2	1.8	1 / 20	1.2334e-04	——	0.1733
		1 / 50	2.2440e-05	1.85235	0.3432
		1 / 80	9.4268e-06	1.84525	0.7176
		1 / 170	2.3297e-06	1.85499	1.7472
0.4	1.6	1 / 20	1.9030e-04	——	0.1823
		1 / 50	3.5703e-05	1.82627	0.3588
		1 / 80	1.5021e-05	1.83848	0.6552
		1 / 170	3.6872e-06	1.86454	1.8096
0.8	1.4	1 / 20	3.1880e-04	——	0.1911
		1 / 50	5.5882e-05	1.90028	0.3744
		1 / 80	2.2790e-05	1.90643	0.6084
		1 / 170	5.3584e-06	1.92118	1.7940

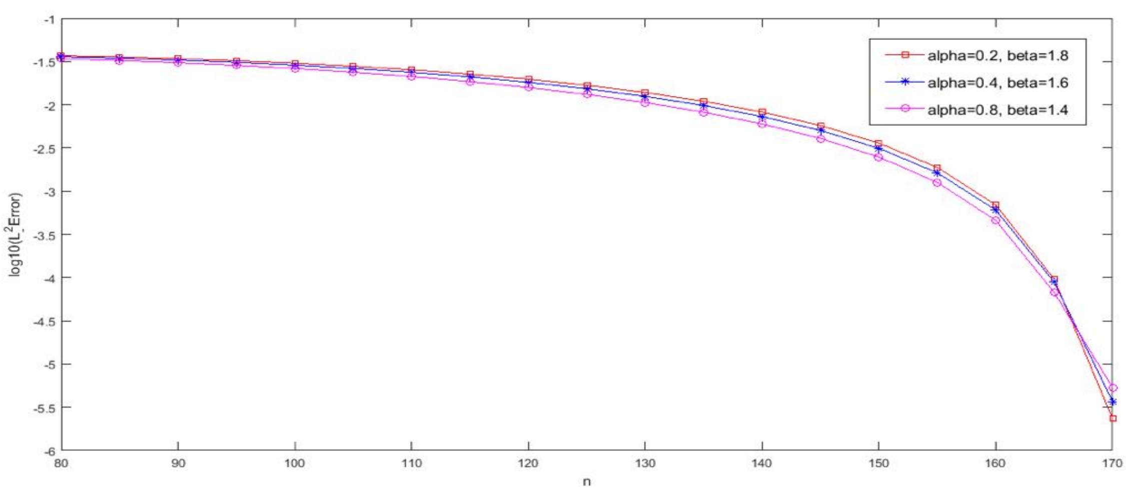

FIGURE 1. $log10(L^2_- Error)$ as a function of n time steps for Ex. (4.1), for different variants of α and β, when $h = \frac{1}{170}$.

The computational results are presented in Table 1 of Example 4.1, which compares the numerical and exact solutions for three different α and β values at $T = 1$. As the time and space step sizes decrease (we assume $\Delta t = h$ for simplicity), the L^2_- errors of the numerical solution also decrease with the same order of accuracy, consistent with Theorem 3.5.

Figure 1 illustrates the comparison of $\log_{10}(L^2_-\text{Error})$ as a function of the time step number n of our proposed scheme at $T = 1$, with $h = \frac{1}{170}$, for the aforementioned cases. At $T = 1$, Figure 3 displays 3D graphs for the numerical solution with $\alpha = 0.8$, $\beta = 1.4$, when $\Delta t = h = \frac{1}{50}$. For $\alpha = 0.4$, $\beta = 1.6$, and

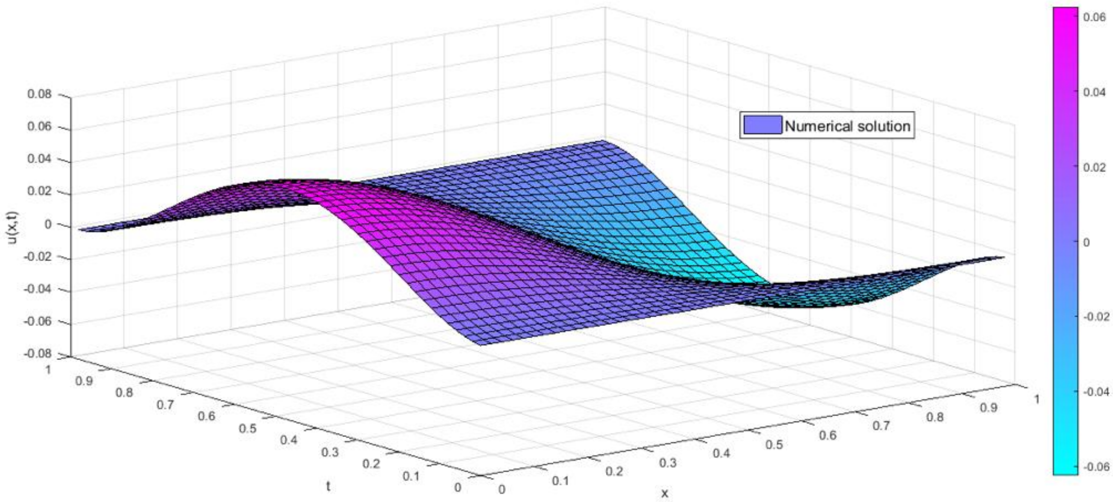

FIGURE 2. Numerical solution for Ex. (4.1),with $\alpha = 0.2$, β =1.8, and $\Delta t = h = \frac{1}{50}$ at T=1.

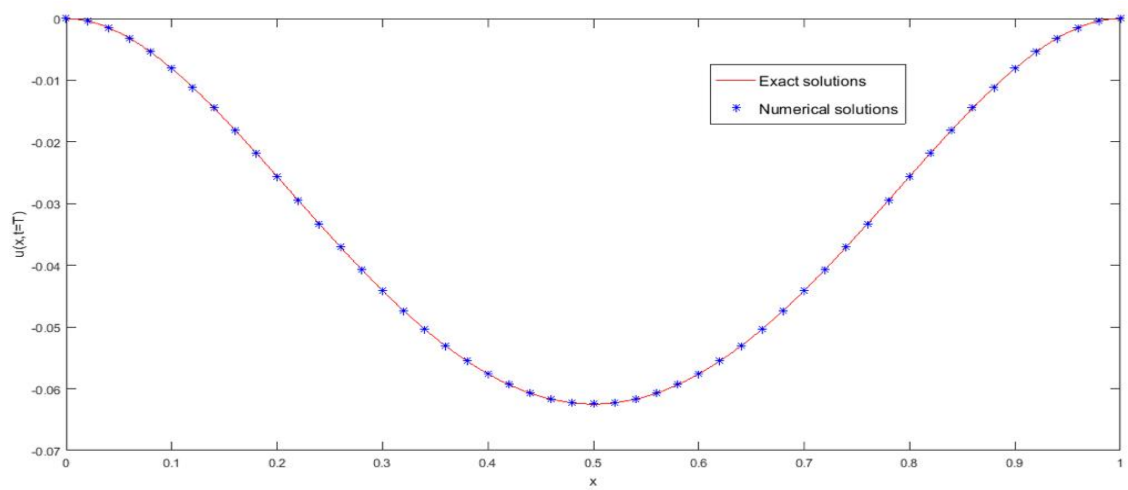

FIGURE 3. Numerical solution of Ex. (4.1),for $\alpha = 0.8$, β =1.4, and $\Delta t = h = \frac{1}{50}$.

$\Delta t = h = \frac{1}{50}$, Figure 3 demonstrates the efficacy of the numerical solution in Example 4.1 against the exact solution $\Phi(x,t) = \cos(\pi t)x^2(x-1)^2$.

Example 4.2. Take the FDE (1.1)-(1.3) problem, with $L = 1$, and $\mathfrak{F}$ presented as follow

$$\begin{aligned}
\mathfrak{F}(x,t) = {} & \frac{x^4(x-1)^4}{1-\alpha}\frac{\sigma\pi}{\sigma^2+\pi^2}\left(\cos(\frac{\pi t}{2}) + \frac{\pi}{2\sigma}\sin(\frac{\pi t}{2}) - \exp(-\sigma t)\right) + \sin\left(\frac{\pi t}{2}\right)\sec\left(\frac{\pi\beta}{2}\right) \\
& \times (12(x^{4-\beta} + (1-x)^{4-\beta})/\Gamma(5-\beta) - 240(x^{5-\beta} + (1-x)^{5-\beta})/\Gamma(6-\beta) \\
& + 2160(x^{6-\beta} + (1-x)^{6-\beta})/\Gamma(7-\beta) - 10080(x^{7-\beta} + (1-x)^{7-\beta})/\Gamma(8-\beta) \\
& + 20160(x^{8-\beta} + (1-x)^{8-\beta})/\Gamma(9-\beta)), \qquad \text{where} \quad \sigma = \alpha/(1-\alpha),
\end{aligned}$$

$\phi(x) = x^4(x-1)^4$, $v(t) = 0$, and $\mathfrak{A} = 1$.

TABLE 2. Example 4.2: The L^2_- errors, L^2_- convergence order and CPU times at T=1.

α	β	$\Delta t = h$	L^2_- error	L^2_- c.order	CPU time(s)
0.3	1.9	1 / 20	1.4838e-05	——	0.0551
		1 / 40	3.6643e-06	2.01769	0.1092
		1 / 100	5.8498e-07	2.00153	1.0608
		1 / 160	2.2852e-07	2.00008	1.9032
0.5	1.5	1 / 20	2.1841e-05	——	0.2415
		1 / 40	5.5389e-06	1.97937	0.4836
		1 / 100	8.9791e-07	1.98683	1.0608
		1 / 160	3.5210e-07	1.99096	1.8564
0.9	1.3	1 / 20	2.3447e-05	——	0.2412
		1 / 40	6.0075e-06	1.96457	0.4836
		1 / 100	9.8015e-07	1.98059	1.0296
		1 / 160	3.8498e-07	1.98699	1.9968

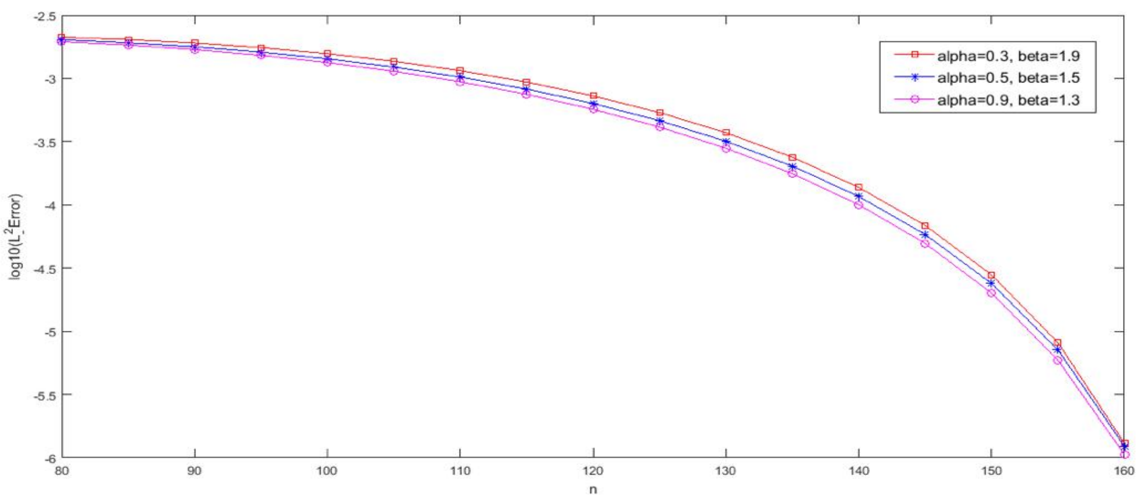

FIGURE 4. $log10(L^2_- Error)$ as a function of n time steps for Ex (4.2), for three cases of α and β, for $h = \frac{1}{170}$ at the time $T = 1$.

The numerical outcomes are presented in Table 2 of Example 4.2, which compares the numerical and exact solutions for three different α and β values at $T = 1$. It is evident that as the space-time step size decreases (assume $\Delta t = h$ for simplicity), the L^2_- errors of the numerical solution also decrease with the same order of accuracy, as dictated by Theorem 3.5.

Figure 4 illustrates the comparison of $\log_{10}(L^2_-\text{Error})$ as a function of the time step number n of our proposed scheme at $T = 1$, with $h = \frac{1}{170}$, for the aforementioned cases. At $T = 1$, Figure 5 describes the numerical solutions obtained

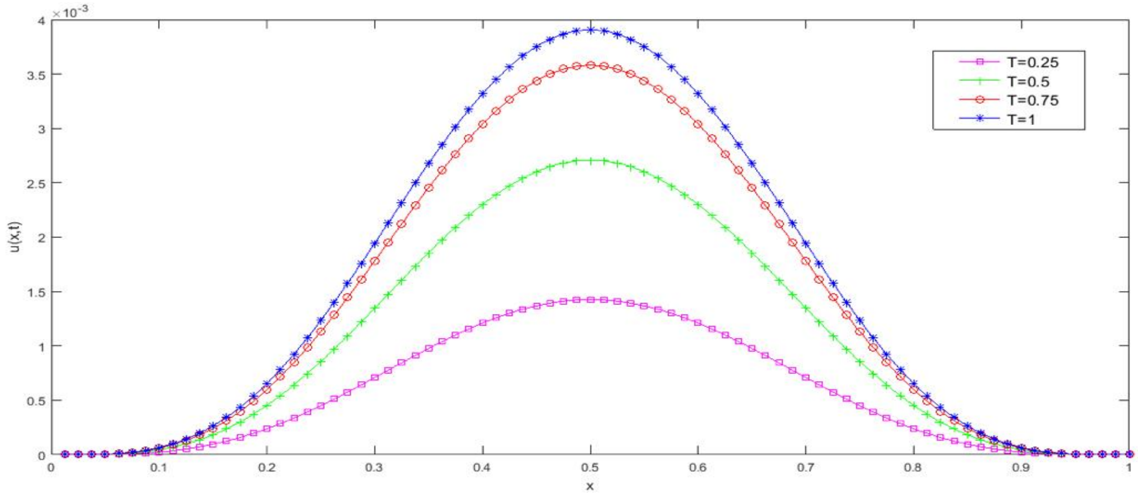

FIGURE 5. Numerical solutions of Ex. (4.2),for $\alpha = 0.5$, β =1.5, and $\Delta t = h = \frac{1}{80}$ at the time T =0.25, 0.5, 0.75, and 1.

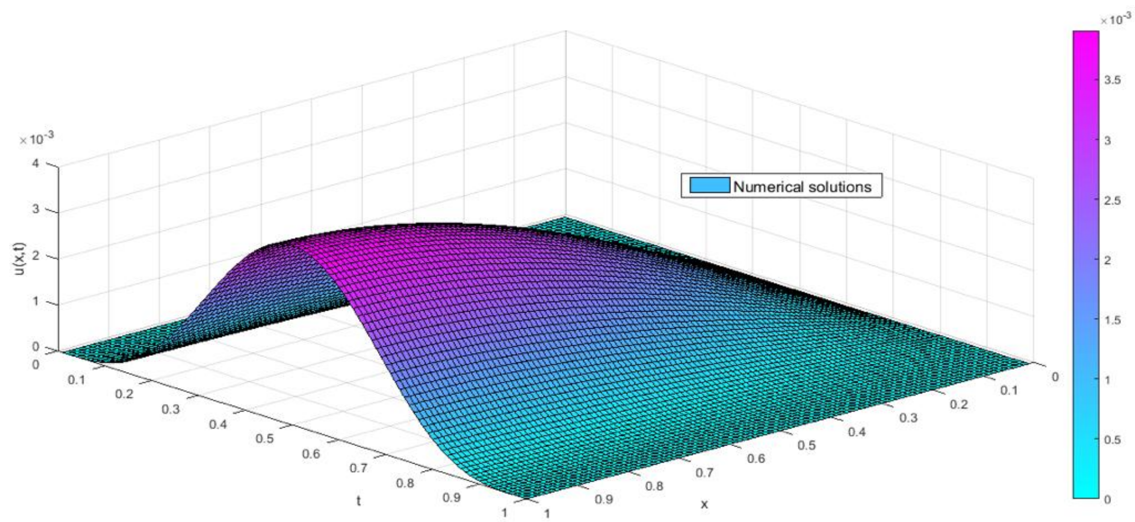

FIGURE 6. Numerical solution of Ex. (4.2), for $\alpha = 0.9$, β =1.3, and $\Delta t = h = \frac{1}{100}$.

for $\alpha = 0.5$, $\beta = 1.5$, at $T = 0.25$, 0.5, 0.75, and 1, for a space-time step size $\Delta t = h = \frac{1}{80}$. For $\alpha = 0.9$, $\beta = 1.3$, and $\Delta t = h = \frac{1}{100}$, the 3D graphs in Figure 6 demonstrate the accuracy of the numerical solutions against the exact solution.

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