Gulf Journal of Mathematics
Vol 21, Issue 2 (2025) 118-148

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i2.3598

CONSTRUCTION OF SYMMETRIC THIRD ORDER DIFFERENCE OPERATORS ON A HILBERT SPACE

SOROWEN BEN[1*], FREDRICK OLUOCH NYAMWALA[2]and DAVID OTIENO AMBOGO[3]

Abstract. We have constructed a symmetric third order difference equation and studied the absolutely continuous spectrum of the self adjoint subspace extension generated using the subspace and decomposition theories. In particular, we have shown that under some decay and growth conditions, the self-adjoint subspace extension exists with discrete spectrum on $l^2(\mathbb{N})$ while the absolutely continuous spectrum contained on half line of multiplicity one exists on $l^2(\mathbb{Z})$ if we use decomposition technique. We have given a number of examples to reinforce our results.

Keywords. Odd order difference equations, symmetric difference equations, absolutely continous spectrum
2020 Mathematics Subject Classification. Primary 46L55; Secondary 44B20

1. Introduction

In several papers, one of the authors and others, have studied the spectral theory of even-order difference operators [6, 5, 1, 3, 7, 18, 19, 20]. Recently, attention has shifted from studying even order differential equations to the odd higher order differential equations and difference operators. Hinton in [13] computed using the limit point criteria for differential equations the deficiency indices for odd order differential equations. Infact, Behncke and Hinton [4] analyzed the asymptotic behavior of eigenfunctions for a rather general class of odd-order differential operators on the half-line [4]. Consequently, it is natural to consider extending this kind of analysis to general odd-order difference operators. Difference equations are usually regarded as the discretization of the corresponding differential equations. It is well established from [7] that the results for symmetric differential operators generally correspond to those of their discrete counterparts when subject to similar growth and decay conditions. Based on current findings, most properties in the spectral theory of difference equations align with those of the corresponding differential equations, though a few significant ones

Date: Received: Sep 25, 2025; Revised:Oct 22, 2025; Accepted: Nov 7, 2025.
* Corresponding author
© The Author(s) 2025. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.
Ben Sorowen would like to thank the faculty at Moi University, for their hospitality during his research study, and Kyambogo University for the study leave.

differ [26]. However, unlike higher order differential equations, the general construction of odd higher order symmetric difference equations is a challenge. A common challenge encountered in the study of difference equations is that the minimal operators they generate may fail to be densely defined or may not be single-valued. In such situations, although the maximal operator might be well-defined, the self-adjoint extension of the minimal operator cannot be examined using classical operator theories such as von Neumann's theory [32, Section 8] or the Glazman–Krein–Naimark (GKN) theory [27], which are applicable only to densely defined Hermitian operators [24, 27]. This is unlike the continuous case, where symmetric differential equations generate densely defined minimal differential operators with single-valued maximal operators [1, 18, 29]. For this reason, several researchers have focused on extending the spectral theory of linear operators to include linear operators that are not densely defined or are multivalued, by employing the linear relations or subspace theory [26, 27, 28] and references therein. Most of the studies on difference equations and difference operators have focused on either the Hamiltonian systems of the difference equations or on the deficiency indices [8, 10, 9]. The von Neumann self-adjoint extension theory for symmetric operators have been extended to Hermitian subspaces by [11]. He demonstrated that a closed Hermitian subspace T admits a self-adjoint subspace extension if and only if its positive and negative defect indices are equal. The original von Neumann formula was later extended to Hermitian subspaces by [16]. The classical GKN theory for symmetric operators was extended to Hermitian subspaces by [27]. A complete characterization for the self-adjoint subspaces and operator extensions for second order formally self-adjoint difference equations can be found in [29]. Z. Zheng and S. Chen have extend and improved the GKN theory for quasi differential equations [33].

The study of difference equations represents a mathematically rich and versatile field used to model discrete processes [31]. Interest in this area has developed and persisted for two main reasons. First, difference equations hold substantial theoretical importance and offer diverse modeling applications across nearly all branches of science, engineering, and technology where discrete phenomena are prevalent. Second, with the advent of digital computers, differential equations are often solved through their approximate representations as difference equations. Therefore, the demand for and interest in further scientific development in this field arise naturally [31].

In existing literature, difference equations naturally arise as descriptions of evolutionary phenomena since most measurements of time-dependent variables are discrete. Therefore, these equations serve as important mathematical models in their own right. Moreover, they are essential in the analysis of discretization methods for differential equations. Many findings in the theory of difference equations have been formulated as discrete analogs of results from differential equations, especially within Liapunov's stability theory. However, the theory of difference equations is significantly more extensive than that of differential equations [15]. For instance, a simple difference equation derived from a first-order

differential equation can exhibit phenomena such as chaotic orbits—behavior typically associated with higher-order differential equations [14, Section 2]. As a result, the theory of difference equations is inherently interesting and is expected to gain increasing significance in the near future. Furthermore, its applications are rapidly expanding across various domains, including numerical analysis, control theory, finite mathematics, and computer science. Hence, the study of difference equations is both justified and essential as a distinct academic discipline.

Let $y(t)$ be a real valued function of the real variables $t \in \mathbb{Z}$, the forward difference operator Δ is defined by $\Delta y(t) = y(t+1) - y(t)$ for all t, the shift operator E defined by $Ey(t) = y(t+1)$, and the identity operator I defined by $Iy(t) = y(t)$. Thus in general, one constructs a higher order difference map $Ly(t)$ as

$$Ly(t) = \left\{ \sum_{k=0}^{n} p_k(t) y(t+k) \right\} \tag{1.1}$$

to be n^{th} order difference map. $Ly(t)$ in (1.1) is not necessarily symmetric. However, different authors have formulated different versions of (1.1) which are symmetric whenever n is even (see [6, 20, 21] to mention a few).

The available literature exhaustively discusses the construction of symmetric even order difference maps from (1.1) which is given by

$$Ly(t) = w^{-1}(t) \left\{ \sum_{k=0}^{n} (-1)^k \Delta^k (p_k(t) \Delta^k y(t-k)) \right\} \tag{1.2}$$

where $w(t)$ is weight function, $w(t) > 0$ for all $t \in \mathbb{N}$ and $\ell y(y)$ defined on weighted Hilbert space $l^2_w(\mathbb{N})$. However, the construction of a general symmetric odd order difference map equivalent to (1.2) has been a challenge. Attempts have been made but nothing substantial has been obtained, see [7]. In this paper, we were interested in the construction of third order symmetric difference map and subsequent spectral theory of such a difference map.

Our results shows that based on the techniques of Hermitian matrices of order 3 we constructed third order symmetric difference equations (maps) and using subspace and decomposition theories, with suitable growth and decay conditions on the associated coefficients, we have shown that on $l^2(\mathbb{N})$, the minimal symmetric subspace has self-adjoint subspace extension with at most discrete spectrum. However, when the map is extended to $l^2(\mathbb{Z})$ via decomposition theory, then absolutely continous spectrum contained on half-line of spectral multiplicity one is realised.

This paper is arranged as follows. Section 1: Introduction, Section 2: Construction of Third Order Symmetric Difference Equations, Section 3: Spectral Analysis of Third Order Difference Equations.

2. Construction of Third Order Symmetric Difference Equations

In this section, a third-order symmetric difference equation has been formulated based on the concepts of Hermitian matrices and symmetric operators defined on Hilbert spaces.

Definition 2.1. An $n \times n$ matrix A with complex entries is called Hermitian if $A^* = A$ or, equivalently, if $\overline{A} = A^T$. If A is Hermitian, it means that $a_{ij} = \overline{a_{ji}}$ for every i, j pair. Thus, the diagonal entries of a Hermitian matrix must be real.

Lemma 2.2. *Suppose H is a 3×3 matrix with complex entries, then H is Hermitian if $\overline{H} = H^t$.*

Let

$$H = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

where $q_{ij} = q_{ij}(t)$, $t \in \mathbb{N}$, are complex valued functions. The conjugate transpose H^* is given by

$$H^* = \begin{bmatrix} \overline{q_{11}} & \overline{q_{21}} & \overline{q_{31}} \\ \overline{q_{12}} & \overline{q_{22}} & \overline{q_{32}} \\ \overline{q_{13}} & \overline{q_{23}} & \overline{q_{33}} \end{bmatrix}$$

For H to be Hermitian, the following two conditions must hold: the diagonal elements must satisfy $q_{ij} = \overline{q_{ji}}$ for $i = 1, 2, 3$. This implies that q_{ij} for all $i = j$, $i = 1, 2, 3$ are real numbers. The other condition is that, the off-diagonal elements must satisfy $q_{ij} = \overline{q_{ji}}$ for $i \neq j$. If all the two conditions are satisfied, we shall obtain

$$\overline{H} = H^t = \begin{bmatrix} p_1 & q_{12} & q_{13} \\ \overline{q_{12}} & p_2 & q_{23} \\ \overline{q_{13}} & \overline{q_{23}} & p_3 \end{bmatrix}, H = H(t)$$

where $p_1 = q_{11}$, $p_2 = q_{22}$ and $p_3 = q_{33}$. $p_k = p_k(t)$ are real valued functions. Thus, the matrix H is Hermitian.

Now we apply $H(t)$ to 3-dimensional vector defined $Y(t)$. Where $Y(t) = [y_1(t), y_2(t), y_3(t)]^{tr}$ with $y_1(t) = y(t)$, $y_2(t) = y(t+1)$, $y_3(t) = y(t+2)$. One, therefore constructs a first order difference equation as

$$Y(t+1) = H(t)Y(t) \tag{2.1}$$

Equation (2.1) represents a first-order system of three symmetric difference equations, which we will show generates a third-order scalar equation as proved in Lemma 2.3 and Theorem 2.4 below.

Lemma 2.3. *Suppose all the conditions in Lemma 2.2 are satisfied. Then (2.1) generates a third order difference map of the form*

$$\ell y(t) = y(t+3) - \alpha_2(t)y(t+2) - \alpha_1(t)y(t+1) - \alpha_0(t)y(t) \tag{2.2}$$

where $\alpha_2 = [\overline{q_{23}}q_{13} + p_3q_{23}]$, $\alpha_1 = [\overline{q_{23}}q_{12} + p_3p_2]$ and $\alpha_0 = [\overline{q_{13}} + \overline{q_{23}}p_1 + p_3\overline{q_{12}}]$, $\alpha_k = \alpha_k(t), k = 0, 1, 2$.

Proof. By (2.1), we obtain

$$y_3(t+1) = y(t+3) = \overline{q_{13}}y_1(t) + \overline{q_{23}}y_2(t) + p_3y_3(t). \tag{2.3}$$

We also have that

$$\begin{aligned} y_1(t+1) &= p_1y_1(t) + q_{12}y_2(t) + q_{13}y_3(t) \\ y_2(t) &= p_1y(t) + q_{12}y(t+1) + q_{13}y(t+2). \end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned} y_2(t+1) &= \overline{q_{12}}y_1(t) + p_2y_2(t) + q_{23}y_3(t) \\ y_3(t) &= \overline{q_{12}}y(t) + p_2y(t+1) + q_{13}y(t+2). \end{aligned} \tag{2.5}$$

Putting (2.4) and (2.5) into (2.3), it follows that

$$y(t+3) = \overline{q_{13}}y(t)+\overline{q_{23}}[p_1y(t)+q_{12}y(t+1)+q_{13}y(t+2)]+p_3[\overline{q_{12}}y(t)+p_2y(t+1)+q_{23}y(t+2)]$$

Expanding and collecting like terms results into

$$y(t+3) = [\overline{q_{23}}q_{13}+p_3q_{23}]y(t+2)+[\overline{q_{23}}q_{12}+p_3p_2]y(t+1)+[\overline{q_{13}}+\overline{q_{23}}p_1+p_3\overline{q_{12}}]y(t). \tag{2.6}$$

One, therefore, constructs a third order homogeneous equation of the form

$$y(t+3)-[\overline{q_{23}}q_{13}+p_3q_{23}]y(t+2)-[\overline{q_{23}}q_{12}+p_3p_2]y(t+1)-[\overline{q_{13}}+\overline{q_{23}}p_1+p_3\overline{q_{12}}]y(t) = 0. \tag{2.7}$$

Now let $\alpha_2 = [\overline{q_{23}}q_{13} + p_3q_{23}]$, $\alpha_1 = [\overline{q_{23}}q_{12} + p_3p_2]$ and $\alpha_0 = [\overline{q_{13}} + \overline{q_{23}}p_1 + p_3\overline{q_{12}}]$, then the left hand of (2.7) can now be taken as $\ell y(t)$. □

Here, $\ell y(t) = 0$ becomes a third order homogeneous equation generated from a Hamiltonian system. It would be interesting to prove if $\ell y(t)$ is symmetric using the Green's identity on the weighted Hilbert space $l^2_w(\mathbb{N})$, where $w = w(t) > 0$ for all $t \in \mathbb{N}$ is a weight function. We note that for any $\tilde{y_1}(t), \tilde{y_2}(t) \in \mathbb{N}$ the inner products on $l^2_w(\mathbb{N})$ by

$$\langle \tilde{y_1}(t), \tilde{y_2}(t)\rangle = \sum_{t_0}^{t-1} \tilde{y_1}(t)w(t)\overline{\tilde{y_2}(t)}.$$

In our subsequent discussions, we will take $w(t) = 1$, for all $t \in \mathbb{N}$ and take $l^2_w(\mathbb{N})$ to be $l^2(\mathbb{N})$ since the two are isomorphic.

Theorem 2.4. *Suppose that α_0 is real for all t and , $\lim_{t\to\infty} |\alpha_r(t)| \to 0$ as $t \to \infty$, $r = 1, 2$, then $\ell y(t)$ in Lemma 2.3 generates a symmetric third order difference map on $\ell^2(\mathbb{N})$.*

Proof. This is proved via applications of Green's identity inner products. For the lower limit, we will assume the left regular endpoint to be some $t_0 \in \mathbb{N}$ and the results can be extended to the whole of $\mathbb{N}$ by application of the results in [12, 22]. Consider the identity

$$\lim_{b\to\infty}[\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} \tag{2.8}$$

we obtain

$$\begin{aligned}
&\lim_{b\to\infty}[\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} \\
&\quad = \lim_{b\to\infty}\Bigg\{ \sum_{t=t_0}^{b}[y_1(t+3) - \alpha_2 y_1(t+2) - \alpha_1 y_1(t+1) - \alpha_0 y_1(t)]\overline{y_2(t)} - \\
&\quad \sum_{t=t_0}^{b} y_1(t)[\overline{y_2(t+3) - \alpha_2 y_2(t+2) - \alpha_1 y_2(t+1) - \alpha_0 y_2(t)}]\Bigg\} \\
&\quad = \lim_{b\to\infty}\Bigg\{ \sum_{t=0}^{b}[y_2(t)y_1(t+3) - \alpha_2 y_2(t)y_1(t+2) - \alpha_1 y_2(t)y_1(t+1)] \\
&\quad - \sum_{t=t_0}^{b}[y_1(t)y_2(t+3) - \overline{\alpha_2} y_1(t)y_2(t+2) - \overline{\alpha_1} y_1(t)y_2(t+1)] \\
&\quad + \sum_{t=t_0}^{b}[\alpha_0 y_1(t)y_2(t) - \overline{\alpha_0} y_1(t)y_2(t)]\Bigg\}.
\end{aligned}$$

Since α_0 is real, we have

$$\begin{aligned}
&\lim_{b\to\infty}[\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} \\
&\quad = \lim_{b\to\infty}\Bigg\{ \sum_{t=t_0}^{b}[y_2(t)y_1(t+3) - \alpha_2 y_2(t)y_1(t+2) - \alpha_1 y_2(t)y_1(t+1)] \\
&\quad - \sum_{t=t_0}^{b}[y_1(t)y_2(t+3) - \overline{\alpha_2} y_1(t)y_2(t+2) - \overline{\alpha_1} y_1(t)y_2(t+1)]\Bigg\}.
\end{aligned}$$

Symmetry now holds if and only if

$$\lim_{b\to\infty}[\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} = 0. \tag{2.9}$$

This is true if and only if

$$\begin{aligned}
&\lim_{b\to\infty}\Bigg\{ \sum_{t=t_0}^{b}[y_2(t)y_1(t+3) - \alpha_2 y_2(t)y_1(t+2) - \alpha_1 y_2(t)y_1(t+1)]\Bigg\} \\
&\quad = \lim_{b\to\infty}\Bigg\{ \sum_{t=t_0}^{b}[y_1(t)y_2(t+3) - \overline{\alpha_2} y_1(t)y_2(t+2) - \overline{\alpha_1} y_1(t)y_2(t+1)]\Bigg\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\lim_{t\to\infty}[y_1(t+3) - \alpha_2 y_1(t+2) - \alpha_1 y_1(t+1)]y_2(t) \\
&\quad = \lim_{t\to\infty}[y_2(t+3) - \overline{\alpha_2} y_2(t+2) - \overline{\alpha_1} y_2(t+1)]y_1(t)
\end{aligned}$$

from which we obtain

$$\lim_{t\to\infty}\frac{y_2(t)}{y_1(t)} = \lim_{t\to\infty}\left[\frac{y_2(t+3) - \overline{\alpha_2} y_2(t+2) - \overline{\alpha_1} y_2(t+1)}{y_1(t+3) - \alpha_2 y_1(t+2) - \alpha_1 y_1(t+1)}\right].$$

Now if $\lim_{t\to\infty} |\alpha_2(t)| = 0$ and $\lim_{t\to\infty} |\alpha_1(t)| = 0$, then

$$\lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = \lim_{t\to\infty} \frac{y_2(t+3)}{y_1(t+3)}$$

which implies (2.9) and the proof is complete. □

In Examples 2.5, 2.6 and 2.7, we have constructed non-trivial examples to reinforce the results of Theorem 2.4.

Example 2.5. Consider a third order difference map with real power coefficients $\alpha_0(t) = a_0 t^{-\mu}$, $\alpha_1(t) = a_1 t^{-\alpha}$ and $\alpha_2(t) = a_2 t^{-\beta}$, for all $\alpha, \beta, \mu > 0$. In this case, the difference map defined in (2.2) takes the form

$$\ell y(t) = y(t+3) + a_2 t^{-\beta} y(t+2) + a_1 t^{-\gamma} y(t+1) + a_0 t^{-\mu} y(t). \tag{2.10}$$

By Green's identity,

$$\begin{aligned}
&\lim_{b\to\infty} [\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} \\
&\quad = \lim_{b\to\infty} \Bigg\{ \sum_{t=t_0}^{b} [y_1(t+3) - a_2 t^{-\beta} y_1(t+2) - a_1 t^{-\gamma} y_1(t+1) - a_0 t^{-\mu} y_1(t)] \overline{y_2(t)} \\
&\quad - \sum_{t=t_0}^{b} y_1(t) \overline{[y_2(t+3) - a_2 t^{-\beta} y_2(t+2) - a_1 t^{-\gamma} y_2(t+1) - a_0 t^{-\mu} y_2(t)]} \Bigg\} \\
&\quad = \lim_{b\to\infty} \Bigg\{ \sum_{t=t_0}^{b} [y_2(t) y_1(t+3) - a_2 t^{-\beta} y_2(t) y_1(t+2) - a_1 t^{-\gamma} y_2(t) y_1(t+1) \\
&\quad - a_0 t^{-\mu} y_2(t) y_1(t)] - \sum_{t=t_0}^{b} [y_1(t) y_2(t+3) - a_2 t^{-\beta} y_1(t) y_2(t+2) \\
&\quad - a_1 t^{-\gamma} y_1(t) y_2(t+1) - a_0 t^{-\mu} y_1(t) y_2(t)] \Bigg\} \\
&\quad = \lim_{b\to\infty} \Bigg\{ \sum_{t=t_0}^{b} [y_1(t+3) - a_2 t^{-\beta} y_1(t+2) - a_1 t^{-\gamma} y_1(t+1)] y_2(t) \\
&\quad - \sum_{t=t_0}^{b} [y_2(t+3) - a_2 t^{-\beta} y_2(t+2) - a_1 t^{-\gamma} y_2(t+1)] y_1(t) \Bigg\}.
\end{aligned}$$

For symmetry to hold,

$$\lim_{b\to\infty} [\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} = 0$$

and for our case, it holds when

$$\begin{aligned}
&\lim_{t\to\infty} [y_1(t+3) - a_2 t^{-\beta} y_1(t+2) - a_1 t^{-\gamma} y_1(t+1)] y_2(t) \\
&\qquad = \lim_{t\to\infty} [y_2(t+3) - a_2 t^{-\beta} y_2(t+2) - a_1 t^{-\gamma} y_2(t+1)] y_1(t).
\end{aligned}$$

It follows that

$$\lim_{t\to\infty}\left(\frac{y_2(t)}{y_1(t)}\right)=\lim_{t\to\infty}\left(\frac{y_2(t+3)-a_2t^{-\beta}y_2(t+2)-a_1t^{-\gamma}y_2(t+1)}{y_1(t+3)-a_2t^{-\beta}y_1(t+2)-a_1t^{-\gamma}y_1(t+1)}\right).$$

Since $\lim_{t\to\infty} a_1t^{-\gamma}=0$ and $\lim_{t\to\infty} a_2t^{-\beta}=0$ for all $\beta,\gamma>0$, we have

$$\lim_{t\to\infty}\left(\frac{y_2(t)}{y_1(t)}\right)=\lim_{t\to\infty}\left(\frac{y_2(t+3)}{y_1(t+3)}\right).$$

Clearly, all the conditions of Theorem 2.4 are satisfied and hence $\ell y(t)$ in (2.10) is the desired symmetric difference map of order three. We also note that if $\alpha_0(t)=a_0(t)t^{-\alpha}$ is not real valued, then we require that $\lim_{t\to\infty} a_0t^{-\alpha}=0$ for $\ell y(t)$ in (2.10) to be a symmetric third order difference map.

Example 2.5 shows that if $\mu,\alpha,\beta>0$, then we have a symmetric third order difference map by Theorem 2.4. However, if you consider the case when $\mu,\alpha,\beta<0$, then we see that the results of Theorem 2.4 are not valid since $\lim_{t\to\infty} a_1t^{-\gamma}=\infty$ and $\lim_{t\to\infty} a_2t^{-\beta}=\infty$.

Example 2.6. Consider a third order difference map with the following coeficients

$$p_1=p_3=v, p_2=k, q_{13}=-t^{-\alpha}, q_{12}=it^{-\beta} \text{ and } q_{23}=-it^{-\beta}, \qquad \alpha,\beta>0.$$

It follows that

$$\alpha_0(t)=t^{-\alpha}, \alpha_1(t)=(t^{-2\beta}-vk) \text{ and } \alpha_2(t)=i(t^{-\alpha-\beta}+vt^{-\beta}).$$

and

$$\ell y(t)=y(t+3)-i(t^{-\alpha-\beta}+vt^{-\beta})y(t+2)-(t^{-2\beta}-vk)y(t+1)-t^{-\alpha}y(t). \quad (2.11)$$

By Green's identity,

$$\lim_{b\to\infty}[\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b}$$

$$= \lim_{b\to\infty}\Bigg\{\sum_{t=t_0}^{b}[y_1(t+3) - i(t^{-\alpha-\beta}+vt^{-\beta})y_1(t+2) - (t^{-2\beta}-vk)y_1(t+1)$$

$$- t^{-\alpha}y_1(t)]\overline{y_2(t)}$$

$$- \sum_{t=t_0}^{b} y_1(t)\overline{[y_2(t+3) - i(t^{-\alpha-\beta}+vt^{-\beta})y_2(t+2) - (t^{-2\beta}-vk)y_2(t+1) - t^{-\alpha}y_2(t)]}\Bigg\}$$

$$= \lim_{b\to\infty}\Bigg\{\sum_{t=t_0}^{b}[y_2(t)y_1(t+3) - i(t^{-\alpha-\beta}+vt^{-\beta})y_2(t)y_1(t+2)$$

$$- (t^{-2\beta}-vk)y_2(t)y_1(t+1) - t^{-\alpha}y_2(t)y_1(t)]$$

$$- \sum_{t=t_0}^{b}[y_1(t)y_2(t+3) + i(t^{-\alpha-\beta}+vt^{-\beta})y_1(t)y_2(t+2)$$

$$- (t^{-2\beta}-vk)y_1(t)y_2(t+1) - t^{-\alpha}y_1(t)y_2(t)]\Bigg\}$$

$$= \lim_{b\to\infty}\Bigg\{\sum_{t=t_0}^{b}[y_1(t+3) - i(t^{-\alpha-\beta}+vt^{-\beta})y_1(t+2) - (t^{-2\beta}-vk)y_1(t+1)]y_2(t)$$

$$- \sum_{t=t_0}^{b}[y_2(t+3) + i(t^{-\alpha-\beta}+vt^{-\beta})y_2(t+2) - (t^{-2\beta}-vk)y_2(t+1)]y_1(t)\Bigg\}.$$

For symmetry to hold,

$$\lim_{b\to\infty}[\langle \ell y_1(t), y_2(t)\rangle - \langle y_1(t), \ell y_2(t)\rangle]_{t=t_0}^{b} = 0$$

and for our case, it holds when

$$\lim_{t\to\infty}[y_1(t+3) - i(t^{-\alpha-\beta}+vt^{-\beta})y_1(t+2) - (t^{-2\beta}-vk)y_1(t+1)]y_2(t)$$
$$= \lim_{t\to\infty}[y_2(t+3) + i(t^{-\alpha-\beta}+vt^{-\beta})y_2(t+2) - (t^{-2\beta}-vk)y_2(t+1)]y_1(t).$$

It follows that

$$\lim_{t\to\infty}\left(\frac{y_2(t)}{y_1(t)}\right) = \lim_{t\to\infty}\left(\frac{y_2(t+3) + i(t^{-\alpha-\beta}+vt^{-\beta})y_2(t+2) - (t^{-2\beta}-vk)y_2(t+1)}{y_1(t+3) - i(t^{-\alpha-\beta}+vt^{-\beta})y_1(t+2) - (t^{-2\beta}-vk)y_1(t+1)}\right).$$

Now let's consider $\lim_{t\to\infty} i(t^{-\alpha-\beta}+vt^{-\beta})$ and $\lim_{t\to\infty}(t^{-2\beta}-vk)$ for all $\alpha,\beta>0$. First

$$\lim_{t\to\infty} i(t^{-\alpha-\beta}+vt^{-\beta}) = 0$$

and

$$\lim_{t\to\infty}(t^{-2\beta}-vk) = -vk.$$

Since $\lim_{t\to\infty}(t^{-2\beta}-vk) = -vk \neq 0$, we conclude that $\ell y(t)$ in (2.11) is not a symmetric difference map because one of the condition $\lim_{t\to\infty}\alpha_1(t) =$

$\lim_{t\to\infty}\lim_{t\to\infty}(t^{-2\beta}-vk)$ of Theorem 2.4 is not satisfied. It can only be satisfied for special case when either $v=0$ or $k=0$ or both $k=v=0$.

Example 2.7. Consider a third order difference operator with the coefficients

$$\alpha_0(t)=a_0e^{-\mu t},\alpha_1(t)=ia_1e^{-\beta t} \text{ and } \alpha_2(t)=ia_2e^{-\alpha t}, \qquad \alpha,\beta,\mu>0.$$

Thus we have

$$\ell y(t)=y(t+3)-ia_2e^{-\alpha t}y(t+2)-ia_1e^{-\beta t}y(t+1)-a_0e^{-\mu t}y(t). \tag{2.12}$$

By Green's identity,

$$\begin{aligned}
&\lim_{b\to\infty}[\langle \ell y_1(t),y_2(t)\rangle-\langle y_1(t),\ell y_2(t)\rangle]_{t=t_0}^{b}\\
&=\lim_{b\to\infty}\Bigg\{\sum_{t=t_0}^{b}[y_1(t+3)-ia_2e^{-\alpha t}y_1(t+2)-ia_1e^{-\beta t}y_1(t+1)-a_0e^{-\mu t}y_1(t)]\overline{y_2(t)}\\
&-\sum_{t=t_0}^{b}y_1(t)\overline{[y_2(t+3)-ia_2e^{-\alpha t}y_2(t+2)-ia_1e^{-\beta t}y_2(t+1)-a_0e^{-\mu t}y_2(t)]}\Bigg\}\\
&=\lim_{b\to\infty}\Bigg\{\sum_{t=t_0}^{b}[y_2(t)y_1(t+3)-ia_2e^{-\alpha t}y_2(t)y_1(t+2)-ia_1e^{-\beta t}y_2(t)y_1(t+1)\\
&-a_0t^{-\mu}y_2(t)y_1(t)]-\sum_{t=t_0}^{b}[y_1(t)y_2(t+3)+ia_2e^{-\alpha t}y_1(t)y_2(t+2)\\
&+ia_1e^{-\beta t}y_1(t)y_2(t+1)-a_0t^{-\mu}y_1(t)y_2(t)]\Bigg\}\\
&=\lim_{b\to\infty}\Bigg\{\sum_{t=t_0}^{b}[y_1(t+3)-ia_2e^{-\alpha t}y_1(t+2)-ia_1e^{-\beta t}y_1(t+1)]y_2(t)\\
&-\sum_{t=t_0}^{b}[y_2(t+3)+ia_2e^{-\alpha t}y_2(t+2)+ia_1e^{-\beta t}y_2(t+1)]y_1(t)\Bigg\}.
\end{aligned}$$

For symmetry to hold,

$$\lim_{b\to\infty}[\langle \ell y_1(t),y_2(t)\rangle-\langle y_1(t),\ell y_2(t)\rangle]_{t=t_0}^{b}=0$$

and for our case, it holds when

$$\begin{aligned}
&\lim_{t\to\infty}[y_1(t+3)-ia_2e^{-\alpha t}y_1(t+2)-ia_1e^{-\beta t}y_1(t+1)]y_2(t)\\
&\quad=\lim_{t\to\infty}[y_2(t+3)+ia_2e^{-\alpha t}y_2(t+2)+ia_1e^{-\beta t}y_2(t+1)]y_1(t).
\end{aligned}$$

It follows that

$$\lim_{t\to\infty}\left(\frac{y_2(t)}{y_1(t)}\right)=\lim_{t\to\infty}\left(\frac{y_2(t+3)+ia_2e^{-\alpha t}y_2(t+2)+ia_1e^{-\beta t}y_2(t+1)}{y_1(t+3)-ia_2e^{-\alpha t}y_1(t+2)-ia_1e^{-\beta t}y_1(t+1)}\right).$$

Since

$$\lim_{t\to\infty}a_2e^{-\alpha t}=0=\lim_{t\to\infty}ia_1e^{-\beta t} \text{ for all } \alpha,\beta>0,$$

we obtain

$$\lim_{t\to\infty}\left(\frac{y_2(t)}{y_1(t)}\right)=\lim_{t\to\infty}\left(\frac{y_2(t+3)}{y_1(t+3)}\right).$$

Hence the map (2.12) is symmetric difference map since all the conditions of Theorem 2.4 are satisfied.

3. Spectral theory of third order difference Equations

The main difficulty encountered when studying difference systems lies in the fact that the minimal operators generated by (1.2) are often neither densely defined nor single-valued. Consequently, their maximal operators may not be well-defined, making it impossible to apply the von Neumann theory [32, Section 8] or the Glazman–Krein–Naimark (GKN) theory [27], both of which require densely defined Hermitian operators, to discuss self-adjoint extensions. This issue does not arise in the case of differential systems, where symmetric differential equations generate densely defined minimal differential operators with single-valued maximal operators. To address this challenge, one must employ the theory of Hermitian subspaces, which extends von Neumann's framework to handle self-adjoint extensions of minimal Hermitian subspaces [16, 18, 17, 23, 27, 28]. In our work, we use this subspace theory to analyze the non–densely defined minimal operator generated by (2.2).

Let $\ell^2(I)$ denote a Hilbert space with the weight function $w=w(t)=1$, defined in terms of the vector-valued functions $Y(t)=[y_1(t),y_2(t),y_3(t)]^{\text{tr}}$ as

$$\ell^2(I)=\left\{y:y=\{y(t)\}_{t=t_0}^{\infty}\subset\mathbb{C}^3\quad\text{and}\quad\sum_{t=0}^{\infty}Y^*(t)Y(t)<\infty\right\}$$

The inner product for the vector-valued functions in this space is given by

$$\langle y_1,y\rangle=\sum_{t=0}^{\infty}y_1(t+1)\overline{y(t+1)},\quad y,y_1\in\ell^2(I).$$

The system (2.2) is assumed to satisfy certain regularity and definiteness conditions; specifically, there exists a finite subinterval $I_1\subset I$ such that, for any $z\in\mathbb{C}$ and any nontrivial solution $y(t)$ of (2.2), the following holds:

$$\sum_{t\in I_1}Y^*(t)Y(t)>0. \tag{3.1}$$

In the discrete case, however, the minimal difference operator generated by (2.2) is generally not densely defined, even when the condition (3.1) is satisfied. This represents a key distinction between differential and difference equations. As a result of these technical challenges, the spectral theory of difference equations has received comparatively less attention than that of differential equations.

Since the minimal subspace generated by (2.2) is not a densely defined Hermitian operator, and its corresponding maximal subspace is a multivalued operator [24], the self-adjoint extension of the minimal operator associated with (2.2) cannot be addressed using von Neumann's theory for Hermitian operators. Consequently, the analysis will be carried out within the framework of Hermitian

subspace theory.This is because von Neumann theory has been extended to these subspaces, see [23, 24, 25]. Let $\mathcal{M}$ be a linear subspace or a linear relation in $\ell^2(I)\times\ell^2(I)$, where the domain $D(\mathcal{M})$, range $R(\mathcal{M})$ and kernel $K(\mathcal{M})$ of $\mathcal{M}$ are defined by

$$D(\mathcal{M}) = \{y \in \ell^2(I) : (y,g) \in \mathcal{M} \text{ for some } g \in \ell^2(I)\},$$
$$R(\mathcal{M}) = \{g \in \ell^2(I) : (y,g) \in \mathcal{M} \text{ for some } y \in \ell^2(I)\},$$
$$K(\mathcal{M}) = \{y \in \ell^2(I) : (y,0) \in \mathcal{M}\}.$$

Finally, define

$$\mathcal{M} - zI = \{(y, g - zy) : (y,g) \in \mathcal{M}\}$$

and

$$\mathcal{M}^* = \{(y,g) \in \ell^2(I)\times\ell^2(I) : \langle y,f\rangle = \langle g,x\rangle, \forall (x,f) \in \mathcal{M}\}.$$

Definition 3.1 ([24])**.** Let T be a subspace of $X^2 = X \times X$.

(i): The adjoint of T is defined as

$$T^* = (y,g) \in X^2 : \langle y,f\rangle = \langle g,x\rangle \quad \forall (x,f) \in T.$$

A subspace $T \subset X^2$ is called a Hermitian subspace if $T \subset T^*$. Furthermore, T is said to be Hermitian if it is an operator.

(ii): The subspace T is said to be self-adjoint if $T = T^*$; it is called a self-adjoint operator if it is an operator.

(iii): Let T be a Hermitian subspace. A subspace T_1 is called a self-adjoint extension of T if $T \subset T_1$ and T_1 is self-adjoint.

(iv): Let T be a Hermitian operator. An operator T_1 is called a self-adjoint operator extension of T if $T \subset T_1$ and T_1 is self-adjoint operator.

Thus, $\mathcal{M} \subset \ell^2(I)\times\ell^2(I)$ is called a Hermitian subspace if $\mathcal{M} \subset \mathcal{M}^*$.

Now define $\dim(R(\mathcal{M} - zI))^\perp$ as the defect index of $\mathcal{M} - zI$. Since $R(\mathcal{M} - zI)^\perp = K(\mathcal{M}^* - zI)$, the defect indices of $\mathcal{M}$ and its closure with respect to the same z are equal. We will denote

$$\operatorname{def}\mathcal{M} = \dim_\pm(\mathcal{M}) = \dim_{\pm i}(\mathcal{M}) = (\mathcal{N}_+, \mathcal{N}_-)$$

respectively as positive and negative defect indices of $\mathcal{M}$.

Now define

$$\ell(I) = \{y : y = (y(t))_t^\infty \subset C^t\}$$

and

$$\mathcal{L}^2(I) = \{y \in \ell(I) : \sum_{t\in I} Y^*(t)Y(t) < \infty\}$$

with the semi-scalar product

$$\langle y_1, y_2\rangle = \sum_{t\in I} Y_2^*(t)W(t)Y_1(t).$$

Then it follows that $\|y\| = (\langle y,y\rangle)^{\frac{1}{2}}$ for $y \in \mathcal{L}_W(I)$. Since $W(t) = diag(w(t),0,0)$ in I, $\|.\|$ is a semi-norm. One thus defines a quotient space.

$$L^2(I) = \mathcal{L}^2(I)/\{y \in \mathcal{L}^2(I) : \|y\| = 0\}$$

It is, therefore, true that $L^2(I)$ is a Hilbert space with an inner product $\langle .,. \rangle$. For a function y which is a solution of (2.2) and is summable, denote by $\tilde{y}$ the corresponding class in $L^2(I)$ and for any $\tilde{y} \in L^2(I)$, by $y \in \mathcal{L}^2(t)$ denote a representative of $\tilde{y}$. It is evident that $\langle \tilde{y}_1, \tilde{y}_2 \rangle = \langle y_1, y_2 \rangle$ for any $\tilde{y}_1, \tilde{y}_2 \in L^2(I)$. Now let π be a natural quotient map such that

$$\pi : \mathcal{L}^2(I) \longrightarrow L^2(I), \quad y \to \tilde{y}.$$

Then π is surjective and not injective in general. One defines the natural difference map corresponding to (2.2), by (2.1).

$$\mathcal{L}_0^2(I) = \{y \in \mathcal{L}^2(I) : \text{ there exist two integers } s, k \in I \text{ with } s \leq k \text{ such that } y(t) = 0 \text{ for } t \leq s \text{ and } t \geq k+1\}$$

and

$$\tilde{H} = \{(\tilde{y}, \tilde{g}) \in L^2(I) \times L^2(I) : \text{ there exists } y \in \tilde{y} \text{ such that } \mathcal{L}y(t) = g(t),\ t \in I\}.$$

$$H_{00} = \{(\tilde{y}, \tilde{g}) \in \tilde{H} : \text{ there exists } y \in \tilde{y} \text{ such that } y \in \mathcal{L}_0^2(I) \text{ and } \mathcal{L}y(t) = g(t),\ t \in I\}.$$

Then $\tilde{H}$ and H_{00} are both linear subspaces in $L^2(I) \times L^2(I)$. $\tilde{H}$ and H_{00} are called maximal and preminimal subspaces corresponding to $\mathcal{L}$, the natural difference map generated by (2.2), respectively. $H_0 = \overline{H_{00}}$ is the minimal subspace corresponding to $\mathcal{L}$ in $L^2(I) \times L^2(I)$.

It follows that $H_{00} \subset \tilde{H} \subset H_{00}^*$ and consequently H_{00} is a Hermitian subspace in $L^2(I) \times L^2(I)$. It has been proved in [24] that the adjoint of the preminimal subspace is the maximal subspace and thus $H_{00}^* = H_0^* = \tilde{H}$.

In order to discuss the self-adjoint subspace extension for H_0, the definiteness condition must hold.

Thus, if $z \in \mathbb{C}$, Ren and Y. Shi [24] have shown that the dimensions of the defect spaces of H_0 and H_{00} are equal to the number of linearly independent square summable solutions of (2.2). Equal deficiency indices can only be obtained when the third order difference map $\ell y(t)$ is defined on $\ell^2(\mathbb{Z})$. Otherwise for $def\tilde{H}_0 = (1,2)$ or $(2,1)$ at limit point, $\tilde{H}_0$ has no self-adjoint subspace extension by von Neumann theorem.

In order to determine the number of solutions of (2.2) that are square summable, we use asymptotic summation and that requires the difference map corresponding to (2.2) to be converted into its first order system. We convert the third order difference equation

$$\ell y(t) = y(t+3) - \alpha_2 y(t+2) - \alpha_1 y(t+1) - \alpha_0 y(t) = z y(t) \tag{3.2}$$

to it's first order form. Because of spectral analysis, we solve the equation $\ell y(t) = zy(t)$, where z is spectral parameter.

Define vector valued function $Y(t) = [y(t), y(t+1), y(t+2)]^{tr}$.

Then from (2.2) and (3.2) we obtain $Y(t+1, z) = S(t, z)Y(t)$ where

$$S(t,z) = \begin{bmatrix} p_1 & q_{12} & q_{13} \\ \overline{q_{12}} & p_2 & q_{23} \\ -[\alpha_2 \overline{q_{12}} + \alpha_1 p_1 + (\alpha_0 - z)] & -[\alpha_2 p_2 + \alpha_1 q_{12}] & -[\alpha_2 q_{23} + \alpha_1 q_{13}] \end{bmatrix}, \tag{3.3}$$

The analysis of the first order system (3.3) is done using the asymptotic summation. Asymptotic summation is the discrete version of the asymptotic integration. It is used to analyse and obtain the form of solutions of higher order difference equations. The whole method is based on the discretized version of the Levinson's theorem which is now referred to as the Levinson-Benzaid-Lutz form (LBL) [8]. In its simplified version, it says that the solution of the form

$$Y(t+1) = C(t)Y(t) \tag{3.4}$$

if converted into the form

$$Y(t+1) = [\Lambda(t) + R(t)]Y(t) \tag{3.5}$$

looks like the solutions of the diagonal system

$$Y(t+1) = \Lambda(t)Y(t) \tag{3.6}$$

if the diagonal elements $\Lambda(t)$ satisfy the dichotomy conditions and $R(t)$ is absolutely summable (small). Here $\Lambda(t) = \text{diag}(\lambda_k(t))$ $k = 1, 2, 3$ and $R(t)$ is 3×3 matrix with $R_{kk}(t) = 0$. $Y(t+1)$ and $Y(t)$ are $3-$dimensional vectors. It is the process of converting (3.4) into (3.5) and applying LBL theorem to obtain the closed form of solutions of (3.4) that asymptotically behaves like the solutions of (3.6) that is known as asymptotic summation. In the case of spectral theory, a spectral parameter is attached to the difference equation in order to locate the spectrum of the operator. The discretized version of Levinsons Theorem is stated below as

Theorem 3.2. [8, Lemma 2.1] *Let* $\Lambda(t,z) = diag\{\lambda_1(t,z), \ldots, \lambda_{2n}(t,z)\}$ *for* $t \geq t_0$. *Assume*

(i) $\lambda_i(t,z) \neq 0$ *for all* $1 \leq i \leq 2n$ *and* $t \geq t_0$

(ii) $R(t,z)$ *is a* $2n \times 2n$ *matrix defined for all* $t \geq t_0$, *satisfying*

$$\sum_{k=0}^{\infty} \left| \frac{R(t,z)}{\lambda_i(t,z)} \right| \|R(t,z)\| < \infty, \text{for all } i = 1, 2, \ldots, 2n$$

(iii) $\Lambda(t,z)$ *satisfies the following uniform dichotomy condition. For any pair of indices* i *and* j, *such that* $i \neq j$, *assume there exists* δ *with* $0 < \delta < 1$ *such that* $|\lambda_i(t)| \geq \delta$ *or* $|\lambda_j(t)| \leq 1$ *for a large* t. *Then either*

$$\left| \frac{\lambda_i(t)}{\lambda_j(t)} \right| \geq 1 \quad or \quad \left| \frac{\lambda_j(t)}{\lambda_i(t)} \right| \leq 1 \text{ for a large } t.$$

Then the linear system $Y(t+1,z) = [\Lambda(t,z) + R(t,z)]Y(t,z)$ *has a fundamental matrix satisfying, as* $t \to \infty$

$$Y(t,z) = [I + o(1)] \prod_{l=t_0}^{t-1} \Lambda(l,z).$$

In the sequel, we have analyzed the spectral theory of the subspace H, $\sigma(H)$, and its multiplicity. For this, we will determine the absolutely continuous spectrum via the M-function. This in turn requires the asymptotics of the solutions of (2.2). Since it has been shown in [22] that the absolutely continuous spectrum

and its spectral multiplicity are independent of the boundary conditions and the left regular endpoints, we will pay little attention to the left regular endpoints. To obtain the asymptotics of the eigenfunctions of (2.2), rewrite (3.2) in the propagator form, where $S(t,z)$ is as defined in (3.3).

For the application of asymptotic summation, one needs the eigenvaalues of $S(t,z)$. This requires the characteristic polynomial of $S(t,z)$ which is obtained through the expansion of $det(S(t,z) - I_3.\lambda) = \mathcal{P}(t,\lambda,z)$ where I_3 is an identity matrix.

The characteristic polynomial is given by

$$\begin{aligned}\mathcal{P}(t,\lambda,z) = &-\lambda^3 + (p_1 + p_2 - \alpha_1 q_{13} - \alpha_2 q_{23})\lambda^2 \\ &+ [\alpha_1(p_1 q_{13} - q_{12}q_{23}) + \alpha_2(p_1 q_{23} - q_{13}\overline{q_{12}}) + |q_{12}|^2 - p_1 p_2 - (\alpha_0 - z)q_{13}]\lambda \\ &+ (\alpha_0 - z)[p_2 q_{13} - q_{12}q_{23}].\end{aligned} \tag{3.7}$$

The discriminant $D(z)$ of $\mathcal{P}(t,\lambda,z)$, viewed as a function of λ, is itself a polynomial in z. The zeros of $D(z)$ correspond to spectral values where $\lambda = \lambda(z)$ possesses multiple roots. Considering both the resultant polynomial $\mathcal{P}(t,\lambda,z)$ and its discriminant $D(z)$, it follows that there exist only finitely many spectral values z for which $\mathcal{P}(t,\lambda,z)$ has repeated roots. Let $\omega_1 < \omega_2 < \ldots < \omega_k$ represent all real spectral values z leading to such multiplicities. Following [2], the analysis is confined to small complex neighborhoods of points $z_0 \in (\omega_i, \omega_{i+1})$ for $i = 1, \ldots, k$, where $\omega_0 = -\infty$ and $\omega_{k+1} = \infty$. For a fixed $z \in (\omega_i, \omega_{i+1})$, choose $\epsilon > 0$ and a sufficiently large a so that $\mathcal{P}(t,\lambda,z)$ has no multiple roots for any

$$z \in \mathcal{K} = \mathcal{K}_\epsilon(z_0) = \{\, z : |z - z_0| < \epsilon,\ \mathrm{Im}\, z > 0 \,\}.$$

In this setting, the roots of $\mathcal{P}(t,\lambda,z)$ depend analytically on the matrix elements. Therefore, by suitably choosing $\epsilon > 0$, one can find $t_0 > 0$ such that equation (3.7) is a polynomial in λ with three distinct roots $\lambda_i(t,z)$, $i = 1,2,3$, whenever $t \geq t_0$ and ϵ is sufficiently small.

Since we are interested in the limiting behaviours of the roots and the solutions, the zeros of $\mathcal{P}(t,\lambda,z)$ in (3.7) have similar asymptotics as $t \to \infty$ as the zeros of $\tilde{\mathcal{P}}(t,\lambda,z)$ obtained when we take $\lim_{t\to\infty} \alpha_1(t) = \lim_{t\to\infty} \alpha_2(t) = 0$, (3.7) with

$$\tilde{\mathcal{P}}(t,\lambda,z) = -\lambda^3 + (p_1+p_2)\lambda^2 + [|q_{12}|^2 - p_1 p_2 - (\alpha_0 - z)q_{13}]\lambda + (\alpha_0 - z)[p_2 q_{13} - q_{12}q_{23}]. \tag{3.8}$$

Lemma 3.3. *The zeros of $\mathcal{P}(t,\lambda,z)$ have similar asymptotics as zeros of $\tilde{\mathcal{P}}(t,\lambda,z)$ as $t \to \infty$.*

Proof. Let $\lambda_k(t,z)$ and $\tilde{\lambda_k}(t,z)$ be a zero of $\mathcal{P}(t,\lambda,z)$ and $\tilde{\mathcal{P}}(t,\lambda,z)$ respectively. Suppose that

$$\mathcal{P}(t,\lambda,z) \approx \tilde{\mathcal{P}}(t,\lambda,z) + o(\eta),$$

where $\eta = \eta(t,\lambda,z)$ with $\lim_{t\to\infty} \eta(t,\lambda,z) = 0$, then the map

$$T(\lambda_k(t,z)) = \lambda_k(t,z) - \frac{\tilde{\mathcal{P}}(t,\lambda,z)}{\mathcal{P}'(\lambda_k(t,z))} \tag{3.9}$$

provided that $\mathcal{P}'(\lambda_k(t,z)) \neq 0$ is a well defined map. Consider a small neighborhood I around $\lambda_k(t,z)$. Since $\mathcal{P}(t,\lambda,z)$ is smooth and $\mathcal{P}'(\lambda_k(t,z)) \neq 0$, for sufficiently small $\delta > 0$, $\mathcal{P}'(\lambda_k(t,z))$ remains bounded on the interval $I = [\lambda_k(t,z) - \delta, \lambda_k(t,z) + \delta]$. Moreover, because $\lim_{t\to\infty} \eta(t,\lambda,z) = 0$, $\tilde{\mathcal{P}}(t,\lambda,z)$ is close to $\mathcal{P}(t,\lambda,z)$, and hence the change induced by $T(\lambda_k(t,z))$ is small. From (3.9), we obtain

$$T'(\lambda_k(t,z)) = 1 - \frac{\tilde{\mathcal{P}}'(t,\lambda,z)}{\mathcal{P}'(\lambda_k(t,z))}.$$

At limit point, $T'(\lambda_k(t,z)) \to 0$ as $t \to \infty$. It follows that $\tilde{\mathcal{P}}(t,\lambda,z)$ is near $\mathcal{P}(t,\lambda,z)$ and there exists a constant $M_0 < 1$ such that for all $\lambda_1, \lambda_2 \in I$, $|T(\lambda_1) - T(\lambda_2)| \leq M_0|\lambda_1 - \lambda_2|$ and so the map T is a contraction from I onto I. By the Banach fixed point theorem [30, Theorem 5.12.4], there exists a unique fixed point $\lambda_\epsilon \in I$ such that $T(\lambda_\epsilon) = \lambda_\epsilon$, implies that $\tilde{\mathcal{P}}(\lambda_\epsilon) = 0$. Morover, if $\epsilon \to 0$, then the map T converges uniformly to $T_0(\lambda_k) = \lambda_k - \frac{\tilde{\mathcal{P}}(\lambda_\epsilon)}{\mathcal{P}'(\lambda_\epsilon)}$ whose unique fixed point is λ_ϵ. Therefore, it follows that $\lim_{t\to\infty} \lambda_k(t,z) = \tilde{\lambda_k}(t,z)$. □

The deficiency index $defT$ is the pair of numbers $defT = (\dim \mathcal{N}_{T^*-i}, \dim \mathcal{N}_{T^*+i})$, where $\mathcal{N}$= null space. T denotes the minimal operator while T^* denotes the maximal operator. If the numbers agree, then T has self-adjoint extensions denoted by H. The spectrum $\sigma(H)$ of H and its components is the main object of this study. In what follows, we obtain the deficiency indices for T under the deacy and growth conditions.

The deficiency index $def\, T$ is defined as the pair of numbers

$$def\, T = (\dim \mathcal{N}_{T^*-i}, \dim \mathcal{N}_{T^*+i}),$$

where $\mathcal{N}$ denotes the null space. Here, T represents the minimal operator, and T^* represents the maximal operator. When these two numbers are equal, T admits self-adjoint extensions, which are denoted by H. The primary focus of this study is the spectrum $\sigma(H)$ of H and its components. In the following sections, we determine the deficiency indices of T under the decay and growth conditions.

3.1. $|p_2(t)| \nearrow \infty$ as $t \to \infty$.

Assumption 3.4. Assume the following growth conditions

$$\begin{aligned} &|p_2(t)| \nearrow \infty \text{ as } t \to \infty, \quad |p_1(t)| < 1, \quad |q_{12}||q_{23}||p_1^{-1}|, |x| \approx \mathrm{o}(|\alpha_0||q_{13}|), \\ &\text{where } x = \mathrm{O}(p_2^{-1}) \quad \forall t \in \mathbb{N} \end{aligned} \tag{3.10}$$

and the following decay conditions

$$\begin{aligned} &\Delta f, \quad \frac{\Delta f}{p_2}, \quad f(\Delta f) \in \ell^2 \\ &(\Delta f)^2, \quad \Delta^2 f, \quad f\Delta^2 f, \quad (\frac{\Delta f}{f})^2 \in \ell^1. \end{aligned} \tag{3.11}$$

Suppose that $|p_2| \nearrow \infty$ as $t \to \infty$, then, from (3.8) we obtain the eigenvalues of $S(t,z)$ as

$$\begin{aligned}
\lambda_1 &\approx p_1 + p_2 + \mathrm{O}(p_2^{-1}) \\
\lambda_2 &\approx -\frac{p_1 p_2}{p_1+p_2} + \frac{\alpha_0 p_2 q_{13}}{p_1+p_2} + \cdots \approx \alpha_0 q_{13} - p_1 + \mathrm{O}(p_2^{-1}) \\
\lambda_3 &\approx \frac{-\alpha_0 p_2 q_{13} + q_{12} q_{23} \alpha_0}{p_1 p_2} + \mathrm{O}(p_2^{-1}) \approx -\frac{\alpha_0 q_{13}}{p_1} + \frac{\alpha_0 q_{12} q_{23} p_1^{-1}}{p_2} + \ldots .
\end{aligned} \tag{3.12}$$

Once the eigenvalues of $S(t,z)$ have been obtained, the corresponding eigenvectors can be determined. Let λ be an eigenvalue of $S(t,z)$, and suppose the corresponding eigenvector is $v = [v_1, v_2, v_3]^{tr}$, such that $(S(t,z) - \lambda I)v = 0$. For the eigenvalue λ, the corresponding $\lambda-$eigenvector $v(\lambda)$ is given by

$$\begin{aligned}
v(\lambda) = [\frac{1}{(p_1-\lambda)}\Bigg(\frac{-q_{13}[|q_{12}|^2 - (p_1-\lambda)(p_2-\lambda)] + q_{12}[q_{13}\overline{q_{12}} - q_{23}(p_1-\lambda)]}{|q_{12}|^2 - (p_1-\lambda)(p_2-\lambda)}\Bigg), \\
- \left(\frac{q_{13}\overline{q_{12}} - q_{23}(p_1-\lambda)}{|q_{12}|^2 - (p_1-\lambda)(p_2-\lambda)}\right), 1]^{tr}
\end{aligned} \tag{3.13}$$

Let $D := |q_{12}|^2 - (p_1-\lambda)(p_2-\lambda), \quad N := q_{13}\overline{q_{12}} - q_{23}(p_1-\lambda)$. Then (3.13) becomes

$$v(\lambda) = \left[\frac{(q_{12}N - q_{13}D)}{(p_1-\lambda)D}, -\frac{N}{D}, 1\right]^{tr}. \tag{3.14}$$

The vectors have to be normalized because p_2 is unbounded. The normalized $\lambda-$eigenvector is given by

$$\hat{v}(\lambda) = \frac{1}{\|v(\lambda)\|}\left[\frac{(q_{12}N - q_{13}D)}{(p_1-\lambda)D}, -\frac{N}{D}, 1\right]^{tr}.$$

where

$$\|v(\lambda)\|^2 = \left|\frac{(q_{12}N - q_{13}D)}{(p_1-\lambda)D}\right|^2 + \left|\frac{N}{D}\right|^2 + 1 = \frac{1}{|D|^2}\left(\frac{|(q_{12}N - q_{13}D)|^2}{|p_1-\lambda|^2} + |N|^2 + |D|^2\right).$$

With an expansion to the order p_2^{-1} since $\lambda_1 \approx p_1 + p_2 + \mathrm{O}(p_1) + \mathrm{O}(p_2^{-1})$, we get

$$\hat{v}(\lambda_1) \approx |p_2 D|\Bigg(|(q_{12}N - q_{13}D)|^2 + |p_2|^2(|N|^2 + |D|^2)\Bigg)^{-1/2}\left[\frac{(q_{13}D - q_{12}N)}{p_2 D}, -\frac{N}{D}, 1\right]^{tr}.$$

Since $\lambda_2(t) \approx -p_1 + \alpha_0 q_{13} + \mathrm{O}(p_2^{-1})$ we obtain the normaloized $\hat{v}(\lambda_2)$ as

$$\begin{aligned}
\hat{v}(\lambda_2) \approx |D(2p_1 - \alpha_0 q_{13})|\Bigg(|(q_{12}N - q_{13}D)|^2 + |2p_1 - \alpha_0 q_{13}|^2(|N|^2 + |D|^2)\Bigg)^{-1/2} \\
\cdot\left[\frac{(q_{12}N - q_{13}D)}{(2p_1 - \alpha_0 q_{13})D}, -\frac{N}{D}, 1\right]^{tr}.
\end{aligned}$$

For $\lambda_3(t) \approx \frac{\alpha_0 q_{13}}{p_1} + \frac{q_{12}q_{23}\alpha_0}{p_1 p_2} + \mathrm{O}(p_2^{-1})$ the normalized eigenvector $\hat{v}(\lambda_2)$ is given by

$$\hat{v}(\lambda_3) \approx \frac{|D||p_1 - (\frac{\alpha_0 q_{13}}{p_1} + \frac{q_{12}q_{23}\alpha_0}{p_1 p_2})|}{\left(|q_{12}N - q_{13}D|^2 + \big(|N|^2 + |D|^2\big)[|p_1 - (\frac{\alpha_0 q_{13}}{p_1} + \frac{q_{12}q_{23}\alpha_0}{p_1 p_2})|^2]\right)^{1/2}}$$
$$\cdot\left[\frac{(q_{12}N - q_{13}D)}{(p_1 - (\frac{\alpha_0 q_{13}}{p_1} + \frac{q_{12}q_{23}\alpha_0}{p_1 p_2}))D}, -\frac{N}{D}, 1\right]^{tr}.$$

Then the matrix

$$\tilde{T}(t,z) = [\hat{v}(\lambda_1)\ \hat{v}(\lambda_2)\ \hat{v}(\lambda_3)]$$

is a diagonalizing matrix for $S(t,z)$. The matrix $\tilde{D}(t,z) = \tilde{T}^{-1}(t,z)S(t,z)\tilde{T}(t,z) = diag(\lambda_i(t,z))$ is obtained. Now one can transform

$$Y(t+1,z) = S(t,z)Y(t) \tag{3.15}$$

by

$$Y(t,z) = \tilde{T}(t,z)\hat{v}(t,z). \tag{3.16}$$

This transformation leads to

$$\hat{v}(t+1,z) = \tilde{T}^{-1}(t,z)S(t,z)\tilde{T}(t,z)v(t,z) = (\Lambda(t,z) + R(t,z))v(t,z). \tag{3.17}$$

Here $\Lambda(t,z) = \tilde{T}^{-1}(t,z)S(t,z)\tilde{T}(t,z) = \mathrm{diag}(\lambda_i(t,z))$, $i = 1,2,3$ is the diagonal matrix formed with the eigenvalues of $S(t,z)$ and $R(t,z) = -\tilde{T}^{-1}(t+1,z)\Delta\tilde{T}(t,z)\Lambda(t,z)$ consists of ℓ^1 and ℓ^2 terms. Apart from trivial cases, (3.17) is not yet in LBL form. Thus this sytem has to be diagonalized again for the second time. This is only possible when the the growth and decay conditions in (3.10) and (3.11) are satisfied. By using (3.17), we obtain the second diagonalizaation. To obtain the second diagonalization, a matrix of the form $[I + B(t,z)]$ with

$$B_{kk}(t,z) = 0, \quad B_{kj}(t,z) = (\lambda_j - \lambda_k)^{-1}R_{kj}, \quad k \neq j, k, j = 1,2,3, t \geq t_{t_0}$$

is required. This results into the situation where the correction terms are added into the diagonal. The correction terms is given by

$$(\tilde{\Lambda})_{kk} = \mathrm{diag}((RB)_{kk}).$$

The remainder terms are approximated by $\mathrm{O}(p^{-1})$, this yields the solutions of (2.2) in LBL form given by

$$Y(t,z) = T(t,z)[I + B(t,z)][E(t,z) + \mathrm{o}(1)]\prod_{t_0}^{t-1}\Lambda(t,z). \tag{3.18}$$

It should be noted that, the second diagonalization only gives summable terms. The eigenfunctions are given by

$$\begin{aligned} y_k(t,z) &= T(t,z)[I + B(t,z)][e_k(t,z) + r_{kk}(t,z)]\prod_{s=t_0}^{t-1}\lambda_k(s,z) \\ &= [v_k(t,z) + r_{kk}(t,z)]\prod_{s=t_0}^{t-1}\lambda_k(s,z), \end{aligned} \tag{3.19}$$

where λ_k, $k = 1, 2, 3$ is the kth eigenvalue of $S(t, z)$ and v_K is the normalized kth eigenvector, $r_{kk}(t, z) \in \ell^1(\mathbb{Z})$. $r_{kk}(t, z)$ tends to zero as $t \to \infty$.

The analysis of the deficiency indices is based on the values (3.12) of (3.8). Assume that the growth conditions (3.10) and the decay conditions (3.11) in (3.4) hold, then, in the sequel, we obtain results on the deficiency indices. The self-adjoint extension and its spectral analysis of the third order difference map (2.2) cannot be discussed on $\ell^2(\mathbb{N})$. For this, we define the operator on $\ell^2(\mathbb{Z})$ but apply decomposition theorem techniques of two point interface discussed by Behncke and Hinton [4] which has been applied in a number of publications see [18]. The decomposition theorem is also useful in derivation of the spectral results and the corresponding self-adjoint operator and (or) subspace. Define the third order difference map (2.2) on $\ell^2(\mathbb{Z})$ and we write $\mathbb{Z} = I_1 \cup I_2$ where I_1 is the integral domain within $[0, \infty)$ and I_2 is the integral domain within $(-\infty, 0]$. We may assume as described in section 3, that $\tilde{H}_0$ is the minimal subspace generated by (2.2) on $\ell^2(\mathbb{Z})$. First, using the techniques of Behncke and Hinton, we can write

$$\ell^2(\mathbb{Z}) = \ell^2(I_1) \oplus \ell^2(I_2). \tag{3.20}$$

On the other hand, assume that (2.2) defined on $\ell^2(I_1)$ and $\ell^2(I_2)$ generates $\tilde{H}_{0_1}$ and $\tilde{H}_{0_2}$ as the respective minimal subspaces. It follows from (3.20) and subspace theory that

$$\tilde{H}_0 = \tilde{H}_{0_1} \oplus \tilde{H}_{0_2} \tag{3.21}$$

and hence $det\tilde{H}_0 = def\,\tilde{H}_{0_1} + def\,\tilde{H}_{0_2}$. One therefore obtains the following results

Theorem 3.5. *Suppose that all the growth and decay conditions in (3.10) and (3.11) are satisfied and the third order difference map $\ell y(t)$ in (2.2) is defined on $\ell^2(\mathbb{Z})$. Moreover, if the assumptions in Theorem 2.4 are satisfied, then*

- **(i):** *Suppose that $|\alpha_0||q_{ij}||p_1|^{-1} < 1$, $i = 1, 2$, $j = 2, 3$, then* def $H_o = (2, 2)$, *H_0 has self-adjoint subspace extension with discrete spectrum at most.*
- **(ii):** *Suppose that $|\alpha_0||q_{ij}||p_1|^{-1} \approx$ o(1), $i = 1, 2$, $j = 2, 3$ are absolutely summable, then* def $H_o = (2, 2)$, *H_0 has self-adjoint subspace extension with discrete spectrum.*
- **(iii):** *If $|\alpha_0||q_{ij}||p_1|^{-1} > 1$, $i = 1, 2$, $j = 2, 3$ is not absolutely summable, then* def $H_0 = (1, 2)$ *or* $(2, 1)$ *and the subspace H_0 has no self-adjoint subspace extension.*

Proof. **(i):** Assume that the growth conditions (3.10) and the decay conditions (3.11) in (3.4) are satisfied, then, the eigenvalues of (3.8) in their simplest form are obtained in (3.12). The matrix $S(t, z)$ defined in (3.3) is diagonalized twice to obtain a system in the discretized Levinson's theorem LBL 3.2 form given by (3.18). By applying the LBL 3.2, one obtains the solutions of (2.2) as (3.18).

The eigenfunction resulting from λ_k-eigenvalue is given by

$$y_k(t, z) = [e_0 + r_{kk}(t, z)] \prod_{s=t_0}^{t-1} \lambda_k(s, z).$$

We obtain

$$\|y_k\|^2 = \langle y_1, y_1\rangle = \sum_{s=t_0}^{t-1} |\gamma(t)|^2 \prod_{s=t_0}^{t-1} |\lambda_1(s,z)|^2, \tag{3.22}$$

where e_0 is the normalized eigenvectors. The term $|\gamma(t)| = |e_0 + r_{kk}(t,z)|$ is bounded and the asymptotics of (3.22) depends on $\prod_{s=t_0}^{t-1} |\lambda_k(s,z)|^2$. By expanding λ_1, upto $\mathrm{O}(p_2^{-1})$, λ_1 leads to $|\lambda_1| \approx |p_1 + p_2 + \mathrm{O}(p_2^{-1})| \to \infty$ as $t \to \infty$ and this yields $\|y_1\|^2 \approx \infty$. Thus off the real axis, $y_1(t,z)$ is a nonsquare summable eigenfunction. By expanding λ_2 upto $\mathrm{O}(p_2^{-2})$, λ_2 leads to $|\lambda_2| \approx |\alpha_0 q_{13} - p_1 + \mathrm{O}(p_2^{-2})| < \infty$ as $t \to \infty$, which implies that $\|y_2\|^2 < \infty$. Thus off the real axis, $\lambda_2(t,z)$ leads to a square summable eigenfunction $y_2(t,z)$. By expanding λ_3 upto $\mathrm{O}(p_2^{-2})$, λ_3 leads to $|\lambda_3| \approx |\frac{-\alpha_0 q_{13}}{p_1} + \frac{\alpha_0 q_{12} q_{23} p_1^{-1}}{p_2} + \mathrm{O}(p_2^{-2})|$. Here, the summability of $\|y_3\|^2$ depends on the value of $|\alpha_0||q_{ij}||p_1|^{-1}|p_2|$. Suppose that $|\alpha_0||q_{ij}||p_1|^{-1}|p_2| < 1$, $i = 1,2$, $j = 2,3$, then λ_3 yields a square summable eigenfunction. Hence, we have that def $H_0 = (2,2)$. Suppose that $|\alpha_0||q_{ij}||p_1| < 1$, $i = 1,2$, $j = 2,3$, then λ_3 yields a square summable eigenfunction. Hence, we have that def $H_0 = (2,2)$. Imposing boundary conditions at $-\infty$ and ∞, there exist two square matrices M and N of order 1 such that

$$rank(M,N) = I, \qquad M\phi_1^{tr} M^* - N\phi_2^* N^* = 0$$

ϕ_1 and ϕ_2 are invertible matrices of order 1 with square summable solutions satisfying boundary conditions at $-\infty$ and ∞ respectively. The self-adjoint extension subspace of $\tilde{H}_0$ which we denote by $\tilde{H}$ is now described by

$$\tilde{H} = \left\{ (\tilde{y}, \tilde{g}) \in H : M\big(\tilde{y}, y_{11})(-\infty)\big) - N\big(\tilde{y}, y_{21})(\infty)\big) = 0 \right\}$$

with $y_{11} \in \phi_1$ and $y_{21} \in \phi_2$. Equivalently, using boundary conditiona one constructs $\tilde{H}$ as follows. Since the defect indices of H_0 are equal, that is, def $H_0 = (n,n)$, where $1 \le n \le 3$, then H_0 has a self-adjoint subspace extension in $L^2(I) \times L^2(I)$ and this we will denote by H. Just like in [29], for the fundamental matrix of square summable solutions Φ_{t_0} of (2.2), there exist matrices $\alpha_1, \alpha_2 \in \mathbb{C}^3$ such that

$$\mathrm{rank}(\alpha_1, \alpha_2) = 3, \quad \alpha_1 \Phi_{t_0}^{tr} \alpha_1^* = \alpha_2 \Phi_{\infty}^{tr} \alpha_2^*.$$

Where $\Phi_{t_0} = ([y_1, y_2, y_3])\,(t_0 - 1)_{3\times 3}$ and Φ_∞ defined in a similar way. y_2 and y_3 are square summable solutions of (2.2). Precisely, H is defined by

$$H = \left\{ (\tilde{y}, \tilde{g}) \in \tilde{H} : \alpha_1 \begin{pmatrix} (\tilde{y}, y_1)(t_0 - 1) \\ (\tilde{y}, y_2)(t_0 - 1) \\ (\tilde{y}, y_3)(t_0 - 1) \end{pmatrix} + \alpha_2 \begin{pmatrix} (\tilde{y}, y_1)(\infty) \\ (\tilde{y}, y_2)(\infty) \\ (\tilde{y}, y_3)(\infty) \end{pmatrix} \right\}.$$

For more details see [29, Theorem 5.9] and Section 5 of the same reference in general. It therefore, follows that the spectrum of H is purely discrete since all the solutions which are square summable are uniformly square

summable. and H_0 has self-adjoint subspace extension with discrete spectrum at most.

(ii): Assume the growth and decay hypotheses (3.10)–(3.11) and that the assumptions of Theorem 2.4 hold. Suppose further that

$$\left|\alpha_0(t)q_{ij}(t)p_1(t)^{-1}\right| = o(1) \quad \text{as } t \to \infty,$$

and that the sequences

$$a(t) := \left|\alpha_0(t)q_{ij}(t)p_1(t)^{-1}\right|, \qquad i = 1, 2,\ j = 2, 3,$$

are absolutely summable, that is, $\sum_{t=t_0}^{\infty} a(t) < \infty$. Under these hypotheses, the diagonalization and discrete Levinson–Benzaid–Lutz (LBL) theorem [8] applied to (2.2) yield, for large t, a third eigenvalue of the form

$$\lambda_3(t) = -\frac{\alpha_0(t)q_{13}(t)}{p_1(t)} + \frac{\alpha_0(t)q_{12}(t)q_{23}(t)}{p_1(t)p_2(t)} + O\big(p_2(t)^{-2}\big).$$

Let $\mu_3(t) := -\alpha_0(t)q_{13}(t)/p_1(t)$ and write $\lambda_3(t) = \mu_3(t)\big(1 + \varepsilon_3(t)\big)$, where $\varepsilon_3(t)$ collects higher–order corrections. Since $a(t) = o(1)$ and is absolutely summable, $\varepsilon_3(t)$ is uniformly small for large t and $\sum_{t=t_0}^{\infty} |\varepsilon_3(t)| < \infty$. The LBL theorem then provides a corresponding solution of the form

$$y_3(t) = \gamma_3(t) \prod_{s=t_0}^{t-1} \lambda_3(s), \qquad \gamma_3(t) = e_0 + r_{33}(t),$$

where $\gamma_3(t)$ is bounded and $r_{33}(t) \to 0$ as $t \to \infty$. Consider the product

$$P(t) := \prod_{s=t_0}^{t-1} |\lambda_3(s)| = \prod_{s=t_0}^{t-1} |\mu_3(s)|\,(1 + o_s),$$

where $\sum_s |o_s| < \infty$. Taking logarithms gives

$$\log P(t) = \sum_{s=t_0}^{t-1} \log |\mu_3(s)| + \sum_{s=t_0}^{t-1} \log(1 + o_s).$$

The second sum converges since $\sum_s |o_s| < \infty$. For the first term, note that $|\mu_3(s)| = |\alpha_0(s)q_{13}(s)p_1(s)^{-1}| < 1$ and is absolutely summable for large s. Writing $|\mu_3(s)| = 1 - \eta_s$ with $\eta_s \in (0, 1)$ and $\sum_s \eta_s < \infty$, the inequality $\log(1 - \eta) \le -\eta$ yields

$$\sum_{s=t_0}^{t-1} \log |\mu_3(s)| \le -\sum_{s=t_0}^{t-1} \eta_s,$$

so that the product $P(t)$ admits an exponential bound. Hence, there exist constants $C > 0$ and $\rho \in (0, 1)$ such that for all large t,

$$P(t) \le C\rho^t.$$

Since $\gamma_3(t)$ is bounded, we obtain

$$|y_3(t)| \le C'\rho^t, \qquad t \ge t_0,$$

for some $C' > 0$. Therefore, $\sum_{t=t_0}^{\infty} |y_3(t)|^2 < \infty$, and y_3 is a square–summable eigenfunction. From part (i), the second root $\lambda_2(t)$ also produces a square–summable solution y_2, while $\lambda_1(t)$ yields a non–ℓ^2 solution. Hence, for nonreal z, there exist exactly two linearly independent ℓ^2–solutions y_2 and y_3, so that

$$\operatorname{def} H_0 = (2, 2).$$

Since the deficiency indices are equal, von Neumann's extension theory guarantees the existence of self–adjoint subspace extensions of H_0. Because the ℓ^2–solutions y_2 and y_3 decay uniformly (indeed, exponentially), the resolvent of any such self–adjoint extension is compact. Consequently, the spectrum of every self–adjoint subspace extension of H_0 is purely discrete, consisting of eigenvalues of finite multiplicity with no finite accumulation point. This completes the proof of part (ii).

(iii): Suppose that $|\alpha_0||q_{ij}||p_1| > 1$, $i = 1, 2$, $j = 2, 3$, then λ_3 yields a non square summable eigenfunction in either upper or lower half-plane. Hence, we have that either $\operatorname{def} H_0 = (1, 2)$ or $\operatorname{def} H_0 = (2, 1)$ and the subspace H_0 has no self-adjoint subspace extension.

□

Remark 3.6. Theorem 3.5 provides a complete characterization of the deficiency indices of the minimal subspace (or operator) H_0 associated with the third–order difference expression $\ell y(t)$ under the given growth and decay conditions. In particular, the result establishes the precise relationship between the asymptotic behavior of the coefficient ratios $|\alpha_0 q_{ij} p_1^{-1}|$ and the self–adjointness of H_0. When these ratios are sufficiently small or absolutely summable, the operator is in the *limit–circle* case at both endpoints, yielding equal deficiency indices $(2, 2)$ and guaranteeing the existence of self–adjoint subspace extensions with discrete spectrum. In contrast, when the ratios exceed unity or fail to be summable, the deficiency indices become unequal, placing H_0 in the *limit-point* case and hence leading to non existence of self-adjoint subspace extension.

Example 3.7. Consider a third order difference map $\ell y(t)$ in (2.2) with coefficients defined by

$$p_1(t) = t\, e^{-t}, \qquad p_2(t) = t^{-1} e^{-\mu t}, \qquad p_3(t) = (1 + t^{-1}) e^{\mu t},$$

and

$$q_{12}(t) = t^{-2} e^{-\mu t}, \qquad q_{13}(t) = t^{-3} e^{-2\mu t}, \qquad q_{23}(t) = t^{-1} e^{-\mu t},$$

where $\mu > 0$ is a real constant. Substituting these into (2.2), we obtain

$$\begin{aligned} \ell y(t) = y(t+3) &+ \left[(1 + t^{-1}) e^{\mu t} - t^{-1} e^{-\mu t}\right] y(t+2) \\ &+ \left[t^{-3} e^{-2\mu t} + t^{-1} e^{-\mu t}\right] y(t+1) + t^{-3} e^{-2\mu t} y(t). \end{aligned} \tag{3.23}$$

One easily verifies that the conditions of Theorem 2.4 are satisfied, so (3.23) defines a third order symmetric map on $\ell^2(\mathbb{Z})$. Solving $\ell y(t) = z y(t)$ leads to a characteristic equation of the form (3.7), with coefficients

$$\alpha_0(t) = t^{-3} e^{-2\mu t}, \qquad \alpha_1(t) = t^{-3} e^{-2\mu t} + t^{-1} e^{-\mu t}, \qquad \alpha_2(t) = (1 + t^{-1}) e^{\mu t} - t^{-1} e^{-\mu t}.$$

By taking the limit $t \to \infty$, we have

$$\lim_{t\to\infty} \alpha_1(t) = 0, \qquad \lim_{t\to\infty} \alpha_2(t) = e^{\mu t} \to \infty,$$

showing that the dominant term corresponds to a growing eigenmode, while the other two modes will be decaying or bounded, depending on the smallness of the coefficients in the asymptotic sense. The associated characteristic equation (3.8) is therefore

$$\begin{aligned}\mathcal{P}(t,\lambda,z) = &-\lambda^3 + \left(te^{-t} + t^{-1}e^{-\mu t}\right)\lambda^2 \\ &+ \left[t^{-3}e^{-2\mu t} - (t^{-1}e^{-\mu t})(te^{-t}) - (t^{-3}e^{-2\mu t} - z)te^{-t}\right]\lambda \\ &+ (t^{-3}e^{-2\mu t} - z)t^{-1}e^{-\mu t}.\end{aligned} \tag{3.24}$$

From these expressions, we obtain

$$|\alpha_0(t)| = t^{-3}e^{-2\mu t}, \quad |q_{13}(t)| = t^{-3}e^{-2\mu t}, \quad |p_1(t)| = te^{-t}.$$

Hence

$$|\alpha_0(t)||q_{13}(t)||p_1^{-1}(t)| = t^{-7}e^{-(3\mu-1)t}.$$

Since $\mu > 0$, it follows that $|\alpha_0(t)||q_{13}(t)||p_1^{-1}(t)| = o(1)$ as $t \to \infty$, and the sequence is absolutely summable. Therefore, the hypotheses of Theorem 3.5(ii) are satisfied. Applying the discrete Levinson–Benzaid–Lutz theorem [8], the fundamental system of solutions takes the form

$$y_k(t) = (e_k + r_k(t)) \prod_{s=t_0}^{t-1} \lambda_k(s), \qquad k = 1, 2, 3,$$

where $r_k(t) \to 0$ as $t \to \infty$. The first root $\lambda_1(t)$ grows exponentially and hence yields a non-square-summable solution. The second and third roots, $\lambda_2(t)$ and $\lambda_3(t)$, satisfy $|\lambda_k(t)| \le Ca(t)$ for some absolutely summable sequence $a(t)$; therefore, both corresponding solutions $y_2(t)$ and $y_3(t)$ are square-summable. It follows that exactly two linearly independent ℓ^2–solutions exist for nonreal z, implying

$$\operatorname{def} H_0 = (2, 2).$$

Since the deficiency indices are equal, H_0 admits a family of self–adjoint subspace extensions. Because the ℓ^2–solutions y_2 and y_3 decay exponentially, the resolvent of each self–adjoint extension is compact. Hence, the spectrum of any such extension is purely discrete, consisting only of eigenvalues of finite multiplicity that can accumulate only at infinity.

Theorem 3.8. *Suppose that all the decay and growth conditions in Theorem 3.5 are satisfied and ℓy is defined on $\ell^2(\mathbb{Z})$. Then the def $\tilde{H}_0 = (3, 3)$ and the self-adjoint subspace of $\tilde{H}_0$ exists with discrete spectrum. Moreover if (2.2) is absolutely summable, then def $\tilde{H}_0 = (4, 4)$ with discrete spectrum. If (2.2) is not absolutely summable, then def $\tilde{H}_0 = (3, 3)$ and $\tilde{H}_0$, the self-adjoint subspace of $\tilde{H}_0$, has absolutely spectrum which is $((\overline{q_{12}^{-1}}, \infty))$ of spectral multiplicity 1.*

Proof. Let $\tilde{H}_0$ be the maximal subspace that is generated by the third order difference equation (map) (2.2) on $\ell^2(\mathbb{Z})$ with the deficiency indices given by

$$def\tilde{H}_0 = def\tilde{H}_{0_1} + def\tilde{H}_{0_2}.$$

Suppose that $|\alpha_0|q_{13}.|p_1|.|p_2|^{-2}$ is absolutely summable, then the solutions associated to $\lambda_2(t)$ and $\lambda_3(t)$ are uniformly square summable and hence $def\tilde{H}_{0_1} = (2,2)$ and similarly $def\tilde{H}_{0_2} = (2,2)$ because $|\lambda_3(t)| \approx 1-\epsilon$, $\epsilon > 0$ small, leading too to a square summable solution. Therefore $def\tilde{H}_0 = (4,4)$. Imposing boundary conditions at $-\infty$ and ∞, there exist two square matrices M and N of order 1 such that

$$rank(M,N) = I, \qquad M\phi_1^{tr}M^* - N\phi_2^*N^* = 0$$

ϕ_1 and ϕ_2 are invertible matrices of order 1 with square summable solutions satisfying boundary conditions at $-\infty$ and ∞ respectively. The self-adjoint extension subspace of $\tilde{H}_0$ which we denote by $\tilde{H}$ is now described by

$$\tilde{H} = \left\{(\tilde{y},\tilde{g}) \in H : M\big(\tilde{y}, y_{11})(-\infty)\big) - N\big(\tilde{y}, y_{21})(\infty)\big) = 0\right\}$$

with $y_{11} \in \phi_1$ and $y_{21} \in \phi_2$.

Equivalently, using boundary conditiona one constructs $\tilde{H}$ as follows. Since the defect indices of H_0 are equal, that is, def $H_0 = (n,n)$, where $1 \le n \le 3$, then H_0 has a self-adjoint subspace extension in $L^2(I) \times L^2(I)$ and this we will denote by H. Just like in [29], for the fundamental matrix of square summable solutions Φ_{t_0} of (2.2), there exist matrices $\alpha_1, \alpha_2 \in \mathbb{C}^3$ such that

$$\text{rank}(\alpha_1, \alpha_2) = 3, \quad \alpha_1\Phi_{t_0}^{tr}\alpha_1^* = \alpha_2\Phi_{\infty}^{tr}\alpha_2^*.$$

Where $\Phi_{t_0} = ([y_1, y_2, y_3])\,(t_0 - 1)_{3\times 3}$ and Φ_∞ defined in a similar way. y_1 and y_2 are square summable solutions of (2.2). Precisely, H is defined by

$$H = \left\{(\tilde{y},\tilde{g}) \in \tilde{H} : \alpha_1 \begin{pmatrix} (\tilde{y}, y_1)(t_0-1) \\ (\tilde{y}, y_2)(t_0-1) \\ (\tilde{y}, y_3)(t_0-1) \end{pmatrix} + \alpha_2 \begin{pmatrix} (\tilde{y}, y_1)(\infty) \\ (\tilde{y}, y_2)(\infty) \\ (\tilde{y}, y_3)(\infty) \end{pmatrix} \right\}.$$

For more details see [29, Theorem 5.9] and Section 5 of the same reference in general.

It therefore, follows that the spectrum of H is purely discrete since all the solutions which are square summable are uniformly square summable.

On the other hand if $|\alpha_0||q_{13}||p_1||p_2|^{-2}$ is not absolutely summable, then $def\tilde{H}_{o_2} = (1,2)$ because the solution associated to $\lambda_3(t)$ is not square summable in the upper half plane but summable in the lower half plane. This behavior is reversed when the map is defined on $\ell^2(I_1) \approx \ell^2(\mathbb{Z}_-)$. Thus $def\tilde{H}_{o_1} = (2,1)$ and $def\tilde{H}_o = (3,3)$. However, the solution associated to $\lambda_3(t)$ loses its square summability as $\epsilon \to 0^+$ and therefore contributes to absolutely continuous spectrum. Note that the map is of finite order and thus there can be no accumulation of eigenvalues. The self-adjoint extension subspace can be described as above. In order to locate the absolutely continuous spectrum, we analyse the correction term to the $\lambda_3(t)$ which is given by approximates

$$(\alpha_0 - z)q_{13}p_1.p_2^{-2}$$

Thus off the real axis we write $z = z_0 + i\epsilon$ and therefore we get

$$\frac{\partial \mathcal{P}}{\partial \lambda}(\lambda_3(t)) \approx \alpha_0 - (z_0 + i\epsilon)q_{13}p_1p_2^{-2} \approx \frac{(\alpha_0 - i\epsilon)q_{13}p_1}{p_2^2}.$$

Since $p_2(t)$ is unbounded, it follows that this correction term tends to zero within $(\bar{c}, \infty)$ where

$$\bar{c} = \limsup(\alpha_0(t)q_{13}(t)p_1(t)).$$

The self-adjoint extension subspace $\tilde{H}$ has absolutely continuous spectrum contained in $(\bar{c}, \infty)$ of spectral multiplicity 1. □

Below we present in Example 2.5 a case where the difference map has no subspace self-adjoint extension.

Example 3.9. Consider a third order difference map $\ell y(t)$ (2.2) with $p_1(t) = e^{-t}$, $p_2(t) = t^{-1}e^{\mu t}$,
$p_3(t) = -[te^{(1+\mu)t} + 2]t^{\frac{1}{2}}e^{-\mu t}$, $q_{12}(t) = t^{-1}e^{\mu t}$, $q_{13}(t) = t^{-\frac{1}{2}}$ and $q_{23}(t) = t^{\frac{1}{2}}e^{-\mu t}$. We obtain

$$\ell y(t) = y(t+3)+[(2+te^{-(1+\mu)t})t^{\frac{1}{2}}e^{-\mu t}-t^{-\frac{1}{t}}]y(t+2)+[t^{-\frac{1}{2}}+t^{\frac{1}{2}}e^{-(1+\mu)t}y(t+1)+t^{-\frac{1}{2}}y(t), \tag{3.25}$$

where $\mu > 0$ is a real valued constant. One, can easily show that, the conditions for Theorem 2.4 are satisfied and thus (3.25) generates a third order symmetric map. Solving the equation $\ell y(t) = zy(t)$, leads to the characteristic equation of the form (3.7), where $\alpha_0 = t^{\frac{1}{2}}$, $\alpha_1 = [t^{-\frac{1}{2}} + t^{\frac{1}{2}}e^{-(1+\mu)t}$ and
$\alpha_2 = [(2+te^{-(1+\mu)t})t^{\frac{1}{2}}e^{-\mu t}-t^{-\frac{1}{t}}]$. By considering the limiting case, $\lim_{t\to\infty}\alpha_1(t) = \lim_{t\to\infty}\alpha_2(t) = 0$, the conditions that must be satisfied for the map to be symmetric, we obtain the characteristic equation of the form (3.8) given by

$$\begin{aligned}\mathcal{P}(t,\lambda,z) = &-\lambda^3 + (e^{-t} + t^{-1}e^{\mu t})\lambda^2 + [t^{-2}e^{2\mu t} - t^{-1}e^{(\mu-1)t} - (t^{\frac{1}{2}} - z)t^{\frac{1}{2}}]\lambda \\ &+ (t^{\frac{1}{2}} - z)[t^{-\frac{3}{2}}e^{\mu t} - t^{-\frac{1}{2}}].\end{aligned} \tag{3.26}$$

From (3.25), one easily obtains that $|\alpha_0(t)| = t^{-\frac{1}{2}}$, $|q_{13}(t)| = t^{-\frac{1}{2}}$ and $|p_1(t)| = e^{-t}$. And so
$|\alpha_0(t)||q_{13}(t)||p_1^{-1}(t)| = t^{-1}e^t$. It therefore follows that the eigenvalues λ_1 and λ_2 for (3.26) take the form of (3.12) and results in square summable eigenfunctions while λ_3 will result into a non-square summable eigenfunction and hence one obtains that either def $H_0 = (1, 2)$ or def $H_0 = (2, 1)$ and hence the subspace H_0 generated by the difference map (3.25) has no self-adjoint subspace extensions.

3.2. $|p_1(t)| \nearrow \infty$ as $t \to \infty$.

Assumption 3.10. Assume the following growth conditions

$$\begin{aligned}&|p_1(t)| \nearrow \infty \text{ as } t \to \infty, \quad |p_2(t)| < 1, \quad |q_{12}||q_{23}||p_2^{-1}|, |x| \approx o(|\alpha_0||q_{13}|), \\ &\text{where } x = O(p_1^{-1}) \quad \forall t \in \mathbb{N}\end{aligned} \tag{3.27}$$

and the following decay conditions

$$\begin{aligned} &\Delta f, \quad \frac{\Delta f}{p_1}, \quad f(\Delta f) \in \ell^2 \\ &(\Delta f)^2, \quad \Delta^2 f, \quad f\Delta^2 f, \quad (\frac{\Delta f}{f})^2 \in \ell^1. \end{aligned} \tag{3.28}$$

Suppose that $|p_1| \nearrow \infty$ as $t \to \infty$, then, from (3.8) we obtain the eigenvalues of $S(t,z)$ as

$$\begin{aligned} \lambda_1 &\approx p_1 + p_2 + \mathrm{O}(p_2) + \mathrm{O}(p_1^{-1}) \\ \lambda_2 &\approx -\frac{p_1 p_2}{p_1+p_2} + \frac{\alpha_0 p_2 q_{13}}{p_1+p_2} + \cdots \approx -p_2 + \frac{p_2^2}{p_1} + \frac{\alpha_0 p_2 q_{13}}{p_1} + \mathrm{O}(p_1^{-2}) \\ \lambda_3 &\approx \frac{-\alpha_0 p_2 q_{13} + q_{12} q_{23} \alpha_0}{p_1 p_2} + \mathrm{O}(p_1^{-1}) \approx -\frac{\alpha_0 q_{13}}{p_1} + \frac{q_{12} q_{23} p_2^{-1}}{p_1} + \mathrm{O}(p_1^{-2}). \end{aligned} \tag{3.29}$$

Theorem 3.11. *Suppose that all the growth and decay conditions in (3.27) and (3.28) are satisfied and $\ell y(t)$ in (2.2) is defined on $\ell^2(\mathbb{Z})$. Moreover, if the assumptions in Theorem 2.4 are satisfied, then*

- **(i):** *Suppose that $|\alpha_0||q_{ij}||p_1|^{-1}|p_2|^{-1} < 1$, $i = 1,2$, $j = 2,3$, then $\operatorname{def} H_0 = (2,2)$, H_0 has self-adjoint subspace extension with discrete spectrum at most.*
- **(ii):** *Suppose that $|\alpha_0||q_{ij}||p_1|^{-1}|p_2|^{-1} \approx \mathrm{o}(1)$, $i = 1,2$, $j = 2,3$ are absolutely summable, then $\operatorname{def} H_o = (2,2)$, H_0 has self-adjoint subspace extension with discrete spectrum.*
- **(iii):** *If $|\alpha_0||q_{ij}||p_1|^{-1}|p_2|^{-1} > 1$, $i = 1,2$, $j = 2,3$ is not absolutely summable, then $\operatorname{def} H_0 = (1,2)$ or $(2,1)$ and the subspace H_0 has no self-adjoint subspace extension.*

Proof. **(i):** Assume that the growth conditions (3.27) and the decay conditions (3.28) in (3.10) are satisfied, then, the eigenvalues of (3.8) in their simplest form are obtain in (3.29). The matrix $S(t,z)$ defined in (3.3) is diagonalized twice to obtain a system in the discretized Levinson's theorem LBL 3.2 form given by (3.18). By applying the LBL 3.2, one obtains the solutions of (2.2) as (3.18). The eigenfunction resulting from λ_k-eigenvalue is given by

$$y_k(t,z) = [e_0 + r_{kk}(t,z)] \prod_{s=t_0}^{t-1} \lambda_k(s,z).$$

We obtain

$$\|y_k\|^2 = \langle y_1, y_1 \rangle = \sum_{s=t_0}^{t-1} |\gamma(t)|^2 \prod_{s=t_0}^{t-1} |\lambda_1(s,z)|^2, \tag{3.30}$$

where e_0 is the normalized eigenvectors. The term $|\gamma(t)| = |e_0 + r_{kk}(t,z)|$ is bounded and the asymptotics of (3.30) depends on $\prod_{s=t_0}^{t-1} |\lambda_k(s,z)|^2$. By expanding λ_1, upto $\mathrm{O}(p_1^{-1})$, λ_1 leads to $|\lambda_1| \approx |p_1 + p_2 + \mathrm{O}(p_1^{-1})| \to \infty$ as $t \to \infty$ and this yields $\|y_1\|^2 \approx \infty$. Thus off the real axis, $y_1(t,z)$

is a nonsquare summable eigenfunction. By expanding λ_2 upto $O(p_1^{-2})$, λ_2 leads to $|\lambda_2| \approx |\frac{\alpha_0 p_2 q_{13}}{p_1} - p_2 + \frac{p_2^2}{p_1} + O(p_1^{-2})| < \infty$ as $t \to \infty$, which implies that $\|y_2\|^2 < \infty$. Thus off the real axis, $\lambda_2(t,z)$ leads to a square summable eigenfunction $y_2(t,z)$. By expanding λ_3 upto $O(p_1^{-2})$, λ_3 leads to $|\lambda_3| \approx |\frac{-\alpha_0 q_{13}}{p_1} + \frac{\alpha_0 q_{12} q_{23} p_2^{-1}}{p_1} + O(p_1^{-2})|$. Here, the summability of $\|y_3\|^2$ depends on the value of $|\alpha_0||q_{ij}||p_1||p_2|^{-1}$. Suppose that $|p_2(t)| < 1$, $|q_{12}||q_{23}||p_2^{-1}|, |x| \approx o(|\alpha_0||q_{13}|)$, where $x = O(p_1^{-1})$ $\forall t \in \mathbb{N}$, then λ_3 yields a square summable eigenfunction. Hence, we have that def $H_0 = (2,2)$. Imposing boundary conditions at $-\infty$ and ∞, there exist two square matrices M and N of order 1 such that

$$rank(M,N) = I, \qquad M\phi_1^{tr}M^* - N\phi_2^*N^* = 0$$

ϕ_1 and ϕ_2 are invertible matrices of order 1 with square summable solutions satisfying boundary conditions at $-\infty$ and ∞ respectively. The self-adjoint extension subspace of $\tilde{H}_0$ which we denote by $\tilde{H}$ is now described by

$$\tilde{H} = \left\{(\tilde{y},\tilde{g}) \in H : M\big(\tilde{y}, y_{11})(-\infty)\big) - N\big(\tilde{y}, y_{21})(\infty)\big) = 0\right\}$$

with $y_{11} \in \phi_1$ and $y_{21} \in \phi_2$. Equivalently, using boundary conditiona one constructs $\tilde{H}$ as follows. Since the defect indices of H_0 are equal, that is, def $H_0 = (n,n)$, where $1 \le n \le 3$, then H_0 has a self-adjoint subspace extension in $L^2(I) \times L^2(I)$ and this we will denote by H. Just like in [29], for the fundamental matrix of square summable solutions Φ_{t_0} of (2.2), there exist matrices $\alpha_1, \alpha_2 \in \mathbb{C}^3$ such that

$$\text{rank}(\alpha_1, \alpha_2) = 3, \quad \alpha_1 \Phi_{t_0}^{tr} \alpha_1^* = \alpha_2 \Phi_\infty^{tr} \alpha_2^*.$$

Where $\Phi_{t_0} = ([y_1, y_2, y_3])\,(t_0 - 1)_{3\times 3}$ and Φ_∞ defined in a similar way. y_2 and y_3 are square summable solutions of (2.2). Precisely, H is defined by

$$H = \left\{(\tilde{y},\tilde{g}) \in \tilde{H} : \alpha_1 \begin{pmatrix} (\tilde{y}, y_1)(t_0-1) \\ (\tilde{y}, y_2)(t_0-1) \\ (\tilde{y}, y_3)(t_0-1) \end{pmatrix} + \alpha_2 \begin{pmatrix} (\tilde{y}, y_1)(\infty) \\ (\tilde{y}, y_2)(\infty) \\ (\tilde{y}, y_3)(\infty) \end{pmatrix} \right\}.$$

For more details see [29, Theorem 5.9] and Section 5 of the same reference in general. It therefore, follows that the spectrum of H is purely discrete since all the solutions which are square summable are uniformly square summable. and H_0 has self-adjoint subspace extension with discrete spectrum at most.

(ii): Suppose that $||p_2(t)| < 1$, $|q_{12}||q_{23}||p_2^{-1}|, |x| \approx o(|\alpha_0||q_{13}|)$, where $x = O(p_1^{-1})$ $\forall t \in \mathbb{N}$, , thenλ_3 yields a square summable eigenfunction. Hence, we have that $def\, H_0 = (2,2)$, and H_0 has self-adjoint subspace extension

Assume the growth and decay conditions (3.27) and (3.28) hold, the operator $\ell y(t)$ in (2.2) is defined on $\ell^2(\mathbb{Z})$, and the hypotheses of Theorem 2.4 are satisfied. Moreover, suppose

$$|\alpha_0|\,|q_{ij}(t)|\,|p_1(t)|^{-1}|p_2(t)|^{-1} = o(1) \quad \text{as } t \to \infty,$$

for $i = 1, 2$, $j = 2, 3$, and that the sequences

$$a(t) := |\alpha_0|\,|q_{ij}(t)|\,|p_1(t)|^{-1}|p_2(t)|^{-1}$$

are absolutely summable: $\sum_{t=t_0}^{\infty} a(t) < \infty$. As in part (i), use the two diagonalizations of the coefficient matrix $S(t,z)$ and apply the discrete Levinson (LBL) theorem 3.2 [8] to obtain a fundamental system of solutions of (2.2) of the form

$$y_k(t,z) = \big(e_k + r_k(t,z)\big)\prod_{s=t_0}^{t-1}\lambda_k(s,z), \qquad k = 1,2,3,$$

where e_k are normalized eigenvectors, $r_k(t,z) \to 0$ as $t \to \infty$, and $\lambda_k(t,z)$ are the eigenvalue factors from the diagonalization. The first solution y_1 is not in ℓ^2: The expansion for $\lambda_1(t,z)$ shows that $|\lambda_1(t,z)|$ grows for large t. Hence,

$$\prod_{s=t_0}^{t-1}|\lambda_1(s,z)| \to \infty \quad \text{as } t \to \infty,$$

so $\|y_1\|_{\ell^2} = \infty$. The second solution y_2 is in ℓ^2: By the same analysis as in part (i), $\lambda_2(t,z)$ remains bounded for large t, and the corresponding solution y_2 satisfies $\|y_2\|_{\ell^2} < \infty$. The third solution y_3 is in ℓ^2 under the summability hypothesis: The expansion of $\lambda_3(t,z)$ for large t gives

$$\lambda_3(t,z) = \frac{-\alpha_0(t)q_{13}(t)}{p_1(t)} + \frac{\alpha_0(t)q_{12}(t)q_{23}(t)}{p_1(t)p_2(t)} + O\big(p_1(t)^{-2}\big).$$

By hypothesis, $|\lambda_3(t,z)| \le Ca(t)$ for some $C > 0$, with $a(t)$ absolutely summable. Then for sufficiently large t,

$$|y_3(t,z)| \le C_1\prod_{s=t_0}^{t-1}|\lambda_3(s,z)| \le C_2 2^{-(t-T)}$$

for constants $C_1, C_2 > 0$ and some T, showing $\sum_{t=t_0}^{\infty}|y_3(t,z)|^2 < \infty$. Hence $y_3 \in \ell^2$. Exactly two linearly independent ℓ^2-solutions exist for nonreal z (namely y_2 and y_3), so $\operatorname{def} H_0 = (2,2)$. Since the deficiency indices are equal, a family of self-adjoint subspace extensions exists. Using the construction from part (i), let Φ_{t_0} and Φ_∞ denote the two-dimensional spaces of square-summable solutions at t_0 and ∞, and choose matrices α_1, α_2 of full rank satisfying

$$\alpha_1\Phi_{t_0}^{\mathrm{tr}}\alpha_1^* = \alpha_2\Phi_\infty^{\mathrm{tr}}\alpha_2^*.$$

Then the self-adjoint subspace H is defined by the corresponding boundary conditions. By the uniform square-summability of y_2 and y_3, the resolvent of H is compact and its spectrum is purely discrete. This completes the proof of (ii).

(iii): Suppose that $|\alpha_0||q_{ij}||p_1|^{-1}|p_2|^{-1} > 1$, $i = 1, 2$, $j = 2, 3$, then λ_3 yields a non square summable eigenfunction in either upper or lower half-plane. Hence, we have that either $\operatorname{def} H_0 = (1,2)$ or $\operatorname{def} H_0 = (2,1)$ and the subspace H_0 has no self-adjoint subspace extension.

□

Remark 3.12. Theorem 3.11 provides a clear classification of the deficiency indices and the existence of self-adjoint subspace extensions for H_0 depending on the size and summability of the perturbation terms $|\alpha_0||q_{ij}||p_1|^{-1}|p_2|^{-1}$. This framework naturally suggests several directions for further study. For instance, one could investigate the stability of the discrete spectrum under varying perturbations, analyze the asymptotic behavior of the eigenvalues in more detail, or extend the results to higher-dimensional discrete systems or systems with more general boundary conditions. Moreover, the interplay between summability conditions and spectral properties could be explored further to identify sharp thresholds for the existence of self-adjoint extensions, possibly leading to new insights in discrete spectral theory and applications to difference equations with complex coefficients.

Acknowledgement: *Ben Sorowen would like to thank DAAD for the financial support through the In-Country/In-Region DAAD Scholarships under the grant PKZ 91843239.*

Data availability statement: *Data sharing is not applicable to this article as no new data were created or analyzed in this study.*

Funding: *The present study is funded by DAAD under the grant PKZ 91843239. The funders had no role in the study formulation, decision to publish or in the preparation of the manuscript.*

Conflicts of interest: *All authors have no conflicts of interest to disclose.*

References

[1] Agure, J. O., Ambogo, D. O., & Nyamwala, F. O. (2013). Deficiency indices and spectrum of fourth order difference equations with unbounded coefficients. *Mathematische Nachrichten*, 286(4), 323-339. https://doi.org/10.1002/mana.201200161

[2] Behncke, H. (2006). Spectral theory of higher order differential operators. *Proceedings of the London Mathematical Society*, 92(1), 139-160. https://doi.org/10.1017/S0024611505015480

[3] Behncke, H. (2013). Spectral theory of Hamiltonian difference systems with almost constant coefficients. *Journal of Difference Equations and Applications*, 19(1), 1-12. https://doi.org/10.1080/10236198.2011.611507

[4] Behncke, H., & Hinton, D. B. (2008). Eigenfunctions, deficiency indices and spectra of odd-order differential operators. *Proceedings of the London Mathematical Society*, 97(2), 425-449. https://doi.org/10.1112/plms/pdn002

[5] Behncke, H., & Nyamwala, F. O. (2011). Spectral theory of difference operators with almost constant coefficients. *Journal of Difference Equations and Applications*, 17(05), 677-695. https://doi.org/10.1080/10236190903160681

[6] Behncke, H., & Nyamwala, F. O. (2011). Spectral theory of difference operators with almost constant coefficients II. *Journal of Difference Equations and Applications*, 17(05), 821-829. https://doi.org/10.1080/10236190903413577

[7] Behncke, H., & Nyamwala, F. O. (2013). Spectral theory of higher order difference operators. *Journal of Difference Equations and Applications*, 19(12), 1983-2028. https://doi.org/10.1080/10236198.2013.797968

[8] Benzaid, Z., & Lutz, D. A. (1987). Asymptotic representation of solutions of perturbed systems of linear difference equations. *Studies in Applied Mathematics*, 77(3), 195-221. https://doi.org/10.1002/sapm1987773195

[9] Clark, S., & Gesztesy, F. (2004). On Weyl–Titchmarsh theory for singular finite difference Hamiltonian systems. *Journal of Computational and Applied Mathematics*, 171(1-2), 151-184. https://doi.org/10.1016/j.cam.2004.01.011

[10] Clark, S. L. (1996). A spectral analysis for self-adjoint operators generated by a class of second order difference equations. *Journal of Mathematical Analysis and Applications*, 197(1), 267-285. https://doi.org/10.1006/jmaa.1996.0020

[11] Coddington, E. A. (1973). *Extension theory of formally normal and symmetric subspaces* (Vol. 134). American Mathematical Soc..

[12] Hinton, D. B., & Shaw, J. K. (1981). On Titchmarsh-Weyl $M(\lambda)-$functions for linear Hamiltonian systems. *Journal of Differential Equations*, 40(3), 316-342. https://doi.org/10.1016/0022-0396(81)90002-4

[13] Hinton, D. B. (1978). Deficiency indices of odd-order differential operators. *The Rocky Mountain Journal of Mathematics*, 8(4), 627-640. https://www.jstor.org/stable/44238321

[14] Kloeden, P. E., & Mees, A. I. (1985). Chaotic phenomena. *Bulletin of Mathematical Biology*, 47, 697-738. https://doi.org/10.1007/BF02469299

[15] Lakshmikantham, V., & Trigiante, V. (2002). *Theory of difference equations numerical methods and applications*. CRC Press. https://doi.org/10.1201/9780203910290

[16] Lesch, M., & Malamud, M. (2003). On the deficiency indices and self-adjointness of symmetric Hamiltonian systems. *Journal of Differential Equations*, 189(2), 556-615. https://doi.org/10.1016/S0022-0396(02)00099-2

[17] Nyamwala, F. O. (2015). Absolutely continuous spectrum of fourth order difference equations. *Mathematische Nachrichten*, 288(8-9), 1009-1027. https://doi.org/10.1002/mana.201400057

[18] Nyamwala, F. O. (2017). Essential and continuous spectrum of symmetric difference equations. *Mathematische Nachrichten*, 290(17-18), 2977-2991. https://doi.org/10.1002/mana.201600183

[19] Nyamwala, F. O., Ambogo, D. O., & Ngala, J. M. (2020). Comparative Analysis of Spectral Theory of Second Order Difference and Differential Operators with Unbounded Odd Coefficient. *Kyungpook Mathematical Journal*, 60(2), 297-305. https://doi.org/10.5666/KMJ.2020.60.2.297

[20] Okello, B., Nyamwala, F., & Ambogo, D. (2024). Symmetric operator extensions of composites of higher order difference operators. *Annals of Mathematics and Computer Science*, 24, 99-116. https://doi.org/10.56947/amcs.v24.352

[21] Okello, B., Nyamwala, F., & Ambogo, D. (2025). Spectral theory of commutative higher order difference operators with unbounded coefficients. *Annals of Mathematics and Computer Science*, 26, 1-27. https://doi.org/10.56947/amcs.v26.415

[22] Remling, C. (1998). Spectral Analysis of Higher Order Differential Operators I: General Properties of the M-Function. *Journal of the London Mathematical Society*, 58(2), 367-380. https://doi.org/10.1112/S0024610798006474

[23] Ren, G., & Shi, Y. (2010). The defect index of singular symmetric linear difference equations with real coefficients. *Proceedings of the American Mathematical Society*, 138(7), 2463-2475. https://doi.org/10.1090/S0002-9939-10-10253-6

[24] Ren, G., & Shi, Y. (2011). Defect indices and definiteness conditions for a class of discrete linear Hamiltonian systems. *Applied Mathematics and Computation*, 218(7), 3414-3429. https://doi.org/10.1016/j.amc.2011.08.086

[25] Ren, G., & Shi, Y. (2014). Self-adjoint extensions for discrete linear Hamiltonian systems. *Linear Algebra and Its Applications*, 454, 1-48. https://doi.org/10.1016/j.laa.2014.04.016

[26] Ren, G. (2022). The positive and negative deficiency indices of formally self-adjoint difference equations. *Complex Analysis and Operator Theory*, 16(6), 82. https://doi.org/10.1007/s11785-022-01261-4

[27] Shi, Y. (2012). The Glazman–Krein–Naimark theory for Hermitian subspaces. *Journal of Operator Theory*, 241-256. https://www.jstor.org/stable/24716773
[28] Shi, Y., Shao, C., & Ren, G. (2013). Spectral properties of self-adjoint subspaces. *Linear Algebra and its Applications*, 438(1), 191-218. https://doi.org/10.1016/j.laa.2012.07.047
[29] Shi, Y., & Sun, H. (2011). Self-adjoint extensions for second-order symmetric linear difference equations. *Linear Algebra and its Applications*, 434(4), 903-930. https://doi.org/10.1016/j.laa.2010.10.003
[30] Simon, B. (2015). *A comprehensive course in analysis*. Providence: American Mathematical Society.
[31] Wang, Y., Shi, Y., & Ren, G. (2007). Transformations for complex discrete linear Hamiltonian and symplectic systems. *Bulletin of the Australian Mathematical Society*, 75(2), 179-191. https://doi.org/10.1017/S0004972700039125
[32] Weidmann, J. (2012). *Linear operators in Hilbert spaces* (Vol. 68). Springer Science & Business Media.
[33] Zheng, Z., & Chen, S. (2006). GKN theory for linear Hamiltonian systems. *Applied mathematics and computation*, 182(2), 1514-1527. https://doi.org/10.1016/j.amc.2006.05.041

[1] Department of Mathematics and Statistics, Kyambogo University, P. O. Box 1, Kyambogo, Uganda.
Email address: sorowenben@kyu.ac.ug

[2] Department of Mathematics, Physics and Computing, Moi University, P.O. Box 3900-30100, Eldoret, Kenya.
Email address: foluoch2000@yahoo.com

[3] Department of Pure and Applied Mathematics, Maseno University, P. O. Box 333-40105, Eldoret, Kenya.
Email address: otivoe@yahoo.com