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POSITIVE SOLUTIONS OF A NONLINEAR MULTIPLE POINT FOR ONE DIMENSIONAL p-LAPLACIAN BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we study the existence of positive solutions to the one-dimensional $p-$Laplacian differential equation on a time scale $\mathbb{T}$

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\Delta}+h(t)f(t,u(t),u^{\Delta}(t))=0,$$

subject to the boundary conditions:

$$u^{\Delta}(a)=0,\qquad u\left(b\right)=\sum_{k=1}^{k=m}\alpha_k u(\xi_k)-\sum_{k=1}^{k=n}\delta_k u^{\Delta}(\eta_k),$$

where ϕ_p is $p-$Laplacian operator, i.e., $\phi_p(u)=\left|u\right|^{p-2}u$, $p>1$, for $t\in[a,b]\subset\mathbb{T}, m,n\in\mathbb{N}$, and $\xi_k\in(a,b)$, $0<\alpha_k<1$, for all $k\in\{1,2,\cdots,m\}$, such that $0<\sum_{k=1}^{k=m}\alpha_k<1$, and $\delta_k\geq 0,\eta_k\in(a,b)$, for all $k\in\{1,2,\cdots,n\}$. We show that it has at least one, two, or three positive solutions to the above boundary value problem. We study the existence of the problem one three cases, the first case: f dependence on u, the second case; f dependence on t and u, the last case f dependence on t, u, and the first order delta derivative.

1. Introduction

The study of dynamic on time scales, which has been created in order to unify of differential and difference equations, is an area of mathematics that has recently received much attention, moreover, many results on this issue have bee well documented in Ph.D. thesis of Hilger [8] and the monographs the Bohner and Peterson, [2, 3]. In recent years, many authors have begun to pay attention to the study of boundary-value problems or with $p-$Laplacian dynamic equations on time scales (see: [12, 11, 14, 7, 13, 10]).

In this paper, we consider the existence of positive solutions to the equation on time scales,

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\Delta}+h(t)f(t,u(t),u^{\Delta}(t))=0,\tag{1.1}$$

with the boundary condition

$$u^{\Delta}(a)=0,\qquad u(b)=\sum_{k=1}^{k=m}\alpha_k u(\xi_k)-\sum_{k=1}^{k=n}\delta_k u^{\Delta}(\eta_k),\tag{1.2}$$

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on a time scale $\mathbb{T}$, where $t \in [a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$, $\phi_p(s) = |s|^{p-2} s$, with $\phi_p^{-1} = \phi_q$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $m,n \in \mathbb{N}$ $a < \xi_1 < \xi_2 < \cdots < \xi_m < b$, for $k \in \{1,2,\cdots,m\}$, $a < \eta_1 < \eta_2 < \cdots < \eta_n < b$, $0 < \alpha_k < 1$, for $k \in \{1,2,\cdots,n\}$. Throughout, we suppose that the following conditions are satisfied:

- (C_1) $\alpha := \sum_{k=1}^{k=m} \alpha_k < 1$, and $0 < \alpha_k < 1$, for all $k \in \{1,2,\cdots,m\}$, $\delta_k \geq 0$, for all $k \in \{1,2,\cdots,n\}$.
- (C_2) $f : \mathbb{D} \longrightarrow [0,\infty)$ is continuous, where $\mathbb{D} := \{(t,x,y) \in \mathbb{R}^3 : t \in [a,b]_{\mathbb{T}}, x \in [0,\infty), y \in (-\infty,0]\}$.
- (C_3) $h : [a,b]_{\mathbb{T}} \longrightarrow [0,\infty)$ is continuous, and h is not identically zero. Furthermore h satisfies $0 < \int_a^b h(t)\Delta t < \infty$, and $\int_a^{\xi_m} h(t)\Delta t > 0$.

2. Preliminary

2.1. **The fundamental theory of time scale.** A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. If the time scale equals the real numbers or integer numbers. On any time scale $\mathbb{T}$, we define the forward and backward jump operators by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}, t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a rights-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k := \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, defined $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k := \mathbb{T}$.

Definition 2.1. [2]Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and lets $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $\mathcal{U}$ of t (i.e., $\mathcal{U} = (t-\delta, t+\delta) \cap \mathbb{T}$, for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|, \qquad \text{for all } s \in \mathcal{U}.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

Definition 2.2. [2]A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathcal{C}^1_{rd}(\mathbb{T}, \mathbb{R})$.

Lemma 2.3. [2] *Let $f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$. Then there exists a function F, such that*

$$F \in \mathcal{C}^1_{rd}(\mathbb{T}, \mathbb{R}) \quad \text{and} \quad F^{\Delta}(t) = f(t) \quad \text{for each } t \in \mathbb{T}^k.$$

Definition 2.4. [2] Let $f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$, then we define the delta integral by:

$$\int_a^t f(s)\Delta s = F(t) - F(a),$$

where F is called antiderivative of f.

Lemma 2.5. [4] *$\mathcal{C}_{rd}([a,b]_{\mathbb{T}})$ is a space of Banach for the norm*

$$\|f\| := \max_{t\in[a,b]_{\mathbb{T}}} |f(t)|,$$

where $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$.

Lemma 2.6. [2] *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \longrightarrow \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \longrightarrow \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f^{'}\left(hg^{\sigma}(t) + (1-h)g(t)\right) dh \right\} g^{\Delta}(t). \tag{2.1}$$

2.2. **Fixed Point Theorem.** The following fixed point theorem is fundamental and important to the proofs of our main results.

Theorem 2.7 (Guo-Krasnoselskii's)**.** [12] *Let E be a Banch space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $A : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \longrightarrow K$, be a completely continuous operator such that*

(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; or
(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$

Then A has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Notationally, the cone K has subsets of the form $K(\phi, r) := \{u \in K : \phi(u) < r\}$ for given functional ϕ.

Theorem 2.8 (Avery-Henderson)**.** [1] *Soit K be a cone in a real Banach space E. If η and ϕ are increasing, nonnegative continuous on K; let θ be a nonnegative continuous on K witch $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\phi(u) \le \theta(u) \le \eta(u) \qquad \text{and} \qquad \|u\| \le M\phi(u),$$

for all $u \in \overline{K(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \le \lambda\theta(u), \text{ for all } 0 \le \lambda \le 1 \qquad \text{and} \qquad u \in \partial K(\theta, q),$$

If $A : \overline{K(\phi, r)} \longrightarrow K$ is a completely continuous operator satisfying

(i) $\phi(Au) > r$ for all $u \in \partial K(\phi, r)$
(ii) $\theta(Au) < q$ for all $u \in \partial K(\theta, q)$
(iii) $K(\eta, p) \neq 0$ and $\eta(Au) > p$ for all $u \in \partial K(\eta, p)$

Then A has at least two fixed points u_1 and u_2 such that

$$p < \eta(u_1) \qquad \text{with} \qquad \theta(u_1) < q \qquad \text{and} \qquad q < \theta(u_2) \qquad \text{with } \phi(u_2) < r.$$

Moreover, we take

$$K_r := \{u \in K : \|u\| < r\}, \qquad K(\phi, p, q) := \{u \in K : p \le \phi(u), \|u\| \le q\}.$$

Theorem 2.9 (Leggett-Williams)**.** [1] *Let K_r be a cone in the real Banach space $E, A : \overline{K_r} \longrightarrow \overline{K_r}$ be completely continuous and ϕ be a nonnegative concave functional on K with $\phi(u) \le \|u\|$ for all $u \in \overline{K_r}$. Suppose there exists $0 < p < q < l \le r$ such that the following conditions hold:*

(i) $\{u \in K(\phi, q, l) : \phi(u) > q\} \neq \emptyset$ and $\phi(Au) > q$ for all $u \in K(\phi, q, l)$,

(ii) $\|Au\| < p$ *for all* $\|u\| \leq p$,
(iii) $\phi(Au) > q$ *for* $u \in K(\phi, q, r)$ *with* $\|Au\| > l$.

Then A has a least there fixed point u_1, u_2, and u_3 in $\overline{K_r}$ satisfying:

$$\|u_1\| < p, \qquad \phi(u_2) > q, \qquad p < \|u_3\| \qquad \text{with } \phi(u_3) < r.$$

Let γ and θ be nonnegative continuous convex functionals on a cone K, ν be a nonnegative continuous concave functional on a cone K, and ψ be a nonnegative continuous functional on a cone K. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$\begin{aligned}
&K(\gamma, d) := \{u \in K : \gamma(u) < d\},\\
&K(\gamma, \nu, b, d) := \{u \in K : b \leq \nu(u), \gamma(u) \leq d\},\\
&K(\gamma, \theta, \nu, b, c, d) := \{u \in K : b \leq \nu(u), \theta(u) \leq c, \gamma(u) \leq d\},\\
&R(\gamma, \psi, a, d) := \{u \in K : a \leq \psi(u), \gamma(u) \leq d\}.
\end{aligned}$$

Theorem 2.10. [1] *Let K be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on K, ν be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\nu(u) \leq \psi(u) \qquad \textit{and} \qquad \|u\| \leq M\gamma(u),$$

for all $u \in K(\gamma, d)$. Suppose $A : \overline{K(\gamma, d)} \longrightarrow \overline{K(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < d$ such that

(i) $\{u \in K(\gamma, \theta, \nu, b, c, d) : \nu(u) > b\} \neq \emptyset$ *and* $\nu(Au) > b$ *for* $u \in K(\gamma, \theta, \nu, b, c, d)$;
(ii) $\nu(Au) > b$ *for* $u \in K(\gamma, \nu, b, d)$ *with* $\theta(Au) > c$;
(iii) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Au) < a$ *for* $u \in R(\gamma, \psi, a, d)$ *with* $\psi(u) = a$.

Then A has at least three fixed points $u_1, u_2, u_3 \in \overline{K(\gamma, d)}$, such that

$$\gamma(u_i) \leq d \quad \textit{for} \quad i = 1, 2, 3, \qquad b < \nu(u_1),$$

$$a < \psi(u_2), \quad \textit{with} \quad \nu(u_2) < b \qquad \psi(u_3) < a.$$

3. Main Results

This section is devoted to prove the existence of a positive solution of the nonlinear boundary value problem (1.1) and (1.2). Before starting the main results, we begin with the following lemma.

Lemma 3.1. *If $\alpha \neq 1$, then for $y \in \mathcal{C}_{rd}([a, b]_{\mathbb{T}})$ the boundary-value problem*

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\Delta} + y(t) = 0, \tag{3.1}$$

$$u^{\Delta}(a) = 0, \qquad u(b) = \sum_{k=1}^{k=m} \alpha_k u(\xi_k) - \sum_{k=1}^{k=n} \delta_k u^{\Delta}(\eta_k), \tag{3.2}$$

has the unique solution

$$\begin{aligned} u(t) &= -\int_a^t \phi_q\left(\int_a^\tau y(s)\,\Delta s\right)\Delta\tau + \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau y(s)\,\Delta s\right)\Delta\tau - \\ & \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\alpha_k\int_a^{\xi_k}\phi_q\left(\int_a^\tau y(s)\,\Delta s\right)\Delta\tau + \\ & \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} y(s)\,\Delta s\right). \end{aligned} \tag{3.3}$$

Proof. Let u be as in (3.3). Then the delta derivative of u is given by:

$$u^\Delta(t) = -\phi_q\left(\int_a^t y(s)\,\Delta s\right), \tag{3.4}$$

and

$$\left[\phi_p\left(u^\Delta(t)\right)\right]^\Delta = -y(t)$$

This completes the proof. □

Lemma 3.2. *Let $0<\alpha<1$, If $y\in\mathcal{C}_{rd}([a,b]_\mathbb{T})$, and $y\ge 0$, then unique solution u of (3.1), (3.2) satisfies*

$$u(t)\ge 0, \qquad t\in[a,b].$$

Proof. Let $0<\alpha<1$, if $y\in\mathcal{C}_{rd}([a,b]_\mathbb{T})$, and $y\ge 0$, we have

$$u(t)\ge \frac{\alpha}{1-\alpha}\int_{\xi_m}^b \phi_q\left(\int_a^\tau y(s)\,\Delta s\right)\Delta\tau + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} y(s)\,\Delta s\right).$$

This completes the proof. □

Lemma 3.3. *Let $0<\alpha<1$, if $y\in\mathcal{C}_{rd}([a,b]_\mathbb{T})$, and $y\ge 0$, then unique solution u of (3.1), (3.2) satisfies*

$$u(b)\ge \alpha_k u(\xi_k)\ge \gamma_k\|u\|, \qquad \text{for all } k\in\{1,2,\cdots,m\}, \tag{3.5}$$

and

$$u(\eta_k)\ge \beta_k\|u\|, \qquad \text{for all } k\in\{1,2,\cdots,n\}, \tag{3.6}$$

where $\gamma_k = \alpha_k\frac{b-\xi_k}{b-a}$, for all $k\in\{1,2,\cdots,m\}$, and $\beta_k=\frac{b-\eta_k}{b-a}$, for all $k\in\{1,2,\cdots,n\}$.

Proof. From that fact that $\left[\phi_p\left(u^\Delta(t)\right)\right]^\Delta = -y(t)\le 0$, by (2.1), we have

$$\left[\phi_p\left(u^\Delta(t)\right)\right]^\Delta = (p-1)\,u^{\Delta\Delta}(t)\int_0^1\left\{-\left(hu^\Delta(\sigma(t))+(1-h)\,u^\Delta(t)\right)\right\}^{p-2}dh\le 0.$$

We know the graph of $u(t)$ is concave down on $[a,b]_\mathbb{T}$.

$$u(\xi_k)\ge\frac{b-\xi_k}{b-a}u(a), \qquad \text{for all } k\in\{1,2,\cdots,m\},$$

and

$$u(\eta_k)\ge\frac{b-\eta_k}{b-a}u(a), \qquad \text{for all } k\in\{1,2,\cdots,n\}.$$

Using (3.4) we get $u(t)$ is strictly decreasing on $[a,b]_{\mathbb{T}}$, and by (3.2) we have

$$u(b) \geq \alpha_k u(\xi_k) \geq \alpha_k \frac{b-\xi_k}{b-a} u(a) = \gamma_k \|u\|, \qquad \text{for all } k \in \{1,2,\cdots,m\},$$

and

$$u(\eta_k) \geq \beta_k \|u\|, \qquad \text{for all } k \in \{1,2,\cdots,n\}.$$

The proof is complete. □

For simplification, we note

(1) $H_q(\tau) := \phi_q\left(\int_a^\tau h(s)\Delta s\right)$ and $R_a^b H := \int_a^b H_q(\tau)\Delta\tau$,
(2) $M_1 := R_a^b H + \sum_{k=1}^{k=n} \delta_k H_q(\eta_k)$,
(3) $M_2 := \alpha\gamma R_{\xi_m}^b H + \sum_{k=1}^{k=n} \delta_k \beta_k H_q(\eta_k)$,
(4) $M_3 := \alpha R_{\xi_m}^b H + \sum_{k=1}^{k=n} \delta_k H_q(\eta_k)$,
(5) $M_4 := (2-\alpha) R_a^b H\Delta\tau + \sum_{k=1}^{k=n} \delta_k H_q(\eta_k)$.

3.1. f dependence on u.

Now, we use the following Guo-Krasnoselskii's fixed point Theorem 2.7, for simplification, we note

$$f_0 := \lim_{|u|\longrightarrow 0} \frac{f(u)}{|u|^{p-1}}, \qquad f_\infty := \lim_{|u|\longrightarrow \infty} \frac{f(u)}{|u|^{p-1}}.$$

Theorem 3.4. *Assume $(c_1), (c_2)$ and (c_3) hold. Then the problem boundary (1.1) and (1.2), has at least one positive solution in one of the cases*

(i) $f_0 = 0$, and $f_\infty = \infty$,
(ii) $f_0 = \infty$, and $f_\infty = 0$.

Proof. We wish to show the existence of a positive solution of (1.1) and (1.2). New has a solution $u = u(t)$ if and only if u solves the operator equation

$$\begin{aligned} Au(t) \; : \; &= -\int_a^t \phi_q\left(\int_a^\tau h(s)f(u(s))\Delta s\right)\Delta\tau \\ &+\frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau h(s)f(u(s))\Delta s\right)\Delta\tau \\ &-\frac{1}{1-\alpha}\sum_{k=1}^{k=m}\alpha_k \int_a^{\xi_k} \phi_q\left(\int_a^\tau h(s)f(u(s))\Delta s\right)\Delta\tau + \\ &\frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} h(s)f(u(s))\Delta s\right). \end{aligned} \tag{3.7}$$

Denote

$$K = \left\{ \begin{array}{l} u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : u(t) \geq 0, u \text{ is concave, strictly decreasing on } [a,b] \subset \mathbb{T}, \\ \text{and } u \text{ satisfies } (3.5), (3.6) \end{array} \right\}.$$

It is obvious that K is a cone in $\mathcal{C}_{rd}([a,b]_{\mathbb{T}})$ and we have $AK \subset K$. It is also easy to check that $A : K \longrightarrow K$ is completely continuous.

(i) Suppose then that $f_0 = 0$ and $f_\infty = \infty$. New since $f_0 = 0$, we choose $H_1 > 0$, such that $f(u) \leq [\varepsilon |u|]^{p-1}$, for $0 < u \leq H_1$, where $\varepsilon > 0$, satisfies $M_1\varepsilon \leq 1-\alpha$. Then for $u \in K \cap \partial\Omega_1$, $\Omega_1 = \{u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : \|u\| < H_1\}$, we have

$$\begin{aligned} Au(t) &\leq \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau h(s) f_1(u(s))\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} h(s) f_1(u(s))\Delta s\right) \\ &\leq \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau [\varepsilon|u|]^{p-1} h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} [\varepsilon|u|]^{p-1} h(s)\Delta s\right) \\ &\leq \frac{H_1\varepsilon}{1-\alpha}M_1 \leq \|u\|. \end{aligned}$$

Then $\|Au\| \leq \|u\|$, for all $u \in K \cap \partial\Omega_1$.

On the other hand, by $f_\infty = \infty$, there exists $\widetilde{H}_2 > 0$, such that $f_1(u) > [\theta|u|]^{p-1}$ for $u \geq \widetilde{H}_2$, where $\theta > 0$ satisfies $M_2\theta \geq 1-\alpha$. Set $H_2 = \max\left\{\frac{1}{\gamma}H_1, \frac{1}{\gamma}\widetilde{H}_2\right\}$, $\Omega_2 = \{u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : \|u\| < H_2\}$, then for $u \in K \cap \partial\Omega_2$, we get

$$\begin{aligned} \|Au\| &\geq \frac{\alpha}{1-\alpha}\int_{\xi_m}^b \phi_q\left(\int_a^\tau h(s) f(u(s))\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} h(s) f(u(s))\Delta s\right) \\ &\geq \frac{\alpha\theta\|u\|}{1-\alpha}\gamma\int_{\xi_m}^b \phi_q\left(\int_a^\tau h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{\theta\|u\|}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\beta_k\phi_q\left(\int_a^{\eta_k} h(s)\Delta s\right) \\ &\geq \frac{\theta M_2}{1-\alpha}\|u\| \geq \|u\|. \end{aligned}$$

Therefore, by the first part of the Theorem 2.7, then A has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$, we conclude that problem (1.1) and (1.2) has at least one positive solution u, such that

$$H_1 \leq \|u\| \leq H_2.$$

(ii) Suppose then that $f_0 = \infty$ and $f_\infty = 0$. New since $f_0 = \infty$, we first choose $H_3 > 0$, such that $f(u) > [\lambda|u|]^{p-1}$, for $0 < u < H_3$, satisfies $M_2\lambda \geq 1-\alpha, \Omega_3 = \{u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : \|u\| < H_3\}$, then for $u \in K \cap \partial\Omega_3$, we get

$$\|Au\| \geq \frac{M_2\lambda}{1-\alpha}\|u\| \geq \|u\|.$$

Then $\|Au\| \geq \|u\|$, for all $u \in K \cap \partial\Omega_3$.

Now since $f_\infty = 0$, there exists $\widetilde{H}_4 > 0$, such that $f_1(u) \leq [\varpi |u|]^{p-1}$, for $u > \widetilde{H}_4$, satisfies $M\varpi \leq 1-\alpha$.

Set $H_4 = \max\left\{\frac{1}{\gamma}H_3, \frac{1}{\gamma}\widetilde{H}_4\right\}$, $\Omega_4 = \{u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : \|u\| < H_4\}$, then for $u \in K \cap \partial\Omega_4$, we have

$$\|Au\| \leq \frac{H_4\varpi}{1-\alpha} M_1 \leq \|u\|.$$

By the second part of the Theorem 2.7, then A has a fixed point in $K \cap \left(\overline{\Omega}_4 \backslash \Omega_3\right)$, we conclude that problem (1.1) and (1.2) has at least one positive solution u, such that

$$H_3 \leq \|u\| \leq H_4.$$

□

Corollary 3.5. *Assume* $(c_1), (c_2)$ *and* (c_3) *hold. Then the problem boundary* (1.1) *and* (1.2), *has at least one positive solution in one of the cases*

(*i*) $f_0 = r_1 \in (0,\infty)$, $f_\infty = r_2 \in (0,\infty)$, *and* $M_1 r_1^{q-1} < 1-\alpha, M_2 r_2^{q-1} > 1-\alpha$,
(*ii*) $f_0 = 0$, $f_\infty = r_2$, *and* $M_2 r_2^{q-1} > 1-\alpha$,
(*iii*) $f_0 = r_1 \in (0,\infty)$, *and* $f_\infty = 0$, *and* $M_2 r_1^{q-1} > 1-\alpha$,
(*iv*) $f_0 = \infty$, $f_\infty = r_2 \in (0,\infty)$, *and* $M_1 r_2^{q-1} < 1-\alpha$,
(*v*) $f_0 = r_1 \in (0,\infty)$, $f_\infty = \infty$, *and* $M_1 r_1^{q-1} < 1-\alpha$.

Proof. Suppose that $f_0 = r_1$, there exists $H_1 > 0$, such that $f(u) \leq (r_1+\varepsilon)|u|^{p-1}, 0 < u < H_1$, where $\varepsilon > 0$ satisfies $M_1(\varepsilon + r_1)^{q-1} \leq 1-\alpha$. Then for $u \in K \cap \partial\Omega_1$, $\Omega_1 = \{u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : \|u\| < H_1\}$, we have

$$\begin{aligned} Au(t) &\leq \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau (\varepsilon + r_1)|u|^{p-1} h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k \phi_q\left(\int_a^{\eta_k}(\varepsilon + r_1)|u|^{p-1} h(s)\Delta s\right) \\ &\leq \frac{H_1(\varepsilon + r_1)^{q-1}}{1-\alpha} M_1 \leq \|u\|. \end{aligned}$$

Then $\|Au\| \leq \|u\|$, for all $u \in K \cap \partial\Omega_1$.

On the other hand, by $f_\infty = r_2$, there exists $\widetilde{H}_2 > 0$, such that $f_1(u) > (r_2-\theta)|u|^{p-1}$ for $|u| \geq \widetilde{H}_2$, where $\theta > 0$ satisfies $M_2(r_2-\theta)^{q-1} \geq 1-\alpha$. Set $H_2 = \max\left\{\frac{1}{\gamma}H_1, \frac{1}{\gamma}\widetilde{H}_2\right\}$, $\Omega_2 = \{u \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}) : \|u\| < H_2\}$, then for $u \in K \cap \partial\Omega_2$, we get

$$\begin{aligned} \|Au\| &\geq \frac{\alpha\gamma\|u\|}{1-\alpha}\int_{\xi_m}^b \phi_q\left(\int_a^\tau (r_2-\theta) h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{\|u\|}{1-\alpha}\sum_{k=1}^{k=n}\beta_k\delta_k\phi_q\left(\int_a^{\eta_k}(r_2-\theta) h(s)\Delta s\right) \\ &\geq \frac{(r_2-\theta)^{q-1} M_2}{1-\alpha}\|u\| \geq \|u\|. \end{aligned}$$

Then $\|Au\| \geq \|u\|$, for all $u \in K \cap \partial\Omega_2$. Therefore, by Theorem 2.7, we conclude that problem (1.1) and (1.2) has at least one positive solution u, such that

$$H_1 \leq \|u\| \leq H_2.$$

The proof (ii), (iii), (iv), and (v) are similar to that in (i). □

3.2. **f dependence on t and u.**
Now, we apply the Avery-Henderson Theorem 2.8, to prove the existence of at least two positive solutions of the boundary value problem (1.1) and (1.2).
Let the nonnegative, increasing, continuous functionals, ϕ, θ, and η be defined on the cone K by:

$$\theta(u) := u(\xi_k), \qquad \text{for all } k \in \{1, 2, \cdots, m\},$$

and

$$\phi(u) := u(b), \qquad \eta(u) := u(a).$$

Observe that, for $u \in K$, the increasing of u, we get

$$\phi(u) \leq \theta(u) \leq \eta(u).$$

By (3.5), we have

$$\|u\| = u(a) \leq \frac{1}{\gamma} u(b) = \frac{1}{\gamma}\phi(u).$$

Theorem 3.6. *Suppose there exists positive numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*

(i) $f(s,u) > \phi_p\left(\frac{r(1-\alpha)}{M_3}\right)$ *for* $s \in [a, \eta_n] \cup [\xi_m, b]$ *and* $u \in \left[r, \frac{r}{\gamma}\right]$,

(iii) $f(s,u) < \phi_p\left(\frac{q(1-\alpha)}{M_1}\right)$ *for* $s \in [a, b]$ *and* $u \in \left[\alpha_k q, \frac{\alpha_k}{\gamma_k} q\right]$,

(iv) $f(s,u) > \phi_p\left(\frac{p(1-\alpha)}{M_3}\right)$ *for* $s \in [a, \eta_n] \cup [\xi_m, b]$ *and* $u \in [\gamma p, p]$.

Then the problem boundary (1.1) and (1.2), has at least two positive solutions u_1 and u_2 such that

$$p < u_1(a) \quad \text{with} \quad u_1(\xi_k) < q,$$
$$q < u_2(\xi_k) \quad \text{with} \quad u_2(b) < r.$$

Proof. We defined operator A by:

$$\begin{aligned} Au(t) &= -\int_a^t \phi_q\left(\int_a^\tau h(s) f(s, u(s))\,\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau h(s) f(s, u(s))\,\Delta s\right)\Delta\tau - \\ &\quad \frac{1}{1-\alpha}\sum_{k=1}^{k=m}\alpha_k \int_a^{\xi_k} \phi_q\left(\int_a^\tau h(s) f(s, u(s))\,\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k \phi_q\left(\int_a^{\eta_k} h(s) f(s, u(s))\,\Delta s\right). \end{aligned} \tag{3.8}$$

For $u \in K$, we have $AK \subset K$. It is also easy to check that $A : K \longrightarrow K$ is completely continuous. For any $u \in K$, it is clear that $\theta(0) = 0$, for all $\lambda \in [0,1]$ we get $\theta(\lambda u) = \lambda\theta(u)$, since $0 \in K$ and $p > 0, K(\eta, p) \neq \emptyset$. In the following claim, we verify the remaining conditions of Theorem 2.8.
(i) for all $u \in \partial K(\phi, r)$, then $\phi(Au) > r$, we have

$$r = u(b) \leq u(s) \leq u(a) \leq \frac{r}{\gamma}, \qquad \text{for } s \in [a, \eta_n] \cup [\xi_m, b].$$

Then

$$\begin{aligned}
\phi(Au) &= Au(b) \geq \frac{\alpha}{1-\alpha}\int_{\xi_m}^{b} \phi_q\left(\int_a^{\tau} h(s) f_2(s.u(s))\Delta s\right)\Delta\tau \\
&\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} h(s) f_2(s, u(s)\Delta s\right) \\
&> \frac{\alpha}{1-\alpha}\int_{\xi_m}^{b} \phi_q\left(\int_a^{\tau} \phi_p\left(\frac{r(1-\alpha)}{M_3}\right) h(s)\Delta s\right)\Delta\tau \\
&\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} \phi_p\left(\frac{r(1-\alpha)}{M_3}\right) h(s)\Delta s\right) \\
&\geq \frac{r\alpha}{M_3}\int_{\xi_m}^{b} \phi_q\left(\int_a^{\tau} h(s)\Delta s\right)\Delta\tau + \frac{r}{M_3}\sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} h(s)\Delta s\right) = r.
\end{aligned}$$

(ii) for all $u \in \partial K(\theta, q)$, then $\theta(Au) < q$, we have

$$\alpha_k q = \alpha_k u(\xi_k) \leq u(b) \leq u(s) \leq u(a) \leq \frac{\alpha_k}{\gamma_k} q, \qquad \text{for } s \in [a, b].$$

Then

$$\begin{aligned}
\theta(Au) &= Au(\xi_k) \leq \frac{1}{1-\alpha}\int_a^{b} \phi_q\left(\int_a^{\tau} h(s) f_2(s, u(s)\Delta s\right)\Delta\tau \\
&\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} h(s) f_2(s, u(s)\Delta s\right) \\
&< \frac{1}{1-\alpha}\int_a^{b} \phi_q\left(\int_a^{\tau} \phi_p\left(\frac{q(1-\alpha)}{M_1}\right) h(s)\Delta s\right)\Delta\tau \\
&\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} \phi_p\left(\frac{q(1-\alpha)}{M_1}\right) h(s)\Delta s\right) \\
&\leq \frac{q}{M_1}\left[\int_a^{b} \phi_q\left(\int_a^{\tau} h(s)\Delta s\right) + \sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} h(s)\Delta s\right)\right] = q.
\end{aligned}$$

(iii) for all $u \in \partial K(\eta, p)$, then $\eta(Au) > p$, we have

$$\gamma p = \gamma u(a) \leq u(b) \leq u(s) \leq u(a) = p, \qquad \text{for } s \in [a, \eta_n] \cup [\xi_m, b].$$

Then

$$\begin{aligned}\eta(Au) &> \frac{\alpha}{1-\alpha}\int_{\xi_m}^{b}\phi_q\left(\int_a^{\tau}\phi_p\left(\frac{p(1-\alpha)}{M_3}\right)h(s)\Delta s\right)\Delta\tau \\ &\quad +\frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k}\phi_p\left(\frac{p(1-\alpha)}{M_3}\right)h(s)\Delta s\right) \\ &\geq \frac{pM_3}{M_3}=p.\end{aligned}$$

Therefore, the hypotheses of Theorem 2.8 are satisfied and there exist at least two positive fixed points u_1 and u_2 of A in $\overline{K(\phi,r)}$, we conclude that problem (1.1) and (1.2) has at least two positive solutions u_1 and u_2, such that

$$p<\eta(u_1)=u_1(a) \qquad \text{with} \qquad u_1(\xi_k)=\theta(u_1)<q,$$
$$q<\theta(u_2)=u_2(\xi_k) \qquad \text{with} \qquad \phi(u_2)=u_2(b)<r.$$

□

Corollary 3.7. *If there exists $q>0$ such that the function f satisfies the following conditions:*

(*i*) $\liminf_{u\longrightarrow 0^+}\frac{f(s,u)}{u^{p-1}}>\phi_p\left(\frac{1-\alpha)}{\gamma M_3}\right)$, *for all* $s\in[a,\eta_n]\cup[\xi_m,b]$,

(*ii*) $f(s,u)<\phi_p\left(\frac{q(1-\alpha)}{M_1}\right)$, *for* $s\in[a,b]$ *and* $u\in\left[\alpha_k q,\frac{\alpha_k}{\gamma_k}q\right]$,

(*iii*) $\liminf_{u\longrightarrow\infty}\frac{f(s,u)}{u^{p-1}}>\phi_p\left(\frac{1-\alpha)}{M_3}\right)$, *for all* $s\in[a,\eta_n]\cup[\xi_m,b]$.

Then, the second-order triple-point boundary value problem (1.1) and (1.2), has two positive solutions.

Proof. By (i) of the corollary, then exists $p\in(0,q)$ such that

$$\frac{f(s,u)}{u^{p-1}}>\phi_p\left(\frac{1-\alpha)}{\gamma M_3}\right), \qquad \text{for all } u\in(0,p]\,,\ s\in[a,\eta_n]\cup[\xi_m,b]\,.$$

Then, for all $u\in[\gamma p,p]\,, s\in[a,\eta_n]\cup[\xi_m,b]$, we have

$$f(s,u)>\phi_p\left(\frac{1-\alpha)}{\gamma M_3}u\right)\geq\phi_p\left(\frac{1-\alpha}{M_3}p\right),$$

and (iii) of Theorem 3.6 holds, and by (iii) of the corollary, then exists $r>q$, such that

$$\frac{f(s,u)}{u^{p-1}}>\phi_p\left(\frac{1-\alpha)}{M_3}\right), \qquad \text{for all } u>r, s\in[a,\eta_n]\cup[\xi_m,b]\,.$$

Then, for all $u\in\left[r,\frac{r}{\gamma}\right]\,, s\in[a,\eta_n]\cup[\xi_m,b]$, we have

$$f(s,u)>\phi_p\left(\frac{1-\alpha}{M_3}u\right)\geq\phi_p\left(\frac{(1-\alpha)r}{M_3}\right),$$

and (i) of Theorem 3.6 holds, Then condition of Theorem 3.6 satisfies. □

Using the ideas in the proof Theorem 3.6, we can establish the existence of an arbitrary pair number of positive solution of (1.1) and (1.2).

Corollary 3.8. *Suppose there exists positive numbers*

$$0 < p_1 < q_1 < r_1 < l_k^1 < p_2 < q_2 < r_2 < l_k^2 < \cdots < p_d < q_d < l_k^d, \qquad d \in \mathbb{N},$$

where $l_k^n = \max\left\{\frac{\alpha_k}{\gamma_k} q_n, \frac{r_n}{\gamma_k}\right\}$, *for all* $k \in \{1, 2, \cdots, n\}$, *such that the function* f *satisfies the following conditions:*

(*i*) $f(s,u) > \phi_p\left(\frac{r_i(1-\alpha)}{M_3}\right)$ *for* $s \in [a, \xi]$ *and* $u \in \left[r_i, \frac{r_i}{\gamma}\right]$,

(*ii*) $f(s,u) < \phi_p\left(\frac{q_i(1-\alpha)}{M_1}\right)$ *for* $s \in [a, b]$ *and* $u \in \left[\alpha q_i, \frac{\alpha}{\gamma} q_i\right]$,

(*iii*) $f(s,u) > \phi_p\left(\frac{p_i(1-\alpha)}{M_3}\right)$ *for* $s \in [a, \xi]$ *and* $u \in \left[\frac{\gamma}{\alpha} p_i, p_i\right]$.

Then the problem boundary (1.1) *and* (1.2), *has at least* $2d$ *positive solutions* $u_i, i = \overline{1, 2d}$ *such that*

$$p_i \le \|u_i\| \le \frac{\alpha_k}{\gamma_k} q_i, \qquad \text{and} \qquad \alpha_k q_i \le \|u_{2i}\| \le \frac{r_i}{\gamma_k}, \qquad \text{for all } i = \overline{1, d}.$$

Now, we apply the Leggett-Williams Theorem 2.9 to prove the existence of at least three positive solutions of to boundary value problem (1.1) and (1.2).

Theorem 3.9. *Suppose that there exists* $0 < p < q < \frac{\alpha_k}{\gamma_k} q < r$ *such that*

(*i*) $f(s,u) > \phi_p\left(\frac{(1-\alpha)q}{M_3}\right)$, *for* $s \in [a, \eta_n] \cup [\xi_m, b]$, $u \in \left[\alpha_k q, \frac{\alpha_k}{\gamma_k} q\right]$,

(*ii*) $f(s,u) < \phi_p\left(\frac{p(1-\alpha)}{M_1}\right)$, *for* $s \in [a, b]$, $u \in [0, p]$,

(*iii*) $f(s,u) < \phi_p\left(\frac{r(1-\alpha)}{M_1}\right)$, *for* $s \in [a, b]$, $u \in [0, r]$.

Then the boundary-value problem (1.1) *and* (1.2) *has at least three solutions positive* u_1, u_2, *and* u_3 *satisfying*

$$\|u_1\| < p, \qquad q < u_2(\xi_k), \qquad \|u_3\| > p \qquad \text{with} \qquad u_3(\xi_k) < r.$$

Proof. We define the continuous concave functional $\phi : K \longrightarrow [0, \infty)$ to be $\phi(u) =: u(\xi_k)$, Let operator A defined by (3.8). For all $u \in K$, we have $\phi(u) = u(\xi_k) \le \|u\|$. If $u \in \overline{K}_r$, then $\|u\| \le r$ and (iii) implies

$$f(s,u) < \phi_p\left(\frac{r(1-\alpha)}{M}\right), \qquad \text{for } s \in [a, b], u \in [0, r],$$

we have

$$\begin{aligned} \|Au\| &\le \frac{1}{1-\alpha} \int_a^b \phi_q\left(\int_a^\tau f(s, u(s)) h(s) \Delta s\right) \Delta\tau \\ &\quad + \frac{1}{1-\alpha} \sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} f(s, u(s)) h(s) \Delta s\right) \\ &\le \frac{r}{M_1}\left[\int_a^b \phi_q\left(\int_a^\tau h(s)\Delta s\right) \Delta\tau + \sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_K} h(s)\Delta s\right)\right] = r. \end{aligned}$$

This proves that $A:\overline{K}_r\longrightarrow\overline{K}_r$. Similarly, if $\|u\|\leq p$, by (ii) we have $\|Au\|<p$, it follows that condition (ii) of Theorem 2.9 is satisfied. We new consider condition (i) of Theorem 2.9. Let $v(t)=\frac{\alpha_k}{\gamma_k}q$, for all $t\in[a,b]$, then $v\in K(\phi,q,\frac{\alpha_k}{\gamma_k}q)$ and $\phi(v)=\frac{\alpha_k}{\gamma_k}q\geq q$. For all $u\in K\left(\phi,q,\frac{\alpha_k}{\gamma_k}q\right)$, then $\alpha_k q\leq u(s)\leq\frac{\alpha_k}{\gamma_k}q$, for all $s\in[a,b]$, by (i) we have $f(s,u)>\phi_p\left(\frac{(1-\alpha)q}{M_3}\right)$, for $s\in[a,\eta_n]\cup[\xi_m,b]\,,u\in\left[q,\frac{\alpha_k}{\gamma_k}q\right]$, then

$$\begin{aligned}\phi\left(Au\right) &\geq \frac{\alpha}{1-\alpha}\int_{\xi_m}^{b}\phi_q\left(\int_a^{\tau}\phi_p\left(\frac{(1-\alpha)q}{M_3}\right)h(s)\Delta s\right)\Delta\tau\\ &\quad+\frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k}h(s)\phi_p\left(\frac{(1-\alpha)q}{M_3}\right)\Delta s\right)\\ &> q.\end{aligned}$$

Hence

$$\phi\left(Au\right)>q,\qquad\text{for all}\qquad u\in K\left(\phi,q,\frac{\alpha_k}{\gamma_k}q\right).$$

So that condition (i) of Theorem 2.9, show that condition (iii) Theorem 2.9, for all $u\in K\left(\phi,q,r\right)$, with $\|Au\|>\frac{\alpha_k}{\gamma_k}q$, and we have

$$\phi\left(Au\right)=Au(\xi_k)\geq\frac{\gamma_k}{\alpha_k}\left\|Au\right\|>q.$$

□

Using the ideas in the proof Theorem 3.9, we can establish the existence of an arbitrary odd number of positive solution of (1.1) and (1.2).

Corollary 3.10. *Suppose that there exists*

$$0<p_1<q_1<\frac{\alpha_k}{\gamma_k}q_1<p_2<q_2<\frac{\alpha_k}{\gamma_k}q_2<\cdots<p_d<q_d<\frac{\alpha_k}{\gamma_k}q_d<p_{d+1},\qquad d\in\mathbb{N},$$

such that

(i) $f_2(s,u)>\phi_p\left(\frac{(1-\alpha)q_i}{M_3}\right)$, *for* $s\in[a,\eta_n]\cup[\xi_m,b]\,,u\in\left[\alpha_k q_i,\frac{\alpha_k}{\gamma_k}q_i\right]$,

(ii) $f_2(s,u)<\phi_p\left(\frac{p_i(1-\alpha)}{M_1}\right)$, *for* $s\in[a,b]\,,u\in[0,p_i]$

Then the boundary-value problem (1.1) *and* (1.2) *has at least* $2d+1$ *positive solutions.*

3.3. f dependence on t, u, and u^{Δ}.

Now, we apply the Theorem 2.10 to prove the existence of at least three positive solutions to the boundary value problem (1.1) and (1.2).

We define the cone $\widetilde{K}\subset\mathcal{C}_{rd}^1\left(\mathbb{T},\mathbb{R}\right)$ by:

$$\widetilde{K}:=\left\{u\in\mathcal{C}_{rd}^1\left(\mathbb{T},\mathbb{R}\right)\cap K:u^{\Delta}\leq 0,\ u^{\Delta}\text{ strictly decreasing on }[a,b]_{\mathbb{T}}\right\}.$$

Let the nonnegative continuous concave functional ν, the nonnegative continuous convex functional θ,φ and the nonnegative functional ψ be defined on the cone $\widetilde{K}$ by:

$$\varphi(u)=u(a)-u^{\Delta}(b),\qquad\nu(u)=u(b),$$

and

$$\theta(u) = u(\xi_k), \qquad \psi(u) = u(a).$$

Observe that, for $u \in \widetilde{K}$, the increasing of u and u^{Δ}, we get

$$\varphi(u) = u(a) - u^{\Delta}(b) \geq |u|, \quad \text{and} \quad \nu(u) = u(b) \leq u(a) = \psi(u).$$

Theorem 3.11. *Suppose there exists positive numbers $0 < r < L$ such that the function f satisfies the following conditions:*

(i) $f(s,u,v) \leq \phi_p\left(\frac{(1-\alpha)L}{M_4\gamma_k}\right)$, *for all* $(s,u,v) \in [a,b] \times \left[0, \frac{L}{\gamma_k}\right] \times \left[-\frac{L}{\gamma_k}, 0\right]$,

(ii) $f(s,u,v) > \phi_p\left(\frac{(1-\alpha)L}{M_3}\right)$, *for all* $(s,u,v) \in [a,\eta_n] \cup [\xi_m, b] \times \left[L, \frac{L}{\gamma_k}\right] \times \left[-\frac{L}{\gamma_k}, 0\right]$,

(iii) $f(s,u,v) < \phi_p\left(\frac{(1-\alpha)r}{M_1}\right)$, *for all* $(s,u,v) \in [a,b] \times [0,r] \times \left[-\frac{L}{\gamma_k}, 0\right]$.

Then the boundary-value problem (1.1) and (1.2) has at least three solutions positive u_1, u_2, and u_3 satisfying

$$\|u_i\| \leq \frac{b}{\gamma_k}, \left\|u_i^{\Delta}\right\| \leq \frac{L}{\gamma_k}, \qquad i = 1,2,3, \qquad u_1(b) > L,$$

$$r < u_2(a) \qquad \textit{with} \qquad u_2(b) < L, \qquad u_3(a) < r.$$

Proof. Define the operator A by:

$$\begin{aligned} Au(t) &= -\int_a^t \phi_q\left(\int_a^\tau h(s) f\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right)\Delta\tau + \\ &\quad \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau h(s) f\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right)\Delta\tau - \\ &\quad \frac{1}{1-\alpha}\sum_{k=1}^{k=m} \alpha_k \int_a^{\xi_k} \phi_q\left(\int_a^\tau h(s) f_3\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n} \delta_k \phi_q\left(\int_a^{\eta_k} h(s) f\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right). \end{aligned}$$

It is well known that $A : \widetilde{K} \longrightarrow \widetilde{K}$ is completely continuous and $A\widetilde{K} \subset \widetilde{K}$, each fixed point of in $\widetilde{K}$ is a solution to the problem (1.1) and (1.2). In the following claim, we verify the remaining conditions of Theorem 2.10. If $u \in \overline{\widetilde{K}\left(\varphi, \frac{L}{\gamma_k}\right)}$, then

$$0 \leq u(s) \leq \frac{L}{\gamma_k}, \quad \text{and} \quad -\frac{L}{\gamma_k} \leq u(s) \leq 0, \quad \text{for } s \in [a,b].$$

By condition (i) of Theorem 3.11, we have

$$\begin{aligned}\varphi(Au) &= Au(a) - Au^{\Delta}(b) \\ &\leq \frac{2-\alpha}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau h(s) f_3\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k \phi_q\left(\int_a^{\eta_k} h(s) f_3\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right) \\ &\leq \frac{2-\alpha}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau \phi_p\left(\frac{(1-\alpha)L}{M_4\gamma_k}\right) h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k \phi_q\left(\int_a^{\eta_k} \phi_p\left(\frac{(1-\alpha)L}{M_4\gamma_k}\right) h(s)\Delta s\right) \\ &\leq \frac{L}{\gamma_k}.\end{aligned}$$

This proves that $A : \overline{\widetilde{K}\left(\varphi, \frac{L}{\gamma_k}\right)} \longrightarrow \overline{\widetilde{K}\left(\varphi, \frac{L}{\gamma_k}\right)}$.

We show that consider condition (i) in Theorem 2.10, we take $w(t) = \frac{L}{\gamma_k}$, for $t \in [a,b]$, then $w \in K\left(\varphi, \theta, \nu, L, \frac{L}{\alpha_k}, \frac{L}{\gamma_k}\right)$, and $\nu(w) = \frac{L}{\gamma_k} > L$. Hence

$$\left\{u \in K\left(\varphi, \theta, \nu, L, \frac{L}{\alpha_k}, \frac{L}{\gamma_k}\right) : \nu(u) > L\right\} \neq \emptyset.$$

Consequently, it $u \in K\left(\varphi, \theta, \nu, L, \frac{L}{\alpha_k}, \frac{L}{\gamma_k}\right)$, then

$$L \leq u(b) \leq u(s) \leq u(a) \leq \frac{L}{\gamma_k}, \quad \text{and} \quad -\frac{L}{\gamma_k} \leq u^{\Delta}(s) \leq 0, \quad \text{for } s \in [a,b].$$

By condition (i) of Theorem 3.11, we have

$$f(s,u,v) > \phi_p\left(\tfrac{(1-\alpha)L}{M_1}\right) \quad \text{for } s \in [a,\eta_n] \cup [\xi_m, b].$$

Then

$$\begin{aligned} v(Au) &= Au(b) \geq \frac{\alpha}{1-\alpha}\int_{\xi_m}^b \phi_q\left(\int_a^\tau h(s) f_3\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right)\Delta\tau + \\ &\quad \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} h(s) f_3\left(s, u(s), u^{\Delta}(s)\right)\Delta s\right) \\ &\geq \frac{\alpha}{1-\alpha}\int_{\xi_m}^b \phi_q\left(\int_a^\tau \phi_p\left(\tfrac{(1-\alpha)L}{M_3}\right) h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k} \phi_p\left(\tfrac{(1-\alpha)L}{M_3}\right) h(s)\Delta s\right) \\ &= b.\end{aligned}$$

Therefore, we have

$$v\left(Au\right) > L, \qquad \text{for all } u \in K\left(\varphi, \theta, \nu, L, \frac{L}{\alpha_k}, \frac{L}{\gamma_k}\right),$$

we show that consider condition (ii) in Theorem 2.10, we suppose that $u \in K\left(\varphi, \nu, L, \frac{L}{\gamma_k}\right)$, with

$$\theta(Au) = Au\left(\xi_k\right) > \frac{L}{\alpha_k},$$

then

$$v\left(Au\right) = Au(b) \geq \alpha_k Au\left(\xi_k\right) > \alpha_k \frac{L}{\alpha_k} = L.$$

Thus, the condition (ii) of Theorem 2.10 is satisfied. Finally, we prove (iii) in 2.10 is also satisfied. Since $\psi(0) = 0 < r$, there holds that $0 \notin R(\varphi, \psi, r, \frac{L}{\gamma_k})$. Suppose that $u \in R(\varphi, \psi, r, \frac{L}{\gamma_k})$, then

$$0 \leq u(s) \leq r, \quad \text{and} \quad -\frac{L}{\gamma_k} \leq u^{\Delta}(s) \leq 0, \quad \text{for } s \in [a,b].$$

By condition (iii) of Theorem 3.11, we have $f\left(s,u,v\right) < \phi_p\left(\frac{(1-\alpha)r}{M_1}\right)$, for all $s \in [a,b]$, we have

$$\begin{aligned}\psi\left(Au\right) &= Au(a) < \frac{1}{1-\alpha}\int_a^b \phi_q\left(\int_a^\tau \phi_p\left(\tfrac{(1-\alpha)r}{M_1}\right) h(s)\Delta s\right)\Delta\tau \\ &\quad + \frac{1}{1-\alpha}\sum_{k=1}^{k=n}\delta_k\phi_q\left(\int_a^{\eta_k}\phi_p\left(\tfrac{(1-\alpha)r}{M_1}\right)h(s)\Delta s\right) \\ &= b.\end{aligned}$$

Hence, condition (iii) of Theorem 2.10 is satisfied. Thus, an application of Theorem 2.10 imply the boundary value problem (1.1) and (1.2) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_i\| \leq \frac{L}{\gamma_k}, \ \ \left\|u_i^{\Delta}\right\| \leq \frac{L}{\gamma_k}, \qquad i = 1,2,3, \qquad u_1(b) > L,$$
$$r < u_2(a) \qquad \text{with} \qquad u_2(b) < L, \qquad u_3(a) < r.$$

□

Using the ideas in the proof Theorem 3.11, we can establish the existence of an arbitrary odd number of positive solution of (1.1) and (1.2).

Corollary 3.12. *Suppose that there exists*

$$0 < r_1 < r_2 < \cdots < r_d, \qquad d \in \mathbb{N},$$

such that

(i) $f\left(s,u,v\right) < \phi_p\left(\frac{(1-\alpha)r_i}{M_4}\right)$, *for all* $(s,u,v) \in [a,b] \times \left[0, \frac{r_i}{\gamma_k}\right] \times \left[-\frac{r_i}{\gamma_k}, 0\right]$,

(ii) $f\left(s,u,v\right) > \phi_p\left(\frac{\gamma_k(1-\alpha)r_i}{M_3}\right)$, *for all* $(s,u,v) \in [a,\eta_n] \cup [\xi_m, b] \times \left[r_i, \frac{r_i}{\gamma_k}\right] \times \left[-\frac{r_i}{\gamma_k}, 0\right]$.

Then the boundary-value problem (1.1) *and* (1.2) *has at least* $2d+1$ *positive solutions.*

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