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A UNIQUENESS THEOREM FOR INVERSE PROBLEMS IN QUASILINEAR ANISOTROPIC MEDIA ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We extend the study of inverse boundary value problems for quasilinear anisotropic conductivities from Euclidean domains to compact Riemannian manifolds with boundary. Given boundary voltage and current measurements, represented by the Dirichlet-to-Neumann (DN) map, we investigate whether the quasilinear anisotropic conductivity can be uniquely determined. Our main result establishes uniqueness for quasilinear anisotropic conductivities, where the conductivity tensor is given by a scalar function multiplied by a fixed Riemannian metric. Under natural geometric conditions, such as conformal flatness or boundary rigidity of the underlying manifold, we show that this scalar factor can be uniquely determined from the boundary measurements.

Keywords. Quasilinear, Dirichlet to Neumann map, linearization, Riemannian manifold, conformal diffeomorphism
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1. Introduction and preliminaries

In this paper, we continue our investigation of inverse boundary value problems in quasilinear anisotropic media, initiated in [7, 8], extending the analysis from Euclidean domains to compact Riemannian manifolds with boundary. The primary objective is to establish uniqueness and regularity results for recovering the underlying geometric and conductivity structures from boundary measurements.

Let (M, g) be a compact Riemannian manifold with $C^{2,\alpha}$ boundary where α is between 0 and 1. Consider $A(x,t) = (a_{ij}(x,t))_{n\times n}$ be a positive definite and symmetric matrix function defined on $\overline{M} \times \mathbb{R}$, satisfying

$$A(x,t) \geq \epsilon I, \quad (x,t) \in \overline{M} \times \mathbb{R}, \quad \epsilon > 0,$$

where I denotes the identity matrix. Assume that A is in $C^{1,\alpha}(\overline{M} \times \mathbb{R})$ for α between 0 and 1, then the quasilinear boundary value problem

$$\begin{cases} \nabla_g \cdot A(x,u)\nabla_g u = 0 & \text{in } M \\ u|_{\partial M} = f & \text{on } \partial M, \end{cases} \tag{1.1}$$

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admits a unique solution u in $C^{2,\alpha}(\overline{M})$ for each boundary value f in $C^{2,\alpha}(\partial M)$ (see e.g. [2]).

The corresponding map $\Lambda_A : C^{2,\alpha}(\partial M) \to C^{1,\alpha}(\partial M)$ is called the Dirichlet-to-Neumann map, which is defined by

$$\Lambda_A : f \to \nu \cdot A(x,f)\nabla_g u|_{\partial M} \tag{1.2}$$

where ν denotes the unit outer normal on boundary ∂M and u is the solution of (1.1). Physically, $A(x,t)$ the nonlinear anisotropic conductivity of manifold M, u is the voltage potential, and the DN map $\Lambda_A(f)$ in (1.2) induced current flux at the boundary corresponding to the voltage f.

The inverse problem under consideration is to determine how much information about the coefficient matrix A can be recovered from the knowledge of the Dirichlet-to-Neumann map Λ_A.

As it was proven in [19], the Dirichlet to Neumann map Λ_A remains invariant under any diffeomorphism of M that fixes the boundary ∂M. In fact, let $\Phi : \bar{M} \to \bar{M}$ be a $C^{2,\alpha}(\bar{M})$ diffeomorphism that concise with the identity map on boundary ∂M. If we define

$$(H_\Phi A)(x,t) = \Big[\frac{(D\Phi(x))^T A(x,t)(D\Phi(x))}{|D\Phi(x)|}\Big] \circ \Phi^{-1}(x)$$

satisfies

$$\Lambda_{H_\Phi A} = \Lambda_A.$$

In this case, $D\Phi$ denotes the Jacobian matrix of Φ and $|D\Phi|$ denotes the determinant of $D\Phi$. (Note that since Φ is an identity on the boundary ∂M, Φ is an orientation-preserving map, and thus $|D\Phi| > 0$.)

Thus, in general, it is impossible to recover A from the Dirichlet to Neumann map Λ_A. The best we can hope for is to show that the invariance $H_\Phi A$ is the only obstruction to the uniqueness. In [19], the following conjecture is posed:

Conjecture 1.1. Suppose that A_1 and A_2 be quasilinear coefficient matrices in $C^{1,\alpha}(\bar{M})$ such that $\Lambda_{A_1} = \Lambda_{A_2}$. Then there exists a diffeomorphism $\Phi : \bar{M} \to \bar{M}$ in $C^{2,\alpha}(\bar{M})$ with $\Phi|_{\partial M}$ =identity such that $A_2 = H_\Phi A_1$.

In the linear case, the conjecture has been partially proven for the two-dimensional case and real-analytic case in [11, 20] when A is independent of u. For the quasilinear case, the two-dimensional case, and the real-analytic case, these results were also verified in [19]. We ask the reader to read the survey paper [22] and other recent related works in [9, 16]. The passage from the linear case to the quasilinear case is the use of two linearization procedures. Isakov first introduced the linearization technique [3, 4, 5] and applied it to the quasilinear isotropic case in [18].

One notices that there is no result in the case of the smooth (non-analytic) case with the dimension is greater than or equal to three. In [7], an argument was presented to investigate this case. We studied a special case when $A(x,t)$ is C^∞ with $n \geq 3$ and we proved the following uniqueness result in [7].

Theorem 1.2. *Let $M \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary. Let A_1 and A_2 be two quasilinear coefficient matrices of the form $A_i(x,t) = \gamma_i(x,t)A(x)$, where $A(x)$ is a C^∞ positive definite matrix and $\gamma_i(x,t)$, $i = 1,2$, are positive C^∞ scalar functions on $\bar{M}\times\mathbb{R}$. Assume $\Lambda_{A_1} = \Lambda_{A_2}$ and the conjecture for the linear case holds in the C^∞ class then $\gamma_1(x,t) = \gamma_2(x,t)$ for all (x,t) in $\bar{M} \times \mathbb{R}$.*

Then we notice that in Theorem 1.2, the obstruction created by the diffeomorphism disappears. This is because of the assumption that the unknown quasilinear anisotropic conductivity $A_i(x,u)$ takes the form of $\gamma(x,u)A(x)$, where $A(x)$ is a known linear matrix. In a sense, this problem becomes a quasilinear isotropic problem in an anisotropic medium. The significance of this result is that it generalizes the quasilinear isotropic results proven in [18] to the quasilinear isotropic case involving an anisotropic medium.

Another important aspect of Theorem 1.2 lies in the structure of its proof. In [7], the key idea is to apply a linearization argument together with the assumption that Conjecture 1.1 holds in the linear case, thereby reducing the problem to one involving a conformal diffeomorphism that fixes the boundary. The argument then employs Ferrand's result on the action of conformal diffeomorphism on manifold [13], we were able to show that the conformal diffeomorphism involved in our problem must be an isometry that leaves the boundary fixed, and by the classical Myers-Steenrod Theorem [17], such an isometry must be an identity, from which we can derive the uniqueness result.

However, the use of Ferrand's theorem in our previous approach required assuming that $A(x,t)$ is of class C^∞. As noted in [8], this smoothness condition appears to be a technical requirement rather than an essential one. It is therefore desirable to establish the result under weaker regularity assumptions. In view of the coefficient regularity typically imposed in quasilinear elliptic equations and the requirements of the linearization argument, we consider $A(x,u)$ to be of class $C^{2,\alpha}$. Then we prove the following uniqueness theorem for $C^{2,\alpha}$ smoothness on $A(x,t)$ in [8].

Theorem 1.3. *Let $M \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and the boundary is in the class of $C^{2,\alpha}$. Suppose that A_1 and A_2 be two quasilinear coefficient matrices with $A_i(x,t) = \gamma_i(x,t)A(x)$, where $A(x) \in C^{2,\alpha}$ is a positive definite matrix and $\gamma_i(x,t) \in C^{2,\alpha}$, $i = 1,2$, are positive scalar functions on $\bar{M} \times \mathbb{R}$. Assume $\Lambda_{A_1} = \Lambda_{A_2}$ and the conjecture for the linear case holds, then $\gamma_1(x,t) = \gamma_2(x,t)$ for all (x,t) in $\bar{M} \times \mathbb{R}$.*

Similarly, we notice that in Theorem 1.3, the obstruction created by the diffeomorphism disappears because of the assumption that the unknown quasilinear anisotropic conductivity $A_i(x,u)$ takes the form of $\gamma(x,u)A(x)$ for known linear matrix $A(x)$ in $C^{2,\alpha}$. In this way, the problem essentially reduces to a quasilinear isotropic one within an anisotropic medium. The importance of this result is that it extends the quasilinear isotropic results obtained in [18] to the quasilinear isotropic setting in an anisotropic medium.

A distinctive aspect of Theorem 1.3 lies in the nature of its proof. In [8], the main idea was to use less regularity for the quasilinear coefficient metric $A(x,t)$.

By applying the result of Liimatainen and Salo's [15], we established that the associated conformal diffeomorphism is of class $C^{3,\alpha}$ on the entire manifold M. Subsequently, using Ferrand's original theorem on the action of conformal diffeomorphisms on compact manifolds with only C^1 regularity [14], we demonstrated that this conformal diffeomorphism must in fact be the identity map on M. Thus, our problem reduces to identifying an isometry that fixes the boundary. By the classical Myers–Steenrod Theorem [17], any such an isometry can only be the identity, which consequently establishes the uniqueness result.

The purpose of this paper is to generalize these results from Euclidean domains to compact Riemannian manifolds with boundary. Our main theorem shows that if the conductivity takes the form

$$A(x,t) = \gamma(x,t)g(x)$$

where $g(x)$ is a given Riemannian metric tensor, then the scalar function $\gamma(x,t)$ can be uniquely determined by the Dirichlet-to-Neumann map, provided either (i) (M,g) is conformally flat, or (ii) the anisotropic linear Calderón problem on (M,g) admits uniqueness up to boundary-fixing diffeomorphism. This result generalizes the Euclidean uniqueness theorems in [7, 8] to a geometric setting and highlights the role of conformal geometry in quasilinear inverse problems. We investigate that, assuming Conjecture 1.1 holds in the linear case, one can establish the following uniqueness theorem for the quasilinear coefficients of the form $A(x,t) = \gamma(x,t)g(x)$, where $g(x)$ is a Riemannian metric tensor.

Theorem 1.4. *Suppose (M,g) be a compact Riemannian manifold with class of $C^{2,\alpha}$ for $n \geq 2$. Let A_1 and A_2 be two quasilinear coefficient matrices of the form $A_i(x,t) = \gamma_i(x,t)g(x)$, where $g(x)$ is a $C^{2,\alpha}$ compact Riemannian metric tensor and $\gamma_i(x,t)$, $i = 1,2$, are positive $C^{2,\alpha}$ scalar functions on $\bar{M} \times \mathbb{R}$. Assume $\Lambda_{A_1} = \Lambda_{A_2}$ and the conjecture for the linear case holds, then $\gamma_1(x,t) = \gamma_2(x,t)$ for all (x,t) in $\bar{M} \times \mathbb{R}$.*

This theorem shows that the global uniqueness of the scalar factor $\gamma(x,t)$ can be established on general compact Riemannian manifolds, provided one assumes uniqueness in the corresponding linear anisotropic case. Thus, the quasilinear anisotropic inverse problem is reduced to the well-studied anisotropic media on manifolds. The linear anisotropic Calderón problem on compact Riemannian manifolds has been extensively studied in [10, 11, 21]. Theorem 1.4 shows that, assuming the conjectured uniqueness for the linear anisotropic case holds on (M,g), one can also establish uniqueness for the quasilinear anisotropic conductivities of the form $A(x,t) = \gamma(x,t)g(x)$. This result reduces the nonlinear inverse problem to the corresponding linear case, thereby connecting quasilinear uniqueness to the classical anisotropic media. The conformally flat case plays a special role in inverse problems on Riemannian manifolds. In this framework, the anisotropic conductivity problem can be transformed into an equivalent isotropic problem through a conformal change of variables, which enables the application of classical uniqueness results established for the isotropic case. Accordingly, we obtain the following theorem.

Theorem 1.5. *Let (M,g) be a compact Riemannian manifold with class of $C^{2,\alpha}$ for $n \geq 2$. Suppose A_1 and A_2 be two quasilinear coefficient matrices with $A_i(x,t) = \gamma_i(x,t)g(x)$, where $g(x) \in C^{2,\alpha}$ is a compact Riemannian metric tensor and $\gamma_i(x,t) \in C^{2,\alpha}$ for $i = 1,2$, are positive scalar functions on $\bar{M} \times \mathbb{R}$. Let $\Lambda_{A_1} = \Lambda_{A_2}$ and (M,g) be conformally equivalent to the Euclidean metric, then for all $(x,t) \in \bar{M} \times \mathbb{R}$, $\gamma_1(x,t) = \gamma_2(x,t)$.*

Theorem 1.5 establishes that if (M,g) is conformally equivalent to the Euclidean metric, then the quasilinear scalar factor $\gamma(x,t)$ is uniquely determined by the corresponding Dirichlet-to-Neumann map. Thus, in conformally flat manifolds, the nonlinear inverse problem inherits the same uniqueness property as in the Euclidean isotropic case. This result extends the Euclidean theory to a geometric setting, highlighting the role of conformal geometry in quasilinear problems. The proof of Theorem 1.5 is not difficult. In the second half of this paper, we will present the direct computation to illustrate the proof.

Remark 1.6. For $g(x) = I$, the theorems 1.4 and 1.5 reduce to the isotropic quasilinear results proven in [18]. The assumptions on conformal flatness or linear uniqueness represent two natural geometric frameworks in which the diffeomorphism invariance can be eliminated, allowing for global uniqueness of the quasilinear scalar factor.

2. Proof of Theorem 1.4

Proof. Let (M,g) be a compact $C^{2,\alpha}$ Riemannian manifold with boundary, $n \geq 2$. Suppose two quasilinear coefficients in the class of $C^{2,\alpha}$ is given by

$$A_i(x,t) = \gamma_i(x,t)g(x), \quad \gamma_i \in C^{2,\alpha}(\overline{M} \times \mathbb{R}),\ \gamma_i > 0, \quad i = 1,2, \tag{2.1}$$

with DN map $\Lambda_{A_1(x,u)} = \Lambda_{A_2(x,u)}$, we shall show $\gamma_1(x,t) = \gamma_2(x,t)$ for all (x,t) in $\bar{M} \times \mathbb{R}$. Assume $t \in \mathbb{R}$ be a fixed constant and f be in $C^{3,\alpha}(\partial M)$ class. Consider the maps

$$s \to \Lambda_{A_i(x,u)}(t + sf), \quad i = 1,2$$

is differentiable at $s = 0$ and

$$\frac{d}{ds}|_{s=0}\Lambda_{A_i}(t + sf) = \Lambda_{A_i(\cdot,f)}(f)$$

where $\Lambda_{A_i(\cdot,t)}$ is Dirichlet-to-Neumann map of the linear elliptic operator $\nabla_g \cdot (A_i(x,t)\nabla_g \cdot)$. It was shown in [18] that the above maps are differentiable in s and the derivative is given by

$$\lim_{s\to 0}\frac{1}{s}\Lambda_{A_i(x,u)}(t + sf) = \Lambda_{A_i(x,t)}(f), \quad i = 1,2.$$

Note that since t is a constant, we have $\Lambda_{A_i(x,u)}(t) = 0$. Also, the convergence is in the $H^{\frac{1}{2}}(\partial M)$ topology, and the Dirichlet to Neumann map is $\Lambda_{A_i(x,t)}$ for the linear equation $\nabla_g \cdot A_i(x,t)\nabla_g u = 0$ with t fixed. In the following, we shall use the Dirichlet to Neumann map $\Lambda_{A_i(x,u)}$ for the quasilinear equation and the Dirichlet to Neumann map $\Lambda_{A_i(x,t)}$ for the linearized equation.

Since $\Lambda_{A_1(x,u)} = \Lambda_{A_2(x,u)}$, we thus have

$$\Lambda_{A_1(x,t)} = \Lambda_{A_2(x,t)}$$

for each fixed t. Since we have assumed that the conjecture is true for the linear case, we have that for each t there exists a $C^{3,\alpha}$ diffeomorphism $\Phi^t(x) : M \to M$, fixing the boundary ∂M, such that

$$A_2(x,t) = H_{\Phi^t} A_1(x,t).$$

Then,

$$A_2(x,t) = \Big[\frac{(D\Phi^t(x))^T A_1(x,t)(D\Phi^t(x))}{|D\Phi^t(x)|}\Big] \circ \Phi^{-1}(x). \tag{2.2}$$

Applying (2.1) into (2.2), we have

$$\gamma_2(x,t)g(x) = \frac{(D\Phi^t(x))^T \big(\gamma_1(\Phi^t(x),t)g(\Phi^t(x))\big)(D\Phi^t(x))}{|D\Phi^t(x)|}. \tag{2.3}$$

We shall unify the analysis for the two-dimensional case with that of the three- or higher-dimensional cases. Taking the inverse on both sides of (2.3) implies

$$|D\Phi^t(x)|\gamma_1^{-1}(x,t)g^{-1}(x) = (D\Phi^t(x))\gamma_2^{-1}(\Phi^t(x),t)g^{-1}(\Phi^t(x))(D\Phi^t(x))^T.$$

So that,

$$\Big(|D\Phi^t(x)|\gamma_1^{-1}(x,t)\gamma_2(\Phi^t(x),t)\Big)g^{-1}(x) = (D\Phi^t(x))g^{-1}(\Phi^t(x))(D\Phi^t(x))^T. \tag{2.4}$$

From equation (2.4), it follows

$$g(x) = \left(\frac{\gamma_1(\Phi^t(x),t)}{\gamma_2(x,t)|D\Phi^t(x)|}\right)(D\Phi^t(x))g(\Phi^t(x))(D\Phi^t(x))^T \tag{2.5}$$

Equivalently,

$$g(x)\rho^t(x) = (D\Phi^t(x))g(\Phi^t(x))(D\Phi^t(x))^T, \tag{2.6}$$

where

$$\rho^t(x) = \frac{\gamma_2(x,t)}{\gamma_1(\Phi^t(x),t)}|D\Phi^t(x)|$$

is the scalar conformal factor.

It follows from the above result in (2.6) that $\Phi^t(x)$ is a conformal diffeomorphism of the $C^{2,\alpha}$ Riemannian manifold $(\bar{M}, g^{-1}(x))$ onto itself that fixes the boundary ∂M as $g^{-1}(x)$ defines a Riemannian metric on manifold M. It follows, we show that $\Phi^t(x)$ is identity map on M, i.e.,

$$\Phi^t(x) = x \text{ for all } x \in M \text{ and } t \in \mathbb{R}. \tag{2.7}$$

With this established, equation (2.5) yields $\gamma_1(x,t) = \gamma_2(x,t)$ for all (x,t) establishing uniqueness. Consequently, proving uniqueness now reduces to verifying (2.7), that is, demonstrating that any $C^{3,\alpha}$ conformal diffeomorphism that fixes the boundary ∂M must coincide with the identity map throughout the entire domain M.

In our earlier work [7, 8], we applied Ferrand's theorem [12, 13] to establish the rigidity of conformal diffeomorphisms. However, that argument required assuming the background metric is C^∞, since Ferrand's modern theorem applies only in the smooth category. While this result fully resolves the Lichnerowicz conjecture in both compact and complete cases, its dependence on C^∞ regularity is too strong for the quasilinear problem considered here, where we only assume metrics of class $C^{2,\alpha}$.

In the present paper, we avoid the C^∞ assumption by using Ferrand's original compact case theorem [14], which requires only C^1 regularity. The trade-off is that we must expand our domain into a compact manifold (rather than into a complete one as in [7]), but this adjustment is sufficient for our purposes. The key observation is that we may smoothly extend the metric g^{-1} from the original domain M to a compact C^∞ manifold $\mathcal{M}$ containing M, while preserving the $C^{2,\alpha}$ regularity class. Similarly, the conformal diffeomorphism $\Phi^t(x)$ can be extended by the identity outside M.

To guarantee the higher regularity of $\Phi^t(x)$ across the boundary, instead of invoking Ferrand's regularity result as in [7], we now rely on the theorem of Liimatainen and Salo [15], which shows that a C^1 conformal diffeomorphism between C^r Riemannian manifolds $(r > 1)$ is automatically C^{r+1}. This ensures that the extended $\Phi^t(x)$ belongs to $C^{3,\alpha}$, which is precisely what is needed in our argument.

The novelty of our present approach is that the use of Ferrand's original compact result, together with the Liimatainen–Salo regularity theorem, allows us to remove the unnecessary C^∞ smoothness assumption from [7, 8] and work directly in the natural $C^{2,\alpha}$ category.

Now we use Liimatainen and Salo's result [15] to prove each Φ^t is $C^{3,\alpha}$ (in fact at least $C^{2,\alpha}$) on $\bar{M}$. The result is presented below.

Theorem 2.1. *Let (U, g) and (V, h) be Riemannian manifolds such that $g, h \in C^r$, $r > 1$. If $\phi : U \to V$ is a C^1 diffeomorphism that is conformal in the sense that $\phi^* h = cg$ for some continuous positive function c, then ϕ is a C^{r+1} diffeomorphism from U onto V.*

The diffeomorphism Φ^t is in C^1 shown in [7]. Since Φ^t is conformal and $C^{2,\alpha}$, theorem 2.1 imply that Φ^t is C^{r+1}, r is the Hölder regularity of the metric(here r=2), on $\mathcal{M}$.

We now show that $\Phi^t(x)$ must coincide with the identity map on the entire manifold $\mathcal{M}$. To do this, we use the original result of Ferrand's paper [14] for the actions of conformal diffeomorphism in a compact manifold with regularity of C^1.

To remove the diffeomorphism obstruction we follow the compactification argument used in [7]. Choose a compact manifold M in C^∞ such that

(1) M embeds into $\mathcal{M}$ as an open subset, and
(2) $\mathcal{M}$ is not conformally equivalent (indeed not homeomorphic) to the sphere S^n.

Such a manifold $\mathcal{M}$ can be constructed by attaching a collar and gluing to any compact manifold not homeomorphic to S^n. Extend the metric g from M

to a $C^{2,\alpha}$ Riemannian metric (still denoted by g) on all of $\mathcal{M}$. Extend Φ^t to a homeomorphism $\widetilde{\Phi}^t : \mathcal{M} \to \mathcal{M}$ of $\mathcal{M}$ by defining $\widetilde{\Phi^t} = \Phi^t$ on M and $\widetilde{\Phi^t} = id$ on $\mathcal{M} \setminus M$. By construction $\widetilde{\Phi^t}$ on $\mathcal{M}$ and is conformal on M and identity outside M. Using the regularity, one checks that $\widetilde{\Phi^t}$ is $C^{3,\alpha}$ accros ∂M and thus globally $C^{3,\alpha}$ on $\mathcal{M}$.

Now apply the following classical Ferrand's theorem [14]:

Theorem 2.2. *Let $C(\mathcal{M})$ be the whole conformal group of a C^1 compact Riemannian manifold $\mathcal{M}$ with dimension $n \geq 2$. If $\mathcal{M}$ is not conformally equivalent with S^n, then $C(\mathcal{M})$ is inessential, i.e. can be reduced to a group of isometries by a conformal change of metric.*

Remark 2.3. For a historical overview of the work on this problem, we also direct the reader to the introduction of D.V. Alekseevskff [1].

Apply Ferrand's theorem to $(\mathcal{M}, g^{-1})$: there exists a positive conformal factor $\alpha(x)$ in $C^{2,\alpha}$ such that the conformal group of $(\mathcal{M}, \alpha g^{-1})$ equals its isometry group. In particular, $\widetilde{\Phi^t}$ becomes an isometry of $(\mathcal{M}, \alpha g^{-1})$. Restricting back to $\mathcal{M}$, $\widetilde{\Phi^t}$ is therefore an isometry of $(\mathcal{M}, \alpha g^{-1})$ and $\Phi^t|_{\partial M} = id$.

Finally, we show that any isometry of a connected Riemannian manifold that fixes the boundary pointwise must be the identity. There are several standard arguments; we present a short one suited to our regularity class. The Myers–Steenrod theorem [17] implies that the isometry group of a connected C^1 Riemannian manifold is a Lie group of diffeomorphisms; in particular, any isometry is C^1. Consider Φ^t as an isometry of $(\mathcal{M}, \alpha g^{-1})$ which fixes the closed set $\mathcal{M} \setminus M$ pointwise (because we extended $\widetilde{\Phi^t}$ to be identity outside M). By analytic continuation-like arguments for isometries (or simply by the fact that an isometry fixing an open set must be the identity), $\widetilde{\Phi^t}$ is the identity map on $\mathcal{M}$. One quick route: because $\widetilde{\Phi^t}$ fixes the interior of an embedded submanifold of codimension zero, compute its differential at a boundary point — it must be the identity, and the isometry then has derivative identity at some point; by a standard open-closed argument for fixed points of isometries — see Theorem 2.4 in [8] — the isometry must be identity everywhere. Restricting back to M we obtain $\Phi^t = id$. This completes our proof. □

Remark 2.4. It is worth noting that the use of the Myers–Steenrod theorem is not required in the C^1 category, since Theorem 2.4 in [8], which was proved in [6], already establishes the corresponding result under this regularity assumption.

3. Proof of Theorem 1.5

In this section, we investigate uniqueness in the conformally flat case. The argument proceeds through three main steps: linearization of the quasilinear problem, reduction to a scalar isotropic conductivity in Euclidean coordinates, and application of the classical uniqueness result for the isotropic Calderón problem.

Lemma 3.1. *Let $A_i(x,t) = \gamma_i(x,t)g(x)$ with $\gamma_i \in C^{2,\alpha}(\overline{M} \times \mathbb{R})$, $i = 1,2$. If $\Lambda_{A_1} = \Lambda_{A_2}$ (after the linearization), then for each fixed $t \in \mathbb{R}$ we have*

$$\Lambda_{s_1(y,t)} = \Lambda_{s_2(y,t)},$$

where $\Lambda_{s_i(y,t)}$ is the Dirichlet-to-Neumann map for the linear operator $\nabla_g \cdot (s_i(y,t)\nabla_g \tilde{u}_i) = 0$ in $\tilde{M}$.

Proof. Choose the test function $\varphi(x) \in C^\infty(\bar{M})$. Assume u_i for $i = 1,2$ be the solution with boundary value $f \in C^{3,\alpha}(\partial M)$, then the weak form of the linearized equation will have the boundary term,

$$\int_{\partial M} h\, \Lambda_{A_i(x,t)}(f)\, ds = \int_M \nabla\varphi \cdot A_i(x,t)\nabla u_i\, dx$$

where $h = \varphi|_{\partial M}$. By Lemma 3.2, we have

$$\int_M \nabla\varphi \cdot A_i(x,t)\, \nabla u_i\, dx = \int_{\tilde{M}} \nabla_y \tilde{\varphi} \cdot s_i(y,t)\, \nabla_y \tilde{u}_i\, dy \tag{3.1}$$

using change of variables $x \to \Psi(x)$. Here, $\tilde{\varphi} = \varphi \circ \Psi^{-1}$ and $\tilde{u}_i$ is the unique solution to

$$\begin{cases} \nabla \cdot s_i(y,t)\nabla \tilde{u}_i = 0 & \text{in } \tilde{M} \\ \tilde{u}_i|_{\partial \tilde{M}} = \tilde{f} = f \circ \Psi^{-1}(y) \in C^{3,\alpha}(\partial \tilde{M}). \end{cases}$$

Applying the Divergence theorem on (3.1),

$$\int_{\tilde{M}} \nabla_y \tilde{\varphi} \cdot s_i(y,t)\nabla_y \tilde{u}_i\, dy = \int_{\partial \tilde{M}} \tilde{h}\, \Lambda_{s_i(y,t)}(\tilde{f}) ds \tag{3.2}$$

where $\tilde{h}(y) = h \circ \Psi^{-1}(y)$. From (3.1) and (3.2), we get

$$\int_{\partial M} h\, \Lambda_{A_i(x,t)}(f)\, ds = \int_{\partial \tilde{M}} \tilde{h}\, \Lambda_{s_i(y,t)}(\tilde{f})\, ds, \quad i = 1,2,$$

for all $h \in C^\infty(\partial M)$ and $f \in C^{3,\alpha}(\partial M)$.
Since

$$\Lambda_{A_1(x,t)} = \Lambda_{A_2(x,t)},$$

we can write

$$\Lambda_{A_1(x,t)}(f) = \Lambda_{A_2(x,t)}(f).$$

So

$$\int_{\partial \tilde{M}} \tilde{h} \Lambda_{s_1(y,t)}(\tilde{f}) ds = \int_{\partial \tilde{M}} \tilde{h} \Lambda_{s_2(y,t)}(\tilde{f})$$

for all $\tilde{h} \in C^\infty(\partial \tilde{M})$ and $\tilde{f} \in C^{3,\alpha}(\partial \tilde{M})$, and we obtain

$$\Lambda_{s_1(y,t)} = \Lambda_{s_2(y,t)} \text{ in } \tilde{M}.$$

This completes the proof. □

Lemma 3.2. *Suppose (M,g) is conformally equivalent to the Euclidean metric. So, there exists a diffeomorphism $\Psi : M \to \widetilde{M} \subset \mathbb{R}^n$ in $C^{2,\alpha}$ and a positive function $\beta(x)$ such that*

$$(D\Psi(x))^T I (D\Psi(x)) = \beta(x)\, g^{-1}(x).$$

In these coordinates, let t be a constant and $u_i(x,t)$ be a solution to the linearized equation $\nabla \cdot A_i(x,t)\nabla u_i = 0$ with the coefficient $A_i(x,t) = \gamma_i(x,t)g(x)$, $i = 1,2$ and let $\tilde{u}_i(y,t) = u_i(\Psi^{-1}(y),t)$. Then, $\tilde{u}_i(y,t)$ solves the transformed equation $\nabla \cdot s_i(x,t)\nabla\tilde{u}_i = 0$ in $\tilde{M}$ with the scalar coefficient

$$s_i(y,t) = \frac{\gamma_i(\Psi^{-1}(y),t)\,\beta(\Psi^{-1}(y))}{|D\Psi(\Psi^{-1}(y))|}, \qquad y \in \widetilde{M}. \tag{3.3}$$

Moreover, the DN maps transform consistently:

$$\Lambda_{s_i(\cdot,t)}(f\circ\Psi^{-1}) = (\Lambda_{A_i(\cdot,t)}f)\circ\Psi^{-1}, \qquad f \in C^{2,\alpha}(\partial M).$$

Proof. In this case, we choose a test function $\varphi(x) \in C_0^\infty(\bar{M})$. We now consider the weak form of the equation. For each $i = 1,2$, we write

$$\int_M \varphi\ \nabla \cdot A_i(x,t)\nabla u_i\,dx = 0.$$

By the Divergence theorem since $\varphi(x) \in C_0^\infty(\bar{M})$, we have

$$\int_M \nabla\varphi\ A(x,t)\ \nabla u_i\,dx = 0.$$

Substitute the variable x by $\Psi^{-1}(y)$ and dx by $|D\Psi|^{-1}dy$, it follows from the above integral that

$$\int_{\tilde{M}} \frac{(\nabla_x\varphi)\circ\Psi^{-1}\cdot A\circ\Psi^{-1}\cdot(\nabla_x u_i)\circ\Psi^{-1}}{|D\Psi|\circ\Psi^{-1}}\,dy = 0$$

for $\tilde{M} = \Psi(M)$.
As

$$(\nabla_x u_i)\circ\Psi^{-1} = (D\Psi)\circ\Psi^{-1}\nabla_y\tilde{u}_i \quad \text{for} \quad \tilde{u}_i(y,t) = u_i(\Psi^{-1}(y),t),$$

we get

$$\int_{\tilde{M}} \frac{(\nabla_y\tilde{\varphi})\cdot(D\Psi)^T\circ\Psi^{-1}\ (A\circ\Psi^{-1})\ (D\Psi)\circ\Psi^{-1}\ (\nabla_y\tilde{u}_i)}{|D\Psi|\circ\Psi^{-1}}\,dy = 0,$$

where

$$\tilde{\varphi}(y) = \varphi\circ\Psi^{-1}(y).$$

Since $A_i(x,t) = \gamma_i(x,t)g(x)$ and $\beta(x)g^{-1}(x) = (D\Psi(x))I(D\Psi(x))^T$, we get

$$\int_{\tilde{M}} (\nabla_y\tilde{\varphi})\gamma_i(x,t)\beta(x)\frac{(D\Psi(x))^T\big((D\Psi(x))^T\big)^{-1}I\big(D\Psi(x)\big)^{-1}D\Psi(x)}{|D\Psi(x)|}\circ\Psi^{-1}\ \nabla_y\tilde{u}_i\,dy = 0.$$

So we have,

$$\int_{\tilde{M}} \nabla_y\tilde{\varphi}\ s_i(y,t)\ \nabla_y\tilde{u}_i\,dy = 0$$

where

$$s_i(y,t) = \gamma_i(x,t)\beta(x)\frac{1}{|D\Psi(x)|}\circ\Psi^{-1}.$$

Thus,

$$\int_{\tilde{M}} \tilde{\varphi}\ \nabla_y\cdot s_i(y,t)\ \nabla_y\tilde{u}_i\,dy = 0$$

which implies that

$$\nabla_y \cdot s_i(y,t)\ \nabla_y \tilde{u}_i = 0,\ i = 1, 2.$$

This finishes our proof. $\square$

Lemma 3.3. *If $\Lambda_{s_1(\cdot,t)} = \Lambda_{s_2(\cdot,t)}$ for scalar conductivities $s_1, s_2 \in C^{2,\alpha}(\overline{\widetilde{M}})$ with $s_i > 0$, then $s_1 = s_2$ in $\widetilde{M}$.*

This is the classical Calderón uniqueness result in the isotropic case: for $n \geq 3$, it follows from the complex geometric optical construction of Sylvester and Uhlmann [21], and for $n = 2$ from the Isokov and Nachman reconstruction scheme [4].

Completion of the proof of Theorem 1.5.

By Lemma 3.1 we have $\Lambda_{A_1(\cdot,t)} = \Lambda_{A_2(\cdot,t)}$ for each fixed t. Applying Lemma 3.2, this implies $\Lambda_{s_1(\cdot,t)} = \Lambda_{s_2(\cdot,t)}$. By the Lemma 3.3 then yields $s_1(y,t) = s_2(y,t)$ in $\widetilde{M}$. From the explicit relation between s_i & γ_i and (3.3), it follows that

$$\gamma_1(x,t)\beta(x)|D\Psi(x)|^{-1} \circ \Psi^{-1}(y) = \gamma_2(x,t)\beta(x)|D\Psi(x)|^{-1} \circ \Psi^{-1}(y).$$

It follows that

$$\gamma_1(x,t) = \gamma_2(x,t), \qquad (x,t) \in \overline{M} \times \mathbb{R}.$$

This proves Theorem 1.5.

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