Gulf Journal of Mathematics ISSN: 2309-4966
Vol 21, Issue 2 (2025) 13-35 https://doi.org/10.56947/gjom.v21i2.3617

EXISTENCE RESULTS FOR SOME NONCOERCIVE PARABOLIC PROBLEMS WITH LOWER ORDER TERMS AND MEASURE DATA

MOHAMMED ABDELLAOUI[1]*

Abstract. We study the existence and properties of renormalized solutions for a class of nonlinear and noncoercive parabolic problems with lower-order terms and measure data, under suitable assumptions on the operator.

Keywords. A priori estimates; non coercive parabolic problems; lower order terms; measure data
2020 Mathematics Subject Classification. Primary 35B45, 35B65, 47H05; Secondary 28A12

1. Introduction

The present work is devoted to existence results for nonlinear and noncoercive parabolic problems with lower order terms. In particular, our main goal concerns the proof of existence of a renormalized solution for problems

$$\begin{cases} u_t - \operatorname{div}\left[a(t,x,u,\nabla u) + \Phi(t,x,u)\right] + g(t,x,u,\nabla u) = \mu \text{ in } Q := (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, $T > 0$, $\zeta \in \mathbb{R}^N \mapsto a(t,x,s,\zeta) \in \mathbb{R}^N$ is a monotone and coercive operator increases as $|s|^{p-1}$, $s \in \mathbb{R} \mapsto \Phi(t,x,s) \in \mathbb{R}^N$ increases as $|s|^{p-1}$, while $\zeta \in \mathbb{R}^N \mapsto g(t,x,s,\zeta) \in \mathbb{R}$ increases as $|\zeta|^p$ and satisfies a sign condition with respect to s. Here, p is a given exponent greater than 1 and p' its conjugate (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$), $u_0 \in L^1(\Omega)$ and μ is a *Radon* measure with bounded variation over Q that do not charge the sets of zero parabolic p-capacity. The combination of the *noncoercive gradient-dependent term* $g(t,x,u,\nabla u)$ and the *non-divergence lower-order term* $\Phi(t,x,u)$ gives rise to significant analytical difficulties. The lack of coercivity induced by g prevents the direct use of standard energy estimates and monotonicity arguments typically employed in the Leray-Lions setting. At the same time, the presence of $\Phi(t,x,u)$, which appears in the divergence operator, destroys the natural variational structure of the problem and complicates the formulation of weak solutions. These two effects interact in

Date: Received: Sep 28, 2025; Revised: Oct 26, 2025; Accepted: Nov 9, 2025.
* Corresponding author
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a nontrivial way, making it difficult to control the nonlinear terms simultaneously in the absence of coercivity. Moreover, when the right-hand side μ is a measure, the low regularity of the data further aggravates these challenges, since standard compactness and duality tools cannot be directly applied.

There is a clear connection between this problem and the possibility of finding a sense to the solution when the forcing term is a measure. For instance, if $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, problem (1.1) admits a unique solution in an appropriate energy space and in $C(0,T;L^2(\Omega))$ (see [17, 16]); Furthermore, if μ be a bounded Radon measure on Q vanishing on sets of zero parabolic p-capacity, the concepts of entropy and renormalized solutions can be defined in order to guarantee the existence and uniqueness of a solution (see [22, 8, 20, 14] for more details)[1,2]. In our setting, let $\mu \in \mathcal{M}(Q)$ (the space of *Radon* measures vanishing on sets with zero parabolic p-capacity) and $u_0 \in L^1(\Omega)$, a measurable function v is said to be a *renormalized* solution of problem (1.1) if there exists a decomposition $\mu = f - \operatorname{div}(F)$, with $f \in L^1(Q)$ and $F \in L^{p'}(Q)^N$, such that $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $q < p - \frac{N}{N+1}$, $T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ for every $k > 0$, and for every $S \in W^{2,\infty}(\mathbb{R})$ ($S(0) = 0$) such that S' has compact support on $\mathbb{R}$, we have for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ with $\varphi(T,x) = 0$ and $S'(v)\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$

$$\begin{aligned}
&-\int_\Omega S(u_0)\varphi(0)dx - \int_0^T \langle \varphi_t, S(v)\rangle dt + \int_Q S'(v)[a(t,x,\nabla v) + \Phi(t,x,v)]\cdot\nabla\varphi dxdt \\
&\quad + \int_Q S''(v)[a(t,x,v,\nabla v) + \Phi(t,x,v)]\cdot\nabla v\varphi dxdt + \int_Q g(t,x,v,\nabla v)S'(v)\varphi dxdt \\
&= \int_Q S'(v)\varphi d\mu
\end{aligned} \tag{1.2}$$

Moreover,

$$\lim_{n\to+\infty} \frac{1}{n}\int_{\{n\le|v|\le 2n\}} |\nabla v|^p dxdt = 0. \tag{1.3}$$

In fact, such a notion of solution turns out to be stronger than the distributional one and it is very useful in order to face problems that involve irregular data. Indeed, the peculiarity of such solutions is that they do not have finite energy, but a priori estimates show that their truncations belong to the energy space; the idea is then to focus the attention to a family of problems solved by such truncations. Attempts to provide a formulation, different from the weak one, allowing for both existence and uniqueness has been proposed in [5, 12] where the notions of entropy and renormalized solutions[3] has been respectively introduced. Both these

[1]According to [15], these two notions of solutions coincide.

[2]It should be noted that these solutions do not belong to the energy space while their truncates $T_k(u)$ are elements of $L^p(0,T;W_0^{1,p}(\Omega))$ (here $T_k(s) = \max(-k,\min(k,s))$).

[3]Both these definitions ask for solutions in the energy space and use a weak formulation of the equation where nonlinear test functions depending on u are used to the equation on the subset where u is bounded.

approaches are able to get uniqueness provided μ belongs to $\mathcal{M}_0(Q)$ (the space of *Radon* measures vanishing on the sets of zero parabolic p-capacity). With respect to measures, this restriction is directly related to the notion of capacity, as it was proved in [7]. The notion of entropy solution cannot be directly generalized to the case of a general, possibly singular, measure in $\mathcal{M}(Q)$. However, through the notion of renormalized solutions, initially introduced in [13] for first order hyperbolic equations, and later developed in numerous papers (see [6, 18, 8]), extending this concept to general measure data, and some interesting partial uniqueness results are also established under slightly stronger assumptions on the equation. These results suggest that the behavior of the solution's energy on the set where it becomes large plays a key role in the notion of renormalized solution[4].

In the parabolic framework and under the general assumptions that μ and u_0 are assumed to be bounded measures, the existence of a distributional solution was established in [6][5]. Unfortunately, as in the elliptic setting, the insufficient regularity of the solution means that this distributional formulation cannot provide uniqueness [23, 21][6]. In [14, 19, 1], existence and uniqueness results are extended to a more general class of measures which does not charge the sets of zero p-capacity and to singular measures. Precisely, in [14] the authors developed the theory of capacity related to parabolic operators and its relationship with measures using the concept of renormalized solutions. The strategy we use in order to prove our new existence results relies on the combination of both a priori estimates and compactness results in suitable *Sobolev* spaces for solutions of certain approximating problems; so we have to consider "good" assumptions on a, Φ and g and to use test functions in a convenient way, we are also able to prove uniqueness of the solution by assuming a strict monotonicity and independence on the solution. The outline of the paper is as follows: After giving the notations and some preliminary results we consider the assumptions on the operator, the lower order term and the initial/right-hand side data. We study the properties of renormalized solution, obtaining the so-called "a priori estimates", and we investigate the convergence properties of the approximate solutions, and in particular, the asymptotic behavior results. We finally give the main result and its proof (referring to some already published papers for classical results).

Notations: Throughout the paper, $\|\cdot\|_E$ denotes the norm in the Banach space E, and the notation $x_n \rightharpoonup x$ stands for the weak convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x in E. We denote, for any $p \in \mathbb{R}$, $1 < p < N$, $p' = \frac{p}{p-1}$ and $p^\star = \frac{Np}{N-p}$

[4]it should be noted that, in the linear case, the renormalized solution is unique for any measure in the space of *Radon* measures, since it coincides with the corresponding duality solution

[5]through an approach involving problems with regular data combined with compactness arguments

[6]In the linear setting, one can overcome this difficulty by considering the solution in a duality framework; Nonetheless, in the case of nonlinear operators the generalized concept of renormalized and entropy solutions need to be defined to get a well-posed problem, this was done independetntly in [8, 22] (see also [4]) where these two notions where introduced

the conjugate and the Hölder exponents. Moreover, we denote by $W_0^{1,p}(\Omega)$, for $1 < p < \infty$, the usual *Sobolev* space, by $W^{-1,p'}(\Omega)$ its dual; by $\langle \cdot, \cdot \rangle$ we mean either the scalar product on $\mathbb{R}^N$ of the duality between a *Banach* space and its dual (we mean the duality between suitable spaces in which functions are involved). We also denote by $\mathcal{M}_b(Q)$ the space of *Radon* measures on a class $\mathcal{B}(Q)$ of the Borel subsets of Q whose total variation is bounded in Q and by $\mathcal{M}_b^p(Q)$ the subspace of $\mathcal{M}_b(Q)$ whose elements vanish on the sets of zero p-capacity, by $\mathcal{M}_b^{p,+}(Q)$ the nonnegative elements of $\mathcal{M}_b^p(Q)$. Besides, we shall denote by $\varphi_\lambda(s)$ the function $\varphi_\lambda(s) = se^{\lambda s^2}$ that enjoys the following property

$$\forall b = \beta(\varrho) > 0,\ \forall a, c > 0,\ \forall \lambda > \frac{b^2}{4c^2}: \quad a\varphi_\lambda'(s) - b|\varphi_\lambda(s)| \geq 1, \quad \forall s \in \mathbb{R}, \tag{1.4}$$

for an increasing and continuous function $\beta : [0, \infty) \to [0, \infty)$ and any fixed $\varrho > 0$, we notice that property (1.4) imply the following condition on φ_λ:

$$\varphi_\lambda'(s) - 2\sqrt{\lambda}|\varphi_\lambda(s)| \geq \frac{1}{2}, \quad \forall s \in \mathbb{R}. \tag{1.5}$$

The symbol ∇u will denote the gradient of u which is understood, as in [5, Lemma 2.1], to be the measurable function $v : Q \to \mathbb{R}^N$, uniquely defined, up to a set of zero Lebesgue measure, by

$$v\chi_{\{(t,x)\in Q:|u(t,x)|\leq k\}} = \nabla T_k(u) \text{ a.e. in } Q \text{ for any } k > 0,$$

where $T_k(s) : \mathbb{R} \to \mathbb{R}$, with $k > 0$, is the usual truncation function, and $\sigma_k(s)$ is an auxiliary function defined as

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k, \\ -k & \text{if } s \leq -k, \\ k & \text{if } s \geq k. \end{cases}, \quad \sigma_k(s) := (-|s| + k + 1) \vee 0 = \begin{cases} 0 & \text{if } |s| \geq k+1, \\ -s+k+1 & \text{if } 0 \leq s \leq k+1, \\ s+k+1 & \text{if } -k-1 \leq s \leq 0. \end{cases}$$

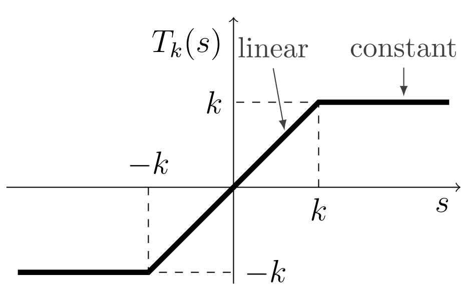


FIGURE 1. The function $T_k(s)$

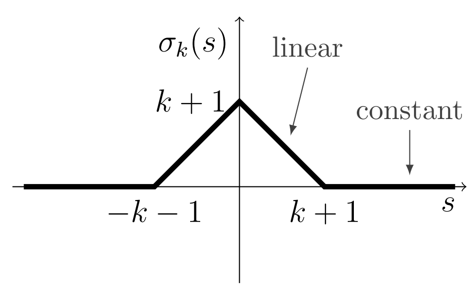


FIGURE 2. The function $\sigma_k(s)$

In what follows, we establish several technical lemmas, starting with a classical embedding result that will be used later.

Lemma 1.1. *Let H be a Hilbert space with dense embeddings $V \hookrightarrow H \hookrightarrow V'$. Suppose that $u \in L^p(0,T;V)$ and that its distributional derivative u_t lies in $L^p(0,T;V')$. It then follows that $u \in C([0,T];H)$.*

Proof. See [13, Chapter XVIII, Theorem 1]. $\square$

As a consequence of this result, the integration by parts formula holds[7]

$$\int_a^b \langle u_t, \varphi\rangle dt + \int_a^b \langle u, \varphi_t\rangle = (u(b), \varphi(b)) - (u(a), \varphi(a)) \tag{1.6}$$

The lemma below extends the integration by parts formula to a more general setting.

Lemma 1.2. *Let*[8] *$f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1-function such that $f(0) = 0$ and f' is zero away from a compact set of $\mathbb{R}$. If $u \in L^p(0, T; W_0^{1,p}(\Omega))$ is such that $u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$ and if $\psi \in C^\infty(\bar{Q})$, then we have*[9]

$$\int_0^T \langle u_t, f(u)\psi\rangle dt = \int_\Omega F(u(T))\psi(T)dx - \int_\Omega F(u(0))\psi(0)dx - \int_Q \psi_t F(u)dxdt. \tag{1.7}$$

Proof. For the proof, we refer the reader to [15] (see also [10]). □

2. Assumptions and statement of intermediary results

Throughout the paper we will consider the following model problem

$$\begin{cases} u_t - \operatorname{div}\left[a(t, x, u, \nabla u) + \Phi(t, x, u)\right] + g(t, x, u, \nabla u) = \mu \text{ in } Q := (0, T) \times \Omega, \\ u(0, x) = u_0(x) \text{ in } \Omega, \quad u(t, x) = 0 \text{ on } (0, T) \times \partial\Omega, \end{cases} \tag{2.1}$$

where $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}(Q)$ with[10]

$$\mathcal{M}(Q) = \left\{\mu : \mathcal{B}(Q) \to \mathbb{R} : |\mu|(Q) < \infty,\ |\mu|(E) = 0,\ \forall E \in \mathcal{B}(Q) \text{ s.t. } \operatorname{cap}_p(E) = 0\right\} \tag{2.2}$$

we recall that the p-capacity of an open set $U \subset Q$ is defined by[11]

$$\operatorname{cap}_p(U) = \inf\{\|u\|_W :\ u \in W,\ u \geq \chi_U \text{ a.e. in } Q\} \tag{2.3}$$

where, as usual, we take $\inf \emptyset = +\infty$. In [14], the authors introduce an alternative notion of parabolic capacity that is equivalent to the original definition, this notion of capacity can also be defined in terms of compact subsets K of Q as[12]

$$\operatorname{cap}(K) = \inf\{\|u\|_W :\ u \in C_c^\infty([0, T] \times \Omega),\ u \geq \chi_U\}. \tag{2.4}$$

Nevertheless, for $1 < p < \infty$, it is possible to provide the existence of a solution of our problem by defining the following functional space

$$\mathcal{T}_0^{1,p}(Q) = \{u \text{ measurable such that } T_k(u) \in W \text{ for every } k > 0\}. \tag{2.5}$$

---

[7] where $\langle \cdot, \cdot\rangle$ is the duality between V and V' and $(\cdot, \cdot)$ the scalar product in H.

[8] Recall that $F(s) = \int_0^s f(\sigma)d\sigma$.

[9] Here we have chosen the continuous representative of u.

[10] Here $\mathcal{B}(Q)$ is the Borel σ-algebra, $|\mu|$ is the total variation of μ and $\operatorname{cap}_p(E)$ is the usual parabolic p-capacity of a set $E \subset Q$ with respect to Q.

[11] here $W = \{u \in L^p(0, T; W_0^{1,p}(\Omega)),\ u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))\}$.

[12] Let us also recall that, if a function u belongs to W and μ is a bounded *Radon* measure such that $\mu(E) = 0$ for every $E \subset Q$ with $\operatorname{cap}_p(E) = 0$, we have that u is measurable with respect to μ and if u is also bounded then u belongs to $L^\infty(Q, \mu)$.

Now, let $a(t,x,s,\zeta):]0,T[\times\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R}^N$ be a *Carathéodory*[13] function such that there exist $\nu\in L^1(Q)$ and $c\in\mathbb{R}^+$ such that

$$a(t,x,s,\zeta)\cdot\zeta\geq\nu(t,x)+c|\zeta|^p \tag{2.6}$$

for a.e. $(t,x)\in Q$ and every $s\in\mathbb{R}^+$ and $\zeta\in\mathbb{R}^N$, and there exist $\mu\in L^{p'}(Q)$ and $c_1,c_2\in\mathbb{R}^+$ such that

$$|a(t,x,s,\zeta)|\leq\gamma(t,x)+c_1|s|^{p-1}+c_2|\zeta|^{p-1} \tag{2.7}$$

$$\langle(a(t,x,s,\zeta)-a(t,x,s,\eta),\zeta-\eta\rangle>0 \tag{2.8}$$

for a.e. (t,x) in Q, for every $s\in\mathbb{R}$ and all $\zeta,\eta\in\mathbb{R}^N$ such that $\zeta\neq\eta$. Under the above assumptions $u\mapsto-\mathrm{div}(a(t,x,u,\nabla u))$ turns out to be a *Leray-Lions* operator acting from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual, see [16]. As far the lower order term is concerned we suppose that $g(t,x,s,\zeta):[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R}^N$ is a *Carathéodory* function such that there exist an increasing and continuous function $\beta:[0,\infty)\to[0,\infty)$ and a nonregular function $d\in L^1(Q)$ satisfying

$$|g(t,x,s,\zeta)|\leq\beta(|s|)\left(|\zeta|^p+d(t,x)\right) \tag{2.9}$$

for a.e. (t,x) in Q and all $s\in\mathbb{R}$ and $\zeta\in\mathbb{R}^N$, and there exists $\varrho\in\mathbb{R}^+$ such that

$$g(t,x,s,\zeta)\cdot s\geq 0 \tag{2.10}$$

for a.e. $(t,x)\in Q$, $\zeta\in\mathbb{R}^N$ and every $s\in\mathbb{R}$ such that $|s|\geq\varrho$. Moreover, $\Phi(t,x,s,\zeta):]0,T[\times\Omega\times\mathbb{R}\to\mathbb{R}^N$ is a *Carathéodory* function such that there exists $b\in L^{\frac{N}{p-1}}(Q)$ for which

$$|\Phi(t,x,s)|\leq b(t,x)(1+|s|^{p-1}) \tag{2.11}$$

for a.e. (t,x) in Q and every $s\in\mathbb{R}$. Let us introduce the following notion of *renormalized* solution which is, by definition, the natural extension of the classical one (see [8, 14]).

Definition 2.1. We say that a measurable function $u\in L^\infty(0,T;L^1(\Omega))$, such that, for every $k>0$, $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))$, is a *renormalized* solution of problem (1.1) if both $a(t,x,u,\nabla u)$ and $\Phi(t,x,u)$ belong to $L^1(0,T;L^1(\Omega))^N$, $g(t,x,u,\nabla u)$ belongs to $L^1(0,T;L^1(\Omega))$, and the following identity holds true

$$\begin{aligned}&-\int_Q\varphi_t\Gamma_k(u)dxdt+\int_Q a(t,x,u,\nabla u)\cdot(\nabla\sigma_k(u)\varphi)dxdt+\int_Q\Phi_n(t,x,u)\cdot\nabla(\sigma_k(u)\varphi)dxdt\\&+\int_Q g(t,x,u,\nabla u)\sigma_k(u)\varphi dxdt=\int_Q\sigma_k(u)\varphi d\mu+\int_\Omega\Gamma_k(u_0)(x)\varphi(0)dx\end{aligned} \tag{2.12}$$

for every $k\in\mathbb{R}^+$ and all $\varphi\in W\cap L^\infty(Q)$ where $\Gamma_k(s)=\int_0^s\sigma_k(\tau)d\tau$. Moreover

$$\lim_{k\to\infty}\frac{1}{k}\int_{\{k\leq|u|\leq 2k\}}a(t,x,u,\nabla u)\cdot\nabla u\psi dxdt=0,\quad\forall\psi\in C_0^0([0,T]\times\Omega). \tag{2.13}$$

[13] I.e., it is continuous with respect to s and ξ for a.e. $(t,x)\in Q$ and measurable with respect to (t,x) for every $s\in\mathbb{R}$ and $\zeta\in\mathbb{R}^N$.

Remark 2.2. Note that the regularity required for the solution is such that any term in (2.12) makes sense. In fact the above formulation is nothing that

$$
\begin{aligned}
&-\int_\Omega S(u_0)\psi(0,x)dx - \int_0^T \langle S(u), \psi_t\rangle dt + \int_Q a(t,x,u,\nabla u)\cdot \nabla u S''(u)\psi dxdt \\
&\quad + \int_Q [a(t,x,u,\nabla u) + \Phi(t,x,u)]\cdot \nabla\psi S'(u)dxdt + \int_Q g(t,x,u,\nabla u)S'(u)\psi dxdt \\
&= \int_Q S'(u)\psi d\mu
\end{aligned}
\tag{2.14}
$$

for every $\psi \in C_0^1([0,T]\times\Omega)$, and for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compactly supported on $\mathbb{R}$ because it is equivalent to consider $S'(u)$ in place of $\sigma_k(u)$ with $S' \in W^{1,\infty}(\mathbb{R})$ having compact support[14], such a condition can be reformulated in a suitable sense adapted to our case; more precisely

$$
\lim_{n\to\infty} \frac{1}{n}\int_{\{n\le |u|\le 2n\}} |\nabla u|^p dxdt = 0. \tag{2.15}
$$

We are also interested in some regularity properties for renormalized solution of (2.1). Thus, let us introduce, for any $n \in \mathbb{N}$, the following approximate problem

$$
\begin{cases}
(u_n)_t - \operatorname{div}[a(t,x,u_n,\nabla u_n) + \Phi_n(t,x,u_n)] + g_n(t,x,u_n,\nabla u_n) = \mu_n(t,x) \text{ in } Q,\\
u_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \quad u_n(0,x) = u_0^n(x) \text{ in } \Omega,
\end{cases}
\tag{2.16}
$$

where $\mu_n = f_n - \operatorname{div}(F_n)$ with f_n is a sequence of smooth functions that converges to f in $L^1(Q)$, F_n is a sequence of smooth functions that converges to F in $L^{p'}(Q)^N$; and u_0^n is a sequence of regular functions that converges to u_0 in $L^1(\Omega)$. Moreover

$$
\Phi_n(t,x,s) = \frac{\Phi(t,x,s)}{1+\frac{1}{n}|\Phi(t,x,s)|}, \quad g_n(t,x,s,\zeta) = \frac{g(t,x,s,\zeta)}{1+\frac{1}{n}|g(t,x,s,\zeta)|}. \tag{2.17}
$$

Note that, thanks to the result of [11] (see also [16]) there exists (at least) a weak solution of (2.16), i.e., a function $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ such that $(u_n)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, both $\Phi_n(t,x,u_n)$ and $a(t,x,u_n,\nabla u_n)$ belongs to $L^1(0,T;L^1(\Omega))^N$ and $g_n(t,x,u_n,\nabla u_n)$ belong to $L^1(0,T;L^1(\Omega))$, and the following identity holds true

$$
\begin{aligned}
&\int_0^T \langle (u_n)_t, \psi\rangle dt + \int_Q [a(t,x,u_n,\nabla u_n) + \Phi(t,x,u_n)]\cdot\nabla\psi dxdt \\
&\quad + \int_Q g_n(t,x,u_n,\nabla u_n)\psi dxdt = \int_Q \psi d\mu_n \ \forall\psi \in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)
\end{aligned}
\tag{2.18}
$$

The estimates presented in the following lemma are standard and, for instance, coincide with those established in [3] (see also [1]).

[14]The fact that S' is compactly supported ensures that all both the first two terms in (2.14) involve only a truncature of u on the set where it is large.

Lemma 2.3. *Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$, $1 < p < \infty$, be defined as before, then there exists a constant $C \in \mathbb{R}^+$ such that*

$$\sup_{t\in[0,T]} \int_\Omega |u_n(t,x)|dx + \int_Q |\nabla T_k(u_n)|^p dxdt \le C(k), \quad \forall n \in \mathbb{N},\ \forall k > 0.$$

Proof. For any $k \ge \varrho$ (with $\varrho > 0$ fixed), let us choose in (2.18) the test function $\psi = \varphi_\lambda(T_k(u_n))\chi_{(0,t)}$, where $\varphi_\lambda(s) = se^{\lambda s^2}$ is defined in (1.4) ($\lambda > 0$ will be fixed later). Thus, we have

$$\begin{aligned}
&\int_0^t \langle (u_n)_t, \varphi_\lambda(T_k(u_n))\rangle dt + \int_0^t \int_\Omega a(t,x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\varphi'_\lambda(T_k(u_n))dxdt \\
&\int_0^t \int_\Omega \Phi_n(t,x,u_n)\cdot\nabla T_k(u_n)\varphi'_\lambda(T_k(u_n))dxdt + \int_0^t \int_\Omega g_n(t,x,u_n,\nabla u_n)\varphi_\lambda(T_k(u_n))dxdt \\
&\quad = \int_0^t \int_\Omega f_n\varphi_\lambda(T_k(u_n))dxdt + \int_0^t \int_\Omega F_n\cdot\nabla T_k(u_n)\varphi'_\lambda(T_k(u_n))dxdt.
\end{aligned} \tag{2.19}$$

Since

$$\begin{cases} \displaystyle\int_{\{|u_n|\ge\varrho\}} g_n(t,x,u_n,\nabla u_n)\varphi_\lambda(T_k(u_n))dxdt \ge 0, \\ \displaystyle\int_0^t \langle (u_n)_t, \varphi_\lambda(T_k(u_n))\rangle dt = \int_\Omega \overline{\varphi}_{\lambda,k}(u_n(t,x))dx - \int_\Omega \overline{\varphi}_{\lambda,k}(u_n(0,x))dx \end{cases}$$

where $\overline{\varphi}_{\lambda,k}(s) = \int_0^s \varphi_\lambda(T_k(\tau))d\tau$ with $\lambda, k > 0$, it is easy to see that

$$\begin{cases} |\overline{\varphi}_{\lambda,k}(s)| \le \varphi_\lambda(s)|s|, \quad \forall s \in \mathbb{R} \\ \displaystyle\varphi_\lambda(k)\int_\Omega |u|dx \le \int_\Omega \overline{\varphi}_{\lambda,k}(u)(t,x)dx + k\varphi_\lambda(k)\mathrm{meas}(\Omega), \end{cases}$$

for all $u \in L^1(\Omega)$ and a.e. $t \in]0,T[$; thus, we obtain

$$\varphi_\lambda(k)\int_\Omega |u_n(t)|dx \le \int_0^t \langle (u_n)_t, \varphi_\lambda(T_k(u_n))\rangle dt + \varphi_\lambda(k)\|u_0^n\|_{L^1(\Omega)} + k\varphi_\lambda(k)\mathrm{meas}(\Omega).$$

Hence, we can write

$$
\begin{aligned}
&\varphi_\lambda(k)\int_\Omega |u_n(t)|dx + \int_Q a(t,x,u_n,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\varphi_\lambda'(T_k(u_n))dxdt\\
&\le \int_Q f_n\varphi_\lambda(T_k(u_n))dxdt + \int_Q F_n\cdot\nabla T_k(u_n)\varphi_\lambda'(T_k(u_n))dxdt + \varphi_\lambda(k)\|u_0^n\|_{L^1(\Omega)} + k\varphi_\lambda(k)\text{meas}(\Omega)\\
&\quad - \int_Q \Phi_n(t,x,u_n)\cdot\nabla T_k(u_n)\varphi_\lambda'(T_k(u_n))dxdt - \int_{\{|u_n|\le\varrho\}} g_n(t,x,u_n,\nabla u_n)\varphi_\lambda(T_k(u_n))dxdt\\
&\le \varphi_\lambda(k)\sup_{n\in\mathbb{N}}\|f_n\|_{L^1(Q)} + \frac{c}{8}\int_{\{|u_n|\le k\}}|\nabla u_n|^p dxdt + c\varphi_\lambda'(k)^{p'}\|F\|^{p'}_{L^{p'}(Q)}\\
&\quad + \varphi_\lambda(k)\|u_0^n\|_{L^1(\Omega)} + k\varphi_\lambda(k)\text{meas}(\Omega)\\
&\quad + \int_{\{|u_n|\le k\}} b(1+k^{p-1})|\nabla u_n|\varphi_\lambda'(T_k(u_n))dxdt + \beta(\varrho)\int_{\{|u_n|\le\varrho\}}(|\nabla u_n|^p + d)|\varphi_\lambda(T_k(u_n))|dxdt\\
&\le \varphi_\lambda(k)\sup_{n\in\mathbb{N}}\|f_n\|_{L^1(Q)} + \frac{c}{4}\int_{\{|u_n|\le k\}}|\nabla u_n|^p dxdt\\
&\quad + C\varphi_\lambda'(k)^{p'}\|F\|^{p'}_{L^{p'}(Q)} + \varphi_\lambda(k)\|u_0^n\|_{L^1(\Omega)} + k\varphi_\lambda(k)\text{meas}(\Omega)\\
&\quad + C(1+k^{p-1})^{p'}\varphi_\lambda'(k)^{p'}\|b\|^{p'}_{L^{p'}(Q)} + \beta(\varrho)\int_{\{|u_n|\le k\}}\frac{1}{c}[a(t,x,u_n,\nabla u_n)\cdot\nabla u_n + |\nu|)\varphi_\lambda(T_k(u_n))dxdt\\
&\quad + \beta(\varrho)\varphi_\lambda(k)\|d\|_{L^1(Q)},
\end{aligned}
\tag{2.20}
$$

Moreover, we have

$$
\begin{aligned}
&\varphi_\lambda(k)\int_\Omega |u_n(t,x)|dx + \frac{1}{2}\int_{\{|u_n|\le k\}}(\nu + c|\nabla u_n|^p)dxdt\\
&\le \int_{\{|u_n|\le k\}} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n\left[\varphi_\lambda'(T_k(u_n)) - \frac{\beta(\varrho)}{c}|\varphi_\lambda(T_k(u_n))|\right]dxdt\\
&\quad + \varphi_\lambda(k)\|u_0^n\|_{L^1(\Omega)} + k\varphi_\lambda(k)\text{meas}(\Omega)\\
&\le \frac{c}{4}\int_{\{|u_n|\le k\}}|\nabla u_n|^p dx + \varphi_\lambda(k)\left[\sup_{n\in\mathbb{N}}\left(\|u_0^n\|_{L^1(\Omega)} + \|f_n\|_{L^1(Q)}\right) + \text{meas}(\Omega)\right]\\
&\quad + \frac{\beta(\varrho)}{c}\varphi_\lambda(k)\left(\|\nu\|_{L^1(Q)} + c\|d\|_{L^1(Q)}\right) + C\left[\varphi_\lambda'(k)^{p'}\|F\|^{p'}_{L^{p'}(Q)} + (1+k^{p-1})^{p'}\varphi_\lambda'(k)^{p'}\|b\|^{p'}_{L^{p'}(Q)}\right].
\end{aligned}
\tag{2.21}
$$

Note that both f_n,ν,d are bounded in $L^1(Q)$, u_0^n is bounded in $L^1(Q)$ and F,b are bounded in $L^{p'}(Q)$, therefore we choose $\lambda > \frac{\beta(\varrho)^2}{4c^2}$ such that (1.4) holds and we deduce that there exists a constant $C(k)>0$ (depending on k) such that

$$
\sup_{t\in[0,T]}\int_\Omega |u_n(t,x)|dx + \int_Q |\nabla T_k(u_n)|^p dxdt \le C(k), \quad \forall k>0. \tag{2.22}
$$

This implies that $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$. Therefore, possibly after extracting a subsequence (not relabeled), $T_k(u_n)$ converges in the weak sense to a function ν_k in $L^p(0,T;W_0^{1,p}(\Omega))$. Furthermore, the sequence (u_n) is bounded in $L^\infty(0,T;L^1(\Omega))$. □

A useful consequence of Lemma 2.3 is the following.

Lemma 2.4. *Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$, $1 < p < N$, be defined as before, then there exists a constant $C \in \mathbb{R}^+$ such that*

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} + \|ln(1+|u_n|)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C(k+1), \quad \forall n \in \mathbb{N},\ \forall k > 0. \tag{2.23}$$

Proof. Let us choose $\psi(u_n)$, with $\psi(s) = \int_0^s \frac{1}{(1+|\tau|)^p} d\tau$, as test function in (2.18) (it is clear that ψ is bounded and Lipschitz continuous, so the sequence $(\psi(u_n))_n$ is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega))$), then by assumption (2.10) and Lemma 2.3 we deduce that

$$\begin{aligned}
&\int_0^T \langle (u_n)_t, \psi(u_n)\rangle dt + \int_Q a(t,x,u_n,\nabla u_n) \cdot \frac{\nabla u_n}{(1+|u_n|)^+} dxdt \\
&\quad + \int_{\{|u_n| \geq \varrho\}} g_n(t,x,u_n,\nabla u_n)\psi(u_n) dxdt \\
&+ \int_{\{|u_n| \langle \varrho\}} g_n(t,x,u_n,\nabla u_n)\psi(u_n) dxdt + \int_Q \Phi_n(t,x,u_n) \frac{\nabla u_n}{(1+|u_n|)^+} dxdt \\
&\quad \leq C\|f_n\|_{L^1(Q)} + \int_Q |F_n| \frac{|\nabla u_n|}{(1+|u_n|)^p} dxdt,
\end{aligned} \tag{2.24}$$

which implies that

$$\begin{aligned}
&\int_0^T \langle (u_n)_t, \psi(u_n)\rangle dt + c\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|^p)} dxdt + \int_Q \frac{\nu(t,x)}{(1+|u_n|)^p} dxdt \\
&\leq C\int_{\{|u_n|<\varrho\}} \beta(\varrho)(|\nabla u_n|^p + d) dxdt + \int_Q b(1+|u_n|^{p-1}) \frac{|\nabla u_n|}{(1+|u_n|)^p} \\
&\quad + C\|f_n\|_{L^1(Q)} + \int_Q |F_n| \frac{|\nabla u_n|}{(1+|u_n|)^p} dxdt.
\end{aligned}$$

First, we deal with the terms of the the right hand side, we have

$$\begin{aligned}
C&\int_{\{|u_n|<\varrho\}} \beta(\varrho)(|\nabla u_n|^p + d) dxdt \leq C\beta(\varrho)\left[(1+|\varrho|)^p \int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^p} dxdt + \|d\|_{L^1(Q)}\right] \\
&\leq C\beta(\varrho)[C(\varrho) + \|d\|_{L^1(Q)}] + \frac{C}{4}\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^p} dxdt
\end{aligned}$$

and, since $\frac{1+|x|^{p-1}}{(1+|x|)^{p-1}} = \frac{1}{(1+|x|)^{p-1}} + \frac{|x|^{p-1}}{(1+|x|)^{p-1}} \leq 2$, we get

$$\begin{aligned}
\int_Q b(1+|u_n|^{p-1}) \frac{|\nabla u_n|}{(1+|u_n|)^p} dxdt &\leq \int_Q |b| \frac{(1+|u_n|^{p-1})}{(1+|u_n|)^{p-1}} \frac{|\nabla u_n|}{(1+|u_n|)} dxdt \\
&\leq 2\int_Q |b| \frac{|\nabla u_n|}{(1+|u_n|)} dxdt \leq C\int_Q |b|^{p'} dxdt + \frac{C}{4}\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^p} dxdt
\end{aligned}$$

observe that $p' = \frac{p}{p-1} < \frac{N}{p-1}$ then, since b belongs to $L^{\frac{N}{p-1}}$, we have $b \in L^{p'}(Q)$ which implies that

$$\int_Q |F_n| \frac{|\nabla u_n|}{(1+|u_n|)^p} dxdt \leq \int_Q |F| \frac{|\nabla u_n|}{(1+|u_n|)} dxdt \leq C\int_Q |F|^{p'} dxdt + \frac{C}{4}\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^p} dxdt,$$

then

$$\int_0^T \langle (u_n)_t, \psi(u_n)\rangle dt + c\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^p}dxdt + \int_Q \frac{\nu}{(1+|u_n|)^p}dxdt$$
$$\leq C\left(\|f_n\|_{L^1(Q)} + \|F_n\|^{p'}_{L^{p'}(Q)} + \|b\|^{p'}_{L^{p'}(Q)}\right) + C\beta(\varrho)[C(\varrho) + \|d\|_{L^1(Q)}] + \frac{3c}{4}\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^p}dxdt,$$

which implies that

$$\begin{aligned}&\int_0^T \langle (u_n)_t, \psi(u_n)\rangle dt + \frac{c}{4}\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|^p)}dxdt + \int_Q \frac{\nu}{(1+|u_n|)^p}dxdt\\ &\leq C\left(\sup_{n\in\mathbb{N}} \|f_n\|_{L^1(Q)}, \|F\|_{L^{p'}(Q)}, \|b\|_{L^{p'}(Q)}, \|d\|_{L^1(Q)}, \varrho\right).\end{aligned} \tag{2.25}$$

To deal with the parabolic term, let us denote by $\overline{\psi}$ (the primitive function of ψ) defined by $\overline{\psi}(s) = \int_0^s \psi(\tau)d\tau$, then by using the integration by parts formula we obtain

$$\int_0^T \langle (u_n)_t, \psi(u_n)\rangle dt = \int_\Omega \overline{\psi}(u_n)(T,x)dx - \int_\Omega \overline{\psi}(u_n(0,x))dx,$$

remark that

$$\psi(s) = \int_0^s \frac{d\tau}{(1+\tau)^p} = \frac{1}{(p-1)}\left[1 - \frac{1}{(1+s)^{p-1}}\right]$$

Hence for $s > 0$ and $p > 2$ and by simple computations[15], we get

$$\int_0^s \psi(\tau)d\tau = \frac{1}{(p-1)}\left[s + \frac{1}{(p-2)(1+s)^{p-2}} - \frac{1}{p-2}\right]$$

which implies that

$$0 \leq \frac{1}{(p-1)}s \leq \int_0^s \psi(\tau)d\tau + \frac{1}{(p-1)(p-2)}$$

thus

$$0 \leq s \leq (p-1)\int_0^s \psi(\tau)d\tau + \frac{1}{p-2}$$

As similar for $s \leq 0$, we finally obtain that

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \leq (p-1)\int_0^T \langle (u_n)_t, \psi(u_n)\rangle dt + \frac{\text{meas}\,(Q)}{p-2} \tag{2.26}$$

which concludes, by combining inequalities (2.25)-(2.26), the desired result. □

Our regularity result concerning the approximate solution is the following one.

[15]As follows

$$\frac{1}{p-1}\int_0^s [1+(1+\tau)^{1-p}]d\tau = \frac{1}{(p-1)}\left[s - \left[\frac{(1+\tau)^{2-p}}{2-p}\right]_0^s\right] = \frac{1}{(p-1)}\left[s - \frac{1}{(2-p)}\left(\frac{1}{(1+s)^{p-2}} - 1\right)\right]$$
$$= \frac{1}{(p-1)}\left[s + \frac{1}{(p-2)} \times \frac{1}{(1+s)^{p-2}} - \frac{1}{p-2}\right]$$

Theorem 2.5. *Assume that $a_n(t,x,s,\zeta)$, $\Phi_n(t,x,s)$ and $g_n(t,x,s,\zeta)$ satisfy assumptions (2.6)-(2.11) and (2.17) and let u_n, u_0^n and μ_n be defined as before, then there exist two positive constants A, B such that*

$$\sup_{t\in[0,T]}\int_\Omega |u_n|dx + \int_Q |\nabla T_k(u_n)|^p dxdt \le A + kB, \quad \forall n\in\mathbb{N},\ \forall k>0. \tag{2.27}$$

Proof. Let us choose $\theta_k^h(u_n) = T_k(u_n) - T_h(u_n)$, with $\varrho \le h < k$ (ϱ is given in (2.10)), as test function in (2.18), and let us denote by $\Theta_k^h(s) = \int_0^s \theta_k^h(\tau)d\tau$ its primitive function which satisfies

$$\Theta_k^h(s) = \begin{cases} 0 & \text{if } |s|\le k, \\ \frac{(s-h)^2}{2} & \text{if } h\le |s|\le k, \\ \frac{(k-h)^2}{2} + (k-h)(|s|-k) & \text{if } |s|\ge k, \end{cases}$$

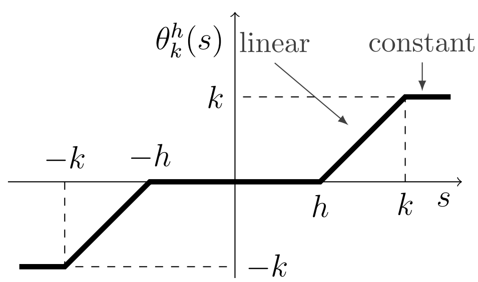


FIGURE 3. The function $\theta_k^h(s)$

we have

$$\int_0^T \langle (u_n)_t, \theta_k^h(u_n)\rangle dtr + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\theta_k^h(u_n)dxdt + \int_Q \Phi_n(t,x,u_n)\cdot\nabla\theta_k^h(u_n)dxdt$$
$$+\int_Q g_n(t,x,u_n,\nabla u_n)\theta_k^h(u_n)dxdt = \int_Q f_n\theta_k^h(u_n)dxdt + \int_Q F_n\cdot\nabla\theta_k^h(u_n)dxdt \tag{2.28}$$

First, we deal with the parabolic term by using the following integration by parts formula

$$\int_0^t \langle (u_n)_t, \theta_k^h(u_n)\rangle dt = \int_\Omega \Theta_k^h(u_n)(t)dx - \int_\Omega \Theta_k^h(u_0^n(x))dx, \tag{2.29}$$

observe that

$$\int_\Omega \Theta_k^h(u_n)(t)dx = \int_{\{|u_n|\le k\}} \Theta_k^h(u_n)(t)dx + \int_{\{|u_n|\ge k\}} \Theta_k^h(u_n)(t)dx = I + J,$$

since

$$|I| \le \int_{\{|u_n|\le k\}} \frac{(k-h)^2}{2}dx \le \frac{(k-h)^2}{2}\ \text{meas}(\Omega) \tag{2.30}$$

and

$$J \ge (k-h)\left[\int_{\{|u_n|\ge k\}} |u_n|dx - k\ \text{meas}(\Omega)\right]$$

then

$$\int_{\{|u_n|\geq k\}} |u_n|dx \leq \frac{1}{(k-h)} J + k \text{ meas}(\Omega), \tag{2.31}$$

In addition, we have

$$\int_\Omega \Theta_k^h(u_0^n)dx \leq k\|u_0^n\|_{L^1(\Omega)}, \tag{2.32}$$

then, since $|\Theta_k^h(s) \leq k|s|$ and using inequalities (2.29)-(2.32), we obtain

$$\int_{\{|u_n|\geq k\}} |u_n|(t)dx \leq \frac{1}{k-h}\left[\int_0^t \langle (u_n)_t, \theta_h^k(u_n)\rangle dt + |I| + \int_\Omega \Theta_k^h(u_0^n)dx\right] + k \text{ meas}(\Omega),$$

which implies that

$$\begin{aligned}\int_{\{|u_n|\geq k\}} |u_n|(t)dx &\leq \frac{1}{k-h}\int_0^t \langle (u_n)_t, \theta_k^h(u_n)\rangle dt + \frac{(k-h)}{2}\text{meas } (\Omega) + \frac{k}{k-h}\|u_0^n\|_{L^1(\Omega)} + k \text{ meas}(\Omega)\\ &\leq \frac{1}{k-h}\int_0^t \langle (u_n)_t, \theta_k^h(u_n)\rangle dt + \frac{k}{k-h}\|u_0^n\|_{L^1(\Omega)} + \frac{3}{2}k \text{ meas}(\Omega)\end{aligned}$$

thus, for a constant $C_0 > 0$ satisfying $\displaystyle\int_0^t \langle (u_n)_t, \theta_k^h(u_n)\rangle dt \leq C_0$ we have

$$\int_{\{|u_n|\geq k\}} |u_n|dx \leq C\left(C_0, h, k, \|u_0\|_{L^1(\Omega)}, \text{meas } (\Omega)\right),$$

and

$$\int_\Omega |u_n|(t)dx = \int_{\{|u_n|\leq k\}} |u_n|dx + \int_{\{|u_n|\geq k\}} |u_n|dx \leq k \text{ meas}(\Omega) + C(C_0, h, k, \cdots),$$

on the other hand, by using assumptions (2.6)-(2.7), (2.9), (2.11) we get

$$\begin{aligned}&\int_\Omega |u_n(t)|dx + C\int_{\{h\leq|u_n|\leq k\}} |\nabla u_n|^p dxdt + \int_{\{h\leq|u_n|\leq k\}} \nu dxdt\\ &\leq k\int_Q |f_n|dxdt + \int_Q F_n\cdot\nabla u_n dxdt + \int_{\{h\leq|u_n|\leq k\}} b(1+|u_n|^{p-1})|\nabla u_n|dxdt + k\int_\Omega |u_0^n|dx\\ &\leq k\left(\|u_0^n)\|_{L^1(\Omega)} + \|f_n\|_{L^1(Q)} + \text{meas}(\Omega)\right) + C(\|F\|^{p'}_{L^{p'}(Q)} + \|b\|^{p'}_{L^{p'}(Q)})\\ &\quad + \frac{c}{4}\int_{\{h\leq|u_n|\leq k\}} |\nabla u_n|^p dxdt + C\int_{\{h\leq|u_n|\leq k\}} b^{p'}|u_n|^p dxdt\\ &\leq k \sup_{n\in\mathbb{N}} \left(\|u_0^n)\|_{L^1(\Omega)} + \|f_n\|_{L^1(Q)}\right) + C(\|F\|^{p'}_{L^{p'}(Q)} + \|b\|^{p'}_{L^{p'}(Q)}) + \frac{c}{4}\int_{\{h\leq|u_n|\leq k\}} |\nabla u_n|^p dxdt\\ &\quad + C\left(\int_{\{h\leq|u_n|\leq k\}} b^{\frac{N}{p-1}} dxdt\right)^{\frac{p}{N}}\left(\int_{\{|u_n|\leq h\}} |\nabla u_n|^p dxdt + \int_{\{|u_n|\geq h\}} |\nabla T_k(u_n)|^p dxdt\right).\end{aligned} \tag{2.33}$$

Now, by means of Lemma 2.4, we have $|\{|u_n|\geq h\}| \leq \frac{C}{(\ln(1+h))^p}$ for every $n\in\mathbb{N}$ and all $h > 0$ and which implies, by choosing $h > \varrho$ s.t $C\left(\displaystyle\int_{\{|u_n|\geq h\}} b^{\frac{N}{p-1}}\right)^{\frac{p}{N}} < \frac{c}{4}$

and using Lemma 2.3, that

$$\begin{aligned}
&\int_\Omega |u_n(t)|dx + \int_{\{h\le |u_n|\le k\}} |\nabla u_n|^p dxdt \\
&\le k \underbrace{\left(\|u_0^n\|_{L^1(\Omega)} + \|f_n\|_{L^1(Q)} + \text{meas}(\Omega)\right)}_{\text{B}} + \\
&\underbrace{C\left(\int_Q |\nu|dxdt + \|F\|^{p'}_{L^{p'}(Q)} + \|b\|^{p'}_{L^{p'}(Q)} + C(h)\|b\|^p_{L^{\frac{N}{p-1}}}\right)}_{\text{A}} \\
&= A + kB,
\end{aligned} \tag{2.34}$$

where $C(h)$ is a constant depending on h, which gives, by Lemma 2.3, the desired result. □

The following results are consequences of the previous theorem, we will use them in the next but we give here their statements for completeness.

Theorem 2.6. *Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ be defined as before such that the estimate (2.27) holds, then there exist a measurable function $u \in \mathcal{T}_0^{1,p}(Q)$, a subsequence of (u_n) (still denoted by (u_n)) and a function $w \in L^r(Q)^N$ for every $1 \le r < \frac{Np-N+p}{(N+1)(p-1)}$ such that*

- (i) *$u_n \to u$ a.e. in Q and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $L^p(Q)^N$ for every $k > 0$;*
- (ii) *$a(t,x,u_n,\nabla u_n)\chi_{\{|u_n|\le k\}} \rightharpoonup w\chi_{\{|u|\le k\}}$ in $L^{p'}(Q)^N$;*
- (iii) *$\{(t,x) \in Q : |u_n| \ge \lambda\} \le \lambda^{-\frac{p*}{p'}} S^{p^*}(A+B)^{\frac{p*}{p}}$ for every $\lambda \ge 1$;*
- (iv) *$|\nabla u|^{p-1} \in L^r(Q)$ and $a(t,x,u_n,\nabla u_n) \rightharpoonup w$ weakly in $L^r(Q)$ for every $1 < p < N$ and all $1 \le r < \frac{Np-N+p}{(N+1)(p-1)}$.*

Proof. The proof is analoguous to the one of [1, Proposition 5.2] and to the one in [2, Proposition 3.12]. □

The estimates presented in the following theorem are necessary in order to prove our main existence result.

Theorem 2.7. *Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ be defined as before, then there exist a suitable constant $C > 0$ such that the following estimate holds*

$$\begin{aligned}
&\sup_{t\in[0,T]} \int_\Omega |u_n(t,x)|dx + \int_{\{k\le u_n\le 2k\}} |\nabla u_n|^p dxdt \\
&\le kC\left[\int_{\{|u_n|\ge k\}} |f_n|dxdt + \left(\int_{\{|u_n|\ge k\}} b^{\frac{N}{p-1}} dxdt\right)^{\frac{p}{N}}\right] \\
&\quad + C\left(\|\nu\|_{L^1(Q)} + \|F\|^{p'}_{L^{p'}(Q)} + \|b\|^{p'}_{L^{\frac{N}{p-1}p'}}\right) + C(k, meas(\Omega), \|u_0\|_{L^1(\Omega)}).
\end{aligned} \tag{2.35}$$

Moreover, the limit function u of u_n satisfies the following asymptotic estimate

$$\lim_{k\to\infty} \frac{1}{k} \int_{\{k\le |u|\le 2k\}} |\nabla u|^p dxdt = 0. \tag{2.36}$$

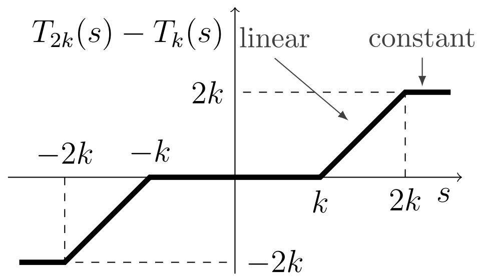


FIGURE 4. The function $T_{2k}(s) - T_k(s)$

Proof. Let $k \geq \varrho$ and let us choose $\psi = T_{2k}(u_n) - T_k(u_n)$ as test function in (2.18), see Figure 4. Thus, by following the same ideas of the previous estimate in Theorem 2.5 we obtain

$$
\begin{aligned}
&\int_0^T \langle (u_n)_t, T_{2k}(u_n) - T_k(u_n)\rangle dt + \int_{\{k\leq |u_n|\leq 2k\}} (c|\nabla u_n|^p + \nu) dxdt \\
&\leq \int_0^T \langle (u_n)_t, T_{2k}(u_n) - T_k(u_n)\rangle dt + \int_Q a(t,x,u_n,\nabla u_n)\cdot \nabla[T_{2k}(u_n) - T_k(u_n)] dxdt \\
&\leq 2k \int_{\{|u_n|\geq k\}} |f_n| dxdt + \frac{c}{4}\int_{\{k\leq |u_n|\leq 2k\}} |\nabla u_n|^p dxdt + C\int_{\{|u_n|\geq k\}} (|F|^{p'} + b^{p'}) dxdt \\
&\quad + C\left(\int_{\{|u_n|\geq k\}} b^{\frac{N}{p-1}} dxdt\right)^{\frac{p}{N}} \left(\int_Q |T_{2k}(u_n)|^{p^*} dxdt\right)^{\frac{N-p}{N}} dxdt \\
&\leq 2k \int_{\{|u_n|\geq k\}} |f_n| dxdt + \frac{c}{k}\int_{\{k\leq |u_n|\leq 2k\}} |\nabla u_n|^p dxdt + C\int_{\{|u_n|\geq k\}} (|F|^{p'} + b^{p'}) dxdt \\
&\quad + C\left(\int_{\{|u_n|\geq k\}} b^{\frac{N}{p-1}} dxdt\right)^{\frac{p}{N}} \left(\int_{\{|u_n|\leq 2k\}} |\nabla u_n|^p dxdt\right) \\
&\leq 2k \int_{\{|u_n|\geq k\}} |f_n| dxdt + \frac{c}{4}\int_{\{k\leq |u_n|\leq 2k\}} |\nabla u_n|^p dxdt + C\int_{\{|u_n|\geq k\}} (|F|^{p'} + b^{p'}) dxdt \\
&\quad + C\left(\int_{\{|u_n|\geq k\}} b^{\frac{N}{p-1}} dxdt\right)^{\frac{p}{N}} (A + 2kB)
\end{aligned}
\tag{2.37}
$$

where the constants A, B are defined in (2.34). Thus, by using the integration by parts formula and the same argument of the proof of Theorem 2.5

$$
\int_0^t \langle (u_n)_t, T_{2k}(u_n) - T_k(u_n)\rangle dx \leq k \ \mathrm{meas}(\Omega) + C(C_0, k, \cdots).
$$

Hence, by the equi-integrability of f_n, Theorem 2.6 (iii) and the semicontinuity of the norm we get for every $k \geq k_\epsilon$ (with a suitable choice $k_\epsilon \in \mathbb{N}$)

$$\begin{aligned}
&\frac{1}{k}\int_{\{k\leq|u_n|\leq 2k\}}|\nabla u|^p dxdt \leq \frac{1}{k}\liminf_{n\to\infty}\int_{\{k\leq|u_n|\leq 2k\}}|\nabla u_n|^p dxdt\\
&\leq C\left[\epsilon+\left(\int_{\{|u_n|\geq k\}} b^{\frac{N}{p-1}}dxdt\right)^{\frac{p}{N}}\right]+\frac{C}{k}\left[\|\nu\|_{L^1(Q)}+\|F\|^{p'}_{L^{p'}(Q)}+\|b\|^{p'}_{L^{p'}()Q}+\|b\|^{p'}_{L^{\frac{N}{p-1}}}\right]\\
&\quad + C(C_0,k,\mathrm{meas}(\Omega),\|u_0\|_{L^1(\Omega)},\cdots),
\end{aligned} \tag{2.38}$$

which conclude the proof of Theorem 2.7. $\square$

The essential key in the proof of our main results is the following theorem.

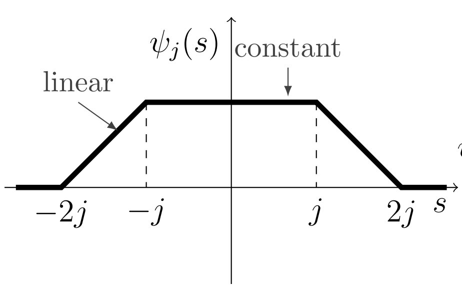
Theorem 2.8. *Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ be defined as before and let u be the limit of a subsequence of (u_n) (still denoted by u_n), then for every $k > 0$*

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{2.39}$$

Proof. Our aim is to prove the following asymptotic estimate

$$\lim_{k\to\infty}\int_{\{|u_n|\leq k,\ |u|\leq k\}}[a(t,x,u_n,\nabla u_n)-a(t,x,u_n,\nabla u)]\cdot\nabla(u_n-u)dxdt = 0. \tag{2.40}$$

The result will readily follow by choosing $\psi_{n,j}(u_n) = \varphi_\lambda(T_{2k+2}(u_n-u))\psi_j(u_n)\sigma_k(u)$ as test function in (2.18) where σ_k is defined in Figure 2, $\varphi_\lambda(s)$ in (1.4) with $\lambda = \frac{b(k+1)^2}{4c^2}$ and $\psi_j(s)$ is defined by



$$\psi_j(s) = \begin{cases} 1 & \text{if } |s| \leq j,\\ 0 & \text{if } |s| \geq 2j,\\ -|s|/j+2 & \text{if } j \leq |s| \leq 2j. \end{cases} \tag{2.41}$$

FIGURE 5. The function $\psi_j(s)$

Note that, by the definition of σ_k and thanks to the fact that $\psi_{n,j} := \varphi_\lambda(T_{2k+2}(u_n - T_{k+1}(u))\psi_j(u_n)\sigma_k(u)$ a.e. in Q, we have $\psi_{n,j} \in L^p(0,T;W_0^{1,p}(\Omega))$ and has the same sign as u_n in the set $\{(t,x) \in Q : |u_n| \geq k+1\} \cap \{(t,x) \in Q : \sigma_k(u) \neq 0\}$ (with

$k > \sigma$); we fix $j > 3k+3$ to obtain

$$\begin{aligned}
&\int_Q [a(t,x,u_n,\nabla u_n) - a(t,x,u_n,\nabla u)] \cdot \nabla T_{2k}(u_n - u)\varphi'_\lambda(T_{2k+2}(u_n-u))\sigma_k(u)dxdt \\
&\leq \int_Q [a(t,x,u_n,\nabla u_n) - a(t,x,u_n,\nabla u)] \cdot \nabla T_{2k+2}(u_n - u)\varphi'_\lambda(T_{2k+2}(u_n-u))\psi_j(u_n)\sigma_k(u)dxdt \\
&\quad + \int_{\{|u_n|\geq k+1\}} g_n(t,x,u_n,\nabla u_n)\psi_{n,j}dxdt \\
&= \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_{n,j}dxdt - \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_j(u_n)\sigma_k(u)\varphi_\lambda(T_{2k+2}(u_n-u))dxdt \\
&\quad - \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla T_{2k+2}(u_n-u)\varphi'_\lambda(T_{2k+2}(u_n-u))\psi_j(u_n)\sigma_k(u)dxdt \\
&\quad + \int_Q g_n(t,x,u_n,\nabla u_n)\psi_{n,j}dxdt - \int_{\{|u_n|\leq k+1\}} g_n(t,x,u_n,\nabla u_n)\psi_{n,j}dxdt \\
&= \int_Q f_n\psi_{n,j}dxdt + \int_Q F_n\cdot\nabla\psi_{n,j}dxdt - \int_{\{|u_n|\leq k+1\}} g_n(t,x,u_n,\nabla u_n)\psi_{n,j}dxdt \\
&\quad - \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla(\psi_j(u_n))\sigma_k(u)\varphi_\lambda(T_{2k+2}(u_n-u))dxdt \\
&\quad - \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla T_{2k+2}(u_n-u)\varphi'_\lambda(T_{2k+2}(u_n-u))\psi_j(u_n)\sigma_k(u)dxdt \\
&\quad - \int_Q \Phi_n(t,x,u_n)\cdot\nabla\psi_{n,j}dxdt - \int_0^T \langle (u_n)_t, \psi_{n,j}\rangle dt,
\end{aligned} \tag{2.42}$$

we notice that assumptions (2.6) and (2.9) imply that

$$\begin{aligned}
&-\int_{\{|u_n|\leq k+1\}} g_n(t,x,u_n,\nabla u_n)\psi_{n,j}dxdt \\
&\leq \beta(k+1)\left[\int_Q d(t,x)|\varphi_\lambda(T_{2k+2}(u_n-u))|dxdt + \int_{\{|u_n|\leq k+1\}} |\nabla u_n|^p|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt\right] \\
&\leq \beta(k+1)\Big[\int_Q d(t,x)|\varphi_\lambda(T_{2k+2}(u_n-u))|dxdt \\
&\quad + \frac{1}{c}\int_{\{|u_n|\leq k+1\}} [a(t,x,u_n,\nabla u_n)\cdot\nabla u_n - \nu]|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt \\
&\leq \beta(k+1)\Big[\int_Q d(t,x)|\varphi_\lambda(T_{2k+2}(u_n-u))|dxdt + \frac{1}{c}\int_Q |\nu\nabla\varphi_\lambda(T_{2k+2}(u_n-u))|dxdt \\
&\quad + \frac{1}{c}\int_{\{|u_n-u|\leq 2k+2\}} [a(t,x,u_n,\nabla u_n) - a(t,x,u_n,\nabla u))]\cdot\nabla(u_n-u)|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt \\
&\quad + \frac{1}{c}\int_{\{|u_n|\leq k+1\}} a(t,x,u_n,\nabla u_n)\cdot\nabla u_n|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt \\
&\quad + \frac{1}{c}\int_{\{|u_n|\leq k+1\}} a(t,x,u_n,\nabla u)\cdot\nabla(u_n-u)|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt\Big],
\end{aligned} \tag{2.43}$$

we first evaluate the term

$$
\begin{aligned}
&\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\psi_j(u_n)\sigma_k(u)\varphi_\lambda(T_{2k+2}(u_n-u))dxdt\\
&\le \varphi_\lambda(2k+2)\frac{1}{j}\int_{\{j\le|u_n|\le 2j\}}\left(\gamma+c_1|u_n|^{p-1}+c_2)|\nabla u_n|^{p-1}\right)|\nabla u_n|dxdt\\
&\le \varphi_\lambda(2k+2)\times\Big[\frac{1}{j}\int_{\{|u_n|\ge j\}}|\gamma|^{p'}ddt+C\frac{1}{j}\int_{\{j\le|u_n|\le 2j\}}|\nabla u_n|^p dxdt\\
&\qquad\qquad +c_1(2j)^{p-1}|\{|u_n|\ge j\}|^{\frac{1}{p'}}\frac{1}{j}\left(\int_{\{j\le|u_n|\le 2j\}}|\nabla u_n|^p dxdt\right)^{\frac{1}{p}}\Big]
\end{aligned}
$$

Using Theorem 2.6 (iii) we get the boundedness of the term $(2j)^{p-1}|\{u_n|\ge j\}|^{\frac{1}{p'}}(\frac{1}{j})^{\frac{1}{p'}}$, and recalling estimate (2.35) of Theorem 2.7 and the fact that the measure of $\{|u_n|\ge j\}$ decreases with j uniformly with respect to h, it is possible to choose $j_\epsilon\in\mathbb{N}$ such that

$$
\left|\int_Q (a(t,x,u_n,\nabla u_n))\cdot\nabla\psi_{n,j_\epsilon}(u_n)\sigma_k(u)\varphi_\lambda(T_{2k+2}(u_n-u))dxdt\right|\le\epsilon,\quad\forall n\in\mathbb{N},
$$

on the other hand, by property (1.5) on $\varphi_\lambda(s)$ and the choice of j_ϵ we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_Q[a(t,x,u_n,\nabla u_n)-a(t,x,u,\nabla u)]\cdot\nabla T_{2k+2}(u_n-u)\sigma_k(u)dxdt\\
&\le\int_Q[a(t,x,u_n,\nabla u_n)-a(t,x,u_n,\nabla u)]\cdot\nabla T_{2k+2}(u_n-u)\sigma_k(u)\\
&\qquad\times\left[\varphi'_\lambda(T_{2k+2}(u_n-u))-\frac{b(k+1)}{c}|\varphi_\lambda(T_{2k+2}(u_n-u))|\right]dxdt\\
&\le\int_Q f_n\psi_{n,j_\epsilon}dxdt+\int_Q F\cdot\nabla\psi_{n,j_\epsilon}dxdt\\
&\quad+\epsilon-\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\sigma_k(u)\psi_{j_\epsilon}(u_n)\varphi_\lambda(T_{2k+2}(u_n-u))dxdt\\
&\quad-\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla T_{2k+2}(u_n-u)\varphi'_\lambda(T_{2k+2}(u_n-u))\psi_{j_\epsilon}(u_n)\sigma_k(u)dxdt\\
&\quad+\beta(k+1)\Big[\int_Q d|\varphi_\lambda(T_{2k+2}(u_n-u))|dxdt+\frac{1}{c}\int_Q|\nu\varphi_\lambda(T_{2k+2}(u_n-u))|dxdt\\
&\quad+\frac{1}{c}\int_{\{|u_n|\le k+1\}}a(t,x,u_n,\nabla u_n)\nabla u|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt\\
&\quad+\frac{1}{c}\int_{\{|u_n|\le k+1\}}a(t,x,u_n,\nabla u)\cdot\nabla(T_{k+1}(u_n)-T_{k+1}(u))|\varphi_\lambda(T_{2k+2}(u_n-u))|\sigma_k(u)dxdt\Big]\\
&\quad-\int_{\{|u_n-u|\le 2k+2\}}\Phi_n(t,x,u,u_n)\cdot\nabla(u_n-u)\varphi'_\lambda(T_{2k+2}(u_n-u))\psi_{j_\epsilon}(u_n)\sigma_k(u)dxdt\\
&\quad-\int_Q\Phi_n(t,x,u_n)\nabla\psi_{j_\epsilon}\sigma_k(u)\varphi_\lambda(u)(T_{2k+2}(u_n-u))dxdt+\int_\Omega\psi_{n,j_\epsilon}dx-\int_0^T\langle(u_n)_t,\psi_{n,j_\epsilon}\rangle dt.
\end{aligned}
\tag{2.44}
$$

Notice that, by definition of ψ_{n,j_ϵ}, we have

$$(\psi_{n,j_\epsilon})_n \text{ is uniformly bounded and } \psi_{n,j_\epsilon} \to 0 \text{ a.e. in } Q,$$

in addition, we have

$$\begin{aligned}\nabla\psi_{n,j_\epsilon} =& \varphi'_\lambda(T_{2k+2}(u_n-u))(\nabla T_{3k+3}(u_n)-\nabla T_{k+1}(u))\chi_{\{|u_n-u|\leq 2k+2\}}\psi_{j_\epsilon}(u_n)\sigma_k(u)\\ &-\varphi_\lambda(T_{2k+2}(u_n-u))\cdot\nabla(T_{2j_\epsilon}(u_n)-T_{j_\epsilon}(u_n))\frac{\operatorname{sign}(u_n)}{j_\epsilon}\sigma_k(u)\\ &+\varphi_\lambda(T_{2k+2}(u_n-u))\psi_{j_\epsilon}(u_n)\cdot\nabla\sigma_k(u)\end{aligned} \tag{2.45}$$

then

$$\nabla\psi_{n,j_\epsilon} \rightharpoonup 0 \text{ weakly in } L^p(Q)^N, \tag{2.46}$$

and

$$\lim_{n\to\infty}\int_0^T\langle(u_n)_t,\psi_{n,j_\epsilon}\rangle dt=0,\quad \lim_{n\to\infty}\int_Q f_n\psi_{n,j_\epsilon}dxdt=0 \text{ and } \lim_{n\to\infty}\int_Q F_n\cdot\nabla\psi_{n,j_\epsilon}dxdt=0,$$

while from the weak convergence of $a(t,x,u_n,\nabla u_n)\psi_{j_\epsilon}(u_n)$ to $w\psi_{j_\epsilon}$ and the $L^p(Q)^N$-strong convergence of $\nabla\sigma_k(u)\varphi_\lambda(T_{2k+2}(u_n-u))$ to zero, we deduce that

$$\lim_{n\to\infty}\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\sigma_k(u)\psi_{j_\epsilon}(u_n)\varphi_\lambda(T_{2k+2}(u_n-u))dxdt=0. \tag{2.47}$$

Now, since $\nabla T_{2k+2}(u_n-u)\varphi'_\lambda(T_{2k+2}(u_n-u))\sigma_k(u)$ weakly converges to zero in $L^p(Q)^N$ and $a(t,x,u_n,\nabla u)\psi_{j_\epsilon}(u_n)$ strongly converges to $a(t,x,u,\nabla u)\psi_{j_\epsilon}$ in $L^{p'}(Q)^N$, the fourth integrals in the right-hand side of (2.44) tends to zero as n goes to infinity, while intergrals between square brackets tends to zero by dominated convergence theorem. Finally, to avoid the integrals involving the function Φ_n (remark that $b\in L^{\frac{N}{p-1}}(Q)$ and Φ is a Carathéodory function) we have

$$\begin{cases}|\Phi_n(t,x,u_n)\psi_{j_\epsilon}(u_n)|\leq \dfrac{|\Phi(t,x,u_n)|}{1+\frac{1}{n}|\Phi(t,x,u_n)|}\psi_{j_\epsilon}(u_n)\leq b(1+(2j_\epsilon)^{p-1}),\\ \dfrac{1}{n}|\Phi_n(t,x,u_n)|\leq\dfrac{1}{k}b(1+(2j_\epsilon)^{p-1})\to 0,\end{cases} \tag{2.48}$$

and thus,

$$\Phi_n(t,x,u_n)\psi_{j_\epsilon}(u_n)\to\Phi(t,x,u)\psi_{j_\epsilon}(u) \text{ a.e. in } Q \text{ and in } L^{\frac{N}{p-1}}(Q)\text{-norm}, \tag{2.49}$$

noticing that $\nabla T_{2k+2}(u_n-u)\sigma_k(u)$ weakly converges to zero in $L^p(Q)^N$, we obtain

$$\lim_{n\to\infty}\int_{\{|u_n-u|\leq 2k+2\}}\Phi_n(t,x,u_n)\cdot\nabla(u_n-u)\varphi'_\lambda(T_{2n+2}(u_n-u))\psi_{j_\epsilon(u_n)}\sigma_k(u)dxdt=0, \tag{2.50}$$

finally, can use the fact that $\Phi_n(t,x,u_n)\varphi_\lambda(T_{2k+2}(u_n-u))\chi_{\{|u_n|\leq 2j_\epsilon\}}$ converges to zero in $L^{\frac{N}{p-1}}(Q)^N$ and the fact that $\nabla\psi_{j_\epsilon}(u_n)\sigma_k(u)+\psi_{j_\epsilon}(u_n)\nabla\sigma_k(u)$ weakly converges to $\nabla\psi_{j_\epsilon}(u)\sigma_k(u)+\psi_{j_\epsilon}(u)\nabla\sigma_k(u)$ in $L^p(Q)^N$ to prove that the last term in (2.44) tends to zero, which concludes the proof of Theorem 2.8. □

A useful consequence of Theorem 2.8 is the following one.

Theorem 2.9. *Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$ be defined as before and let u be the limit of a subsequence of (u_n) (still denoted by u_n), then for every $k>0$*

$$\begin{cases} \nabla u_n \to \nabla u \text{ a.e. in } Q, \\ \nabla T_k(u_n)\chi_{\{|u|\le k\}} \to \nabla T_k(u)\chi_{\{|u|\le k\}} \text{ strongly in } L^p(Q)^N, \quad \forall n,k>0. \end{cases} \tag{2.51}$$

Proof. The proof of Theorem 2.9 follows exactly using the same steps of [1, Proposition 5.2]. □

3. Main result & proof

In this section we are going to prove the existence of a renormalized solutions of problem (1.1). Let us emphasize that we would like to be able to get a uniqueness of a solution of a renormalized type. In principle, according to our assumptions we are not allowed to do that. Anyway, after suitably additional conditions, this fact can be made rigorously by a not easy argument.

Theorem 3.1. *Suppose that $2-\frac{1}{N+1} < p < N$, $u_0 \in L^1(\Omega)$ and μ belongs to $\mathcal{M}_b(Q)$. Then, under assumptions (2.6)-(2.11) there exists a renormalized solution $u \in \mathcal{T}_0^{1,p}(Q)$ of problem (1.1) in the sense of Definition 2.1.*

Proof. Let us choose $\sigma_{k+1}\sigma_k(u)(u_n)\psi$, with $\psi \in W \cap L^\infty(Q)$, as test function in the approximate weak formulation (2.18), we get

$$\begin{aligned} &\int_0^T \langle (u_n)_t, \sigma_{k+1}(u_n)\sigma_k(u)\psi\rangle dt + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt \\ &+ \int_Q \Phi_n(t,x,u_n)\cdot\nabla\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt + \int_Q g_n(t,x,u_n,\nabla u_n)\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt \\ &= \int_Q f_n\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt + \int_Q F_n\cdot\nabla\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt. \end{aligned} \tag{3.1}$$

First, observe that, thanks to the fact that (f_n) converges to f weakly in $L^1(Q)$ and (F_n) converges to F weakly in $L^{p'}(Q)^N$, the last two terms easily pass to their limits as n tends to infinity; actually the only term that give some problems are the others, we can write, thanks to Theorem2.6 (ii) and Theorem 2.9, that

$$\begin{cases} a(t,x,u_n,\nabla u_n)\chi_{\{|u_n|\le k+2\}} \rightharpoonup a(t,x,u,\nabla u)\chi_{\{|u|\le k+2\}} \text{ in } L^{p'}(Q)^N, \\ \nabla(\sigma_k(u)\sigma_{k+1}(u_n)\psi) \to \nabla(\sigma_k(u)\psi) \text{ in } L^p(Q)^N, \end{cases} \tag{3.2}$$

thus,

$$\lim_{n\to\infty}\int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt = \int_Q a(t,x,u,\nabla u)\cdot\nabla\sigma_k(u)\psi dxdt \tag{3.3}$$

on the other hand

$$\begin{cases} |\Phi_n(t,x,u_n)\chi_{\{|u_n|\le k+2\}}| \le b(1+(k+2)^{p-1}) \in L^{\frac{N}{p-1}}(Q), \\ \left|\dfrac{1}{k}\Phi_n(t,x,u_n)\chi_{\{|u_n|\le k+2\}}\right| \le \lim\limits_{n\to\infty}\dfrac{1}{k}b(1+(k+2)^{p-1}) = 0, \end{cases} \tag{3.4}$$

which imply that

$$\Phi_n(t,x,u_n)\chi_{\{|u_n|\le k+2\}} \underset{n\to\infty}{\to} \Phi(t,x,u_n)\chi_{\{|u|\le k+2\}} \text{ a.e. in Q and in } L^{\frac{N}{p-1}}(Q)^N, \tag{3.5}$$

that is,

$$\lim_{n\to\infty}\int_Q \Phi_n(t,x,u_n)\cdot\nabla\sigma_k(u)\psi dxdt = \int_Q \Phi(t,x,u)\cdot\nabla\sigma_k(u)\psi dxdt. \tag{3.6}$$

Now, thanks to the growth assumptions on g and to the fact that g in a *Carathéodory* function we readily have

$$|g_n(t,x,u_n,\nabla u_n)|\chi_{\{|u_n|\le k+2\}}\chi_{\{|u|\le k+1\}} \le \beta(k+2)(d+|\nabla T_{k+2}(u_n)|^p)\chi_{\{|u|\le k+1\}}, \tag{3.7}$$

which is strongly convergent in $L^1(Q)$, that is

$$g_n(t,x,u_n,\nabla u_n)\chi_{\{|u_n|\le k+2\}}\chi_{\{|u|\le k+1\}} \to g_n(t,x,u,\nabla u)\chi_{\{|u|\le k+1\}} \text{ a.e. in } Q \text{ and in } L^1(Q). \tag{3.8}$$

Therefore we deduce that

$$\lim_{n\to\infty}\int_Q g(t,x,u_n,\nabla u_n)\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt = \int_Q g(t,x,u,\nabla u)\sigma_k(u)\psi dxdt, \tag{3.9}$$

and finally, since

$$\lim_{n\to\infty}\int_0^T \langle (u_n)_t, \sigma_{k+1}(u_n)\sigma_k\psi\rangle dt = \int_0^T \langle u_t, \sigma_k(u)\psi\rangle dt, \tag{3.10}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\int_Q f_n(u)\sigma_{k+1}(u_n)\sigma_k\psi dxdt + \int_Q F\cdot\nabla\sigma_{k+1}(u_n)\sigma_k(u)\psi dxdt\\ &= \int_Q f\sigma_k(u)\psi dxdt + \int_Q F\cdot\nabla\sigma_k(u)\psi dxdt,\end{aligned} \tag{3.11}$$

then u satisfies the renormalized formulation (2.12) of Definition 2.1; indeed, the argument of Theorem 2.7 shows that u satisfies the asymptotic estimate (2.13) which concludes the proof of Theorem 3.1. □

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[1] Laboratory of Mathematics and Their Interactions, Faculty of Sciences, Moulay Ismaïl University, B.P. 11201, Zitoune, 50 500, Meknès, Morocco
Email address: m.abdellaoui@umi.ac.ma