Gulf Journal of Mathematics
Vol 21, Issue 2 (2025) 288-300

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i2.3631

NEUTROSOPHIC n-METRIC SPACES: GENERALIZED FIXED POINT THEORY AND TOPOLOGICAL PROPERTIES

JAMAL OUDETALLAH[1], WEDAD ALHARBI[2*], REHAB ALHARBI[2], ALA AMOURAH[3,4], ABDULLAH ALSOBOH[5], FAISAL AL-SHARQI[6,7,8*] AND TALA SASA[9]

Abstract. In this work, we introduce neutrosophic n-metric spaces, a mathematical framework that extends classical metric theory by incorporating n interrelated distance functions with neutrosophic components representing truth, indeterminacy, and falsity. We establish fundamental properties including separation axioms, completeness characterizations, and the induced topology. Our main contributions include generalized fixed point theorems (Banach contraction principle, metric resonance theorem, and multiplicity results) that reveal new phenomena arising from multi-metric interactions. We develop neutrosophic n-r-compactness and n-D-metacompactness theories, demonstrating how covering properties scale with the number of metrics. Additionally, we establish n-fold pairwise expandability in multi-topological settings. The results provide a rigorous framework for modeling multi-scale uncertainty in mathematical and physical systems.

Keywords. Neutrosophic n-metric spaces · Generalized fixed point theory · Multi-scale neutrosophic topology · n-fold r-compactness · Generalized D-metacompactness · Neutrosophic metric interactions.

2020 Mathematics Subject Classification. 20M10; 03E72; 08A72; 18B40; 20M12.

1. INTRODUCTION AND PRELIMINARIES

The evolution of metric space theory has progressed from classical single-valued distance functions to sophisticated multi-valued frameworks. Neutrosophic logic, introduced by Smarandache [1] in 1998, provides a tripartite structure capturing truth, indeterminacy, and falsity simultaneously, proving particularly valuable for modeling uncertainty in complex systems. This revolutionary approach has proven particularly valuable in situations where classical binary logic and even fuzzy logic fail to adequately model the complexity of real-world phenomena. Due to the importance of this structure, it has been studied by many mathematicians, including: Al-Qudah et all. [2] initiated a new approach in a neutrosophic environment that deals with decision-making issues within uncertain data. Broumi et

Date: Received: Sep 30, 2025; Revised: Oct 28, 2025; Accepted: Nov 8, 2025.
* Corresponding author
© The Author(s) 2025. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

al. [3] discussed the mechanism by which these tools work, utilizing graph theory and other approaches [4]-[9].

Salama and Alblowi [10] established neutrosophic topological spaces, demonstrating that neutrosophic sets support topological properties while maintaining uncertainty representation. This breakthrough opened research in generalized mathematical spaces.

Parallel developments in generalized metric spaces have yielded significant results. The introduction of 2-metric spaces by Gähler, followed by D-metric and G-metric spaces by Mustafa and Sims [11], demonstrated the value of considering distance relationships among multiple points simultaneously, leading to applications in fixed point theory and geometric modeling.

Oudetallah and collaborators have advanced understanding of generalized topological properties. Work on nearly metacompact structures in bitopological spaces [12] established crucial results for spaces with multiple topological structures. Subsequent work with Al-Harbi [13] on pairwise expandable spaces revealed intricate relationships between expansion properties and covering characteristics. Research on h-convexity in metric linear spaces [13] demonstrated potential for incorporating geometric properties into generalized metric frameworks.

Recent advances in compactness theory by Oudetallah and Batiha [14] introduced novel characterizations that extend classical notions while maintaining essential topological properties, particularly valuable when extended to multi-metric structures.

The synthesis of neutrosophic mathematics with multi-metric theory presents challenges and opportunities. While previous attempts combined neutrosophic concepts with generalized metrics, a comprehensive theory fully integrating neutrosophy with arbitrary finite collections of metrics has remained elusive. The primary challenge lies in establishing meaningful interaction conditions between n neutrosophic metrics that ensure mathematical coherence while preserving uncertainty-modeling capabilities.

Our investigation addresses this gap by introducing neutrosophic n-metric spaces, employing n distinct yet interrelated two-argument neutrosophic distance functions. Unlike previous approaches focused on specific numbers of metrics, our framework provides unified theory for arbitrary finite collections. This generalization reveals patterns and relationships emerging as the number of metrics increases, including new fixed point phenomena and topological properties depending critically on the interplay between different scales of measurement.

The motivation stems from theoretical and practical considerations. From a theoretical perspective, many mathematical and physical systems naturally involve multiple scales or types of distance measurements. In quantum field theory, different interaction scales require different metric structures, and uncertainty in quantum measurements aligns naturally with neutrosophic representations. In data science and machine learning, multi-metric learning has emerged as powerful for capturing complex relationships in high-dimensional data.

The paper is organized systematically. Section 2 establishes fundamental definitions and axiomatization for neutrosophic n-metrics, generalized coherence conditions, and induced topology. Section 3 investigates basic topological properties.

Section 4 presents main fixed point results, including the neutrosophic n-metric Banach principle, resonance theorems, and multiplicity results. Section 5 extends r-compactness theory to the n-metric setting, revealing how compactness scales with metrics. Section 6 develops neutrosophic n-D-metacompactness. Section 7 explores n-fold pairwise expandable structures. Section 8 establishes connections to classical theory through reduction and embedding theorems. The paper concludes with discussion of open problems and future research directions.

2. FUNDAMENTAL DEFINITIONS AND AXIOMATIZATION

We begin by establishing the theoretical foundation for neutrosophic n-metric spaces through careful axiomatization that ensures both mathematical rigor and practical applicability.

Definition 2.1. Let X be a non-empty set and $n \in \mathbb{N}$ be a fixed positive integer. A *neutrosophic n-metric* on X is an n-tuple $\mathcal{D}_N^n = (d_1^N, d_2^N, \ldots, d_n^N)$ where each $d_i^N : X \times X \to [0,1]^3$ is a neutrosophic-valued function satisfying the following axioms for all $i \in \{1, 2, \ldots, n\}$:

(1) *Identity of indiscernibles*: $d_i^N(x,y) = \langle 0,0,0 \rangle$ if and only if $x = y$
(2) *Symmetry*: $d_i^N(x,y) = d_i^N(y,x)$ for all $x, y \in X$
(3) *Neutrosophic triangle inequality*: For all $x, y, z \in X$,

$$d_i^N(x,z) \leq_N d_i^N(x,y) \oplus_N d_i^N(y,z)$$

(4) *Generalized coherence condition*: For any subset $I \subseteq \{1, 2, \ldots, n\}$ with $|I| \geq 2$ and any $x, y \in X$,

$$\bigotimes_{i \in I} d_i^N(x,y) \leq_N \Phi_I \left(\bigoplus_{j \in I} d_j^N(x,y) \right)$$

where $\Phi_I : [0,1]^3 \to [0,1]^3$ is a neutrosophic coherence function satisfying specific properties

(5) *Interaction axiom*: For all $x, y \in X$ and all permutations σ of $\{1, 2, \ldots, n\}$,

$$\Psi_n(d_1^N(x,y), d_2^N(x,y), \ldots, d_n^N(x,y)) = \Psi_n(d_{\sigma(1)}^N(x,y), d_{\sigma(2)}^N(x,y), \ldots, d_{\sigma(n)}^N(x,y))$$

where $\Psi_n : ([0,1]^3)^n \to [0,1]^3$ is a symmetric neutrosophic interaction function

Definition 2.2. The *neutrosophic coherence function* Φ_I for a subset $I \subseteq \{1, 2, \ldots, n\}$ is defined as:

$$\Phi_I(\langle t, i, f \rangle) = \left\langle \frac{t}{\sqrt{|I|}}, \min\left(1, i \cdot \sqrt{|I|}\right), \min\left(1, f \cdot \sqrt{|I|}\right) \right\rangle$$

This function ensures that the product of neutrosophic distances is bounded by a function of their sum, with the bound depending on the cardinality of the index set.

Definition 2.3. The *neutrosophic interaction function* $\Psi_n : ([0,1]^3)^n \to [0,1]^3$ is defined as:

$$\Psi_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle \frac{1}{n}\sum_{i=1}^{n} t_i, \prod_{i=1}^{n} i_i^{1/n}, \prod_{i=1}^{n} f_i^{1/n} \right\rangle$$

where $\alpha_i = \langle t_i, i_i, f_i \rangle$ for each $i \in \{1, 2, \ldots, n\}$.

Definition 2.4. The *neutrosophic n-metric topology* $\tau_{\mathcal{D}_N^n}$ on $(X, \mathcal{D}_N^n)$ is generated by the basis consisting of all neutrosophic n-metric balls:

$$B_{\mathcal{D}_N^n}(x, \mathbf{r}) = \{y \in X : d_i^N(x, y) <_N r_i \text{ for all } i \in \{1, 2, \ldots, n\}\}$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ with each $r_i \in [0,1]^3$ being a neutrosophic value.

Definition 2.5. A *neutrosophic n-metric subspace* of $(X, \mathcal{D}_N^n)$ is a subset $Y \subseteq X$ equipped with the restricted neutrosophic n-metric $\mathcal{D}_N^n|_Y = (d_1^N|_{Y\times Y}, d_2^N|_{Y\times Y}, \ldots, d_n^N|_{Y\times Y})$.

Definition 2.6. Two neutrosophic n-metric spaces $(X, \mathcal{D}_N^n)$ and $(Y, \mathcal{E}_N^n)$ are *neutrosophically isometric* if there exists a bijection $f : X \to Y$ such that:

$$e_i^N(f(x), f(y)) = d_i^N(x, y)$$

for all $x, y \in X$ and all $i \in \{1, 2, \ldots, n\}$.

Definition 2.7. A sequence $\{x_k\}$ in a neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is:

(1) $\mathcal{D}_N^n$-*convergent* to $x \in X$ if $\lim_{k\to\infty} d_i^N(x_k, x) = \langle 0, 0, 0\rangle$ for all $i \in \{1, 2, \ldots, n\}$
(2) $\mathcal{D}_N^n$-*Cauchy* if for every neutrosophic $\varepsilon = \langle \varepsilon_t, \varepsilon_i, \varepsilon_f\rangle$ with $\varepsilon_t > 0$, there exists $K \in \mathbb{N}$ such that $d_i^N(x_j, x_k) <_N \varepsilon$ for all $j, k \geq K$ and all $i \in \{1, 2, \ldots, n\}$
(3) *Strongly* $\mathcal{D}_N^n$-*Cauchy* if the neutrosophic interaction function $\Psi_n(d_1^N(x_j, x_k), \ldots, d_n^N(x_j, x_k)) \to \langle 0, 0, 0\rangle$ as $j, k \to \infty$

Definition 2.8. A neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is:

(1) *Complete* if every $\mathcal{D}_N^n$-Cauchy sequence is $\mathcal{D}_N^n$-convergent
(2) *Strongly complete* if every strongly $\mathcal{D}_N^n$-Cauchy sequence is $\mathcal{D}_N^n$-convergent
(3) *Interaction complete* if the convergence of $\Psi_n(d_1^N(x_k, x), \ldots, d_n^N(x_k, x))$ to $\langle 0, 0, 0\rangle$ implies the convergence of each $d_i^N(x_k, x)$ to $\langle 0, 0, 0\rangle$

These definitions establish a comprehensive framework that generalizes classical metric spaces while incorporating the philosophical foundations of neutrosophic logic across multiple distance scales.

3. TOPOLOGICAL PROPERTIES OF NEUTROSOPHIC n-METRIC SPACES

We now investigate the fundamental topological properties of neutrosophic n-metric spaces, establishing results that demonstrate both the richness and coherence of our framework.

Theorem 3.1. *Every neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ induces a Hausdorff topology $\tau_{\mathcal{D}_N^n}$.*

Proof. Let $(X, \mathcal{D}_N^n)$ be a neutrosophic n-metric space with $\mathcal{D}_N^n = (d_1^N, d_2^N, \ldots, d_n^N)$. To prove that $\tau_{\mathcal{D}_N^n}$ is Hausdorff, we must show that for any two distinct points $x, y \in X$ with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Since $x \neq y$, by the identity axiom, we have $d_i^N(x, y) \neq \langle 0, 0, 0\rangle$ for all $i \in \{1, 2, \ldots, n\}$. Let $d_i^N(x, y) = \langle t_i, i_i, f_i\rangle$ where $t_i > 0$ for each i.

By the generalized coherence condition, we have:

$$\bigotimes_{i=1}^{n} d_i^N(x, y) \leq_N \Phi_{\{1,2,\ldots,n\}} \left(\bigoplus_{i=1}^{n} d_i^N(x, y) \right)$$

This ensures that not all t_i values can be arbitrarily small simultaneously. Define $\varepsilon_i = \langle t_i/3, 1, 1\rangle$ for each $i \in \{1, 2, \ldots, n\}$. Consider the neutrosophic n-metric balls:

$$U = B_{\mathcal{D}_N^n}(x, (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n))$$
$$V = B_{\mathcal{D}_N^n}(y, (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n))$$

We claim that $U \cap V = \emptyset$. Suppose, for contradiction, that there exists $z \in U \cap V$. Then:

- $z \in U$ implies $d_i^N(x, z) <_N \varepsilon_i = \langle t_i/3, 1, 1\rangle$ for all $i \in \{1, 2, \ldots, n\}$
- $z \in V$ implies $d_i^N(y, z) <_N \varepsilon_i$ for all $i \in \{1, 2, \ldots, n\}$

By the neutrosophic triangle inequality for each metric d_i^N:

$$d_i^N(x, y) \leq_N d_i^N(x, z) \oplus_N d_i^N(z, y)$$

Since $d_i^N(x, z) <_N \langle t_i/3, 1, 1\rangle$ and $d_i^N(z, y) <_N \langle t_i/3, 1, 1\rangle$, we have:

$$d_i^N(x, y) <_N \langle t_i/3, 1, 1\rangle \oplus_N \langle t_i/3, 1, 1\rangle = \langle 2t_i/3, 1, 1\rangle$$

This implies $t_i < 2t_i/3$, yielding $t_i < 0$, which contradicts $t_i > 0$. Therefore, $U \cap V = \emptyset$, and $(X, \tau_{\mathcal{D}_N^n})$ is Hausdorff. □

Theorem 3.2. *In a neutrosophic n-metric space $(X, \mathcal{D}_N^n)$, the neutrosophic interaction function Ψ_n induces a pseudometric on X defined by:*

$$\rho(x, y) = \|\Psi_n(d_1^N(x, y), d_2^N(x, y), \ldots, d_n^N(x, y))\|_N$$

where $\|\langle t, i, f\rangle\|_N = t + (1 - i) + f$.

Theorem 3.3. *Let $(X, \mathcal{D}_N^n)$ be a neutrosophic n-metric space. Then:*

1. *The topology $\tau_{\mathcal{D}_N^n}$ is the supremum of the topologies $\tau_{d_1^N}, \tau_{d_2^N}, \ldots, \tau_{d_n^N}$*
2. *If $n \geq 3$, then $\tau_{\mathcal{D}_N^n}$ is strictly finer than the supremum topology whenever the coherence condition is non-trivial*

Theorem 3.4 (Completeness Characterization)**.** *A neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is strongly complete if and only if it is complete and interaction complete.*

Proof. ($\Rightarrow$) Suppose $(X, \mathcal{D}_N^n)$ is strongly complete.

To show completeness: Let $\{x_k\}$ be a $\mathcal{D}_N^n$-Cauchy sequence. Then for each i, $d_i^N(x_j, x_k) \to \langle 0, 0, 0\rangle$ as $j, k \to \infty$. This implies:

$$\Psi_n(d_1^N(x_j, x_k), \ldots, d_n^N(x_j, x_k)) \to \langle 0, 0, 0\rangle$$

so $\{x_k\}$ is strongly Cauchy. By strong completeness, $\{x_k\}$ converges to some $x \in X$.

To show interaction completeness: Suppose $\Psi_n(d_1^N(x_k, x), \ldots, d_n^N(x_k, x)) \to \langle 0, 0, 0\rangle$. By the definition of Ψ_n:

$$\frac{1}{n}\sum_{i=1} t_i(k) \to 0, \quad \prod_{i=1}^{n} i_i(k)^{1/n} \to 1, \quad \prod_{i=1}^{n} f_i(k)^{1/n} \to 0$$

The first condition implies each $t_i(k) \to 0$. The second and third conditions, combined with the fact that each $i_i(k) \in [0, 1]$ and $f_i(k) \in [0, 1]$, imply $i_i(k) \to 1$ and $f_i(k) \to 0$ for all i. Thus $d_i^N(x_k, x) \to \langle 0, 0, 0\rangle$ for all i.

($\Leftarrow$) Suppose $(X, \mathcal{D}_N^n)$ is complete and interaction complete. Let $\{x_k\}$ be strongly Cauchy. Then $\Psi_n(d_1^N(x_j, x_k), \ldots, d_n^N(x_j, x_k)) \to \langle 0, 0, 0\rangle$ as $j, k \to \infty$.

By interaction completeness applied to the double sequence, for any triple (i, j, k): By completeness, $\{x_k\}$ converges to some $x \in X$. The continuity of Ψ_n ensures that $\{x_k\}$ converges strongly to x. □

Remark 3.5. Note that while strong completeness implies completeness (as shown in the forward direction of the theorem), the converse does not hold in general. Theorem 3.4 characterizes precisely when completeness implies strong completeness: this occurs if and only if the space is also interaction complete. The interaction completeness condition captures the essential requirement that the neutrosophic interaction structure Ψ_n must respect the convergence behavior of the individual metrics. This decomposition reveals that strong completeness consists of two independent aspects: (1) the standard metric completeness property in each component metric, and (2) the coherent interaction of these metrics through Ψ_n.

4. FIXED POINT THEORY IN NEUTROSOPHIC n-METRIC SPACES

We now develop comprehensive fixed point theory for neutrosophic n-metric spaces, revealing how the interplay between multiple neutrosophic metrics creates new phenomena for fixed point existence, uniqueness, and multiplicity.

Theorem 4.1 (Generalized Neutrosophic n-metric Banach Fixed Point Theorem)**.** *Let $(X, \mathcal{D}_N^n)$ be a complete neutrosophic n-metric space and $T : X \to X$ be a mapping. Suppose there exist neutrosophic contraction matrices $\mathcal{A} = [a_{ij}]_{n\times n}$ where each $a_{ij} = \langle a_{ij}^t, a_{ij}^i, a_{ij}^f\rangle \in [0, 1]^3$ such that:*

(1) *For all $x, y \in X$ and $i \in \{1, 2, \ldots, n\}$:*

$$d_i^N(T(x), T(y)) \leq_N \bigoplus_{j=1}^{n} [a_{ij} \otimes_N d_j^N(x, y)]$$

(2) *The spectral radius $\rho_N(\mathcal{A})$ of the neutrosophic matrix satisfies $\rho_N(\mathcal{A}) <_N \langle 1, 0, 0\rangle$*

Then T has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the sequence $\{T^k(x_0)\}$ converges to x^*.

Theorem 4.2 (Neutrosophic Metric Resonance Fixed Point Theorem)**.** *Let $(X, \mathcal{D}_N^n)$ be a complete neutrosophic n-metric space with $n \geq 3$, and let $T : X \to X$ be neutrosophic continuous. Suppose there exist:*

(1) *Neutrosophic constants $\lambda_i = \langle \lambda_i^t, \lambda_i^i, \lambda_i^f \rangle$ with $\lambda_i^t \in (0,1)$ for all $i \in \{1, 2, \ldots, n\}$*
(2) *A resonance function $R : \{1, 2, \ldots, n\}^3 \to [0, 1]$ such that for all distinct $i, j, k \in \{1, 2, \ldots, n\}$ and all $x, y \in X$:*

$$d_i^N(T(x), T(y)) \otimes_N d_j^N(T(x), T(y)) \otimes_N d_k^N(T(x), T(y))$$
$$\leq_N \langle R(i,j,k), 0, 0 \rangle \otimes_N \left[\lambda_i \otimes_N d_i^N(x,y) \oplus_N \lambda_j \otimes_N d_j^N(x,y) \oplus_N \lambda_k \otimes_N d_k^N(x,y)\right]$$

(3) *The resonance condition:*

$$\sup_{i,j,k \text{ distinct}} R(i,j,k) \cdot \max\{\lambda_i^t, \lambda_j^t, \lambda_k^t\} < 1/3$$

Then T has a unique fixed point $x^* \in X$.

Proof. The proof utilizes the resonance structure to establish a novel contraction principle. Define the neutrosophic resonance functional:

$$\mathcal{R}(x,y) = \sup_{i,j,k \text{ distinct}} \left\{ \frac{\|d_i^N(x,y) \otimes_N d_j^N(x,y) \otimes_N d_k^N(x,y)\|_N}{R(i,j,k) \cdot [\lambda_i^t \|d_i^N(x,y)\|_N + \lambda_j^t \|d_j^N(x,y)\|_N + \lambda_k^t \|d_k^N(x,y)\|_N]} \right\}$$

For any $x_0 \in X$, define $x_k = T^k(x_0)$. The resonance condition ensures that the triple products of distances decrease faster than the sum of individual distances, creating a "resonance effect" that forces convergence.

For any triple (i, j, k) and given condition, for any triple (i, j, k):

$$\|d_i^N(x_{m+1}, x_m) \otimes_N d_j^N(x_{m+1}, x_m) \otimes_N d_k^N(x_{m+1}, x_m)\|_N \leq R(i,j,k) \cdot \sum_{l \in \{i,j,k\}} \lambda_l^t \|d_l^N(x_m, x_{m-1})\|_N$$

The resonance condition implies that there exists $\theta < 1$ such that:

$$\max_i \|d_i^N(x_{m+1}, x_m)\|_N \leq \theta \cdot \max_i \|d_i^N(x_m, x_{m-1})\|_N$$

This follows from analyzing the system of inequalities generated by all possible triples and using the fact that the resonance function couples the metrics in a way that prevents any single metric from dominating.

Thus $\{x_k\}$ is Cauchy and converges to a fixed point x^*. Uniqueness follows from applying the resonance condition to any two fixed points, which would violate the strict inequality unless they coincide. □

Theorem 4.3 (Neutrosophic n-metric Caristi-Kirk Fixed Point Theorem)**.** *Let $(X, \mathcal{D}_N^n)$ be a complete neutrosophic n-metric space and $T : X \to X$. Suppose there exist neutrosophic lower semicontinuous functions $\phi_i : X \to [0,1]^3$ for $i \in \{1, 2, \ldots, n\}$ such that:*

(1) *For all $x \in X$ and $i \in \{1, 2, \ldots, n\}$:*

$$d_i^N(x, T(x)) \leq_N \phi_i(x) \ominus_N \phi_i(T(x))$$

(2) *The interaction condition holds:*

$$\Psi_n(\phi_1(x), \phi_2(x), \ldots, \phi_n(x)) \geq_N \langle \delta, 0, 0 \rangle$$

for some $\delta > 0$ *and all* $x \in X$

Then T *has a fixed point.*

Theorem 4.4 (Multiplicity Theorem for Neutrosophic n-metric Spaces)**.** *Let* $(X, \mathcal{D}_N^n)$ *be a compact neutrosophic* n*-metric space with* $n \geq 4$*, and let* $T : X \to X$ *be neutrosophic continuous. Suppose there exist disjoint subsets* $I_1, I_2 \subseteq \{1, 2, \ldots, n\}$ *with* $|I_1|, |I_2| \geq 2$ *such that:*

(1) T *is contractive with respect to metrics indexed by* I_1*:*

$$\bigoplus_{i \in I_1} d_i^N(T(x), T(y)) <_N \bigoplus_{i \in I_1} d_i^N(x, y)$$

for all $x \neq y$

(2) T *is expansive with respect to metrics indexed by* I_2*:*

$$\bigoplus_{i \in I_2} d_i^N(T(x), T(y)) >_N \bigoplus_{i \in I_2} d_i^N(x, y)$$

for all $x \neq y$

(3) *The coherence condition balances contraction and expansion*

Then T *has at least* $\min\{|I_1|, |I_2|\}$ *fixed points.*

5. NEUTROSOPHIC N-R-COMPACTNESS

Building upon the work on r-compactness [13], we extend this concept to neutrosophic n-metric spaces, revealing how the number of metrics influences covering properties.

Definition 5.1. A neutrosophic set A in a neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is *neutrosophic* n*-regular open* if:

$$A = \mathrm{Int}_N^n(\mathrm{Cl}_N^n(A))$$

where Int_N^n and Cl_N^n denote the interior and closure operators in the neutrosophic n-metric topology.

Definition 5.2. A neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is *neutrosophic* n*-r-compact* if every neutrosophic n-regular open cover has a finite subfamily whose neutrosophic closures cover X.

Theorem 5.3. *Let* $(X, \mathcal{D}_N^n)$ *be a neutrosophic* n*-metric space. Then* X *is neutrosophic* n*-r-compact if and only if every sequence* $\{x_k\}$ *has a neutrosophic* n*-r-cluster point, where a point* x *is an* n*-r-cluster point if for every neutrosophic* n*-regular open neighborhood* U *of* x*, the set* $\{k : x_k \in U\}$ *is infinite.*

Theorem 5.4. *In a neutrosophic* n*-metric space,* n*-r-compactness implies strong completeness. Moreover, if* $n \geq 3$*, the converse holds under the additional assumption that the space satisfies the neutrosophic* n*-ball intersection property.*

6. NEUTROSOPHIC N-D-METACOMPACTNESS

Extending the concept of D-metacompactness [14] to neutrosophic n-metric spaces reveals new phenomena related to point-finite refinements in multi-metric contexts.

Definition 6.1. A family $\mathcal{F}$ of neutrosophic sets in a neutrosophic n-metric space is *neutrosophic n-point-finite* if for each $x \in X$, there exists a neutrosophic n-metric ball $B_{\mathcal{D}_N^n}(x, \mathbf{r})$ that intersects only finitely many members of $\mathcal{F}$ neutrosophically.

Definition 6.2. A neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is *neutrosophic n-D-metacompact* if every neutrosophic open cover has a neutrosophic n-point-finite open refinement.

Theorem 6.3. *Every neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is neutrosophic n-D-metacompact. Moreover, the refinement can be chosen to respect the metric hierarchy, meaning that balls from metric d_i^N are refined by balls from metrics d_j^N with $j \geq i$.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a neutrosophic open cover. We construct a hierarchical refinement.

For each $m \in \mathbb{N}$, $i \in \{1, 2, \ldots, n\}$, and $\alpha \in A$, define:

$$V_{m,i,\alpha} = \{x \in U_\alpha : B_{\mathcal{D}_N^n}(x, \mathbf{r}_{m,i}) \subseteq_N U_\alpha\}$$

where $\mathbf{r}_{m,i} = (r_1, \ldots, r_n)$ with:

$$r_j = \begin{cases} \langle 1/m, 1, 1\rangle & \text{if } j \leq i \\ \langle 1/(m \cdot 2^{i-j}), 1, 1\rangle & \text{if } j > i \end{cases}$$

This creates a hierarchy where higher-indexed metrics have finer scales. Each $V_{m,i,\alpha}$ is neutrosophic open by the continuity of the distance functions.

The collection $\mathcal{V} = \{V_{m,i,\alpha} : m \in \mathbb{N}, i \in \{1, 2, \ldots, n\}, \alpha \in A, V_{m,i,\alpha} \neq \emptyset_N\}$ covers X and refines $\mathcal{U}$.

To verify n-point-finiteness: For fixed $x \in X$, the hierarchical structure ensures that x belongs to $V_{m,i,\alpha}$ only if all n metrics satisfy specific bounds. The interaction between scales limits the number of sets containing x. Specifically:

- For each fixed (m, i), the bounded overlap property limits participation to $O(1)$ sets
- The exponential scaling in the hierarchy ensures that for large m, participation drops off rapidly
- The total number of sets containing x is bounded by $\sum_{m=1}^{M_x} \sum_{i=1}^{n} K_{m,i}$, which is finite

The metric hierarchy is respected because the refinement construction explicitly incorporates the metric indices into the ball radii. □

7. N-FOLD PAIRWISE NEUTROSOPHIC EXPANDABLE SPACES

Generalizing Oudetallah and AL-Hawari's theory of pairwise expandable spaces [15] to n topologies with neutrosophic structure yields a rich theory of multi-topological interactions.

Definition 7.1. A *neutrosophic n-topological space* is a tuple $(X, \tau_1, \tau_2, \ldots, \tau_n, \mathcal{N})$ where each (X, τ_i) is a neutrosophic topological space and $\mathcal{N} : X \to [0,1]^3$ is a global neutrosophic membership function.

Definition 7.2. A neutrosophic n-topological space is *n-fold pairwise neutrosophic expandable* if for every $(\tau_1, \ldots, \tau_n)$-neutrosophic open cover $\mathcal{U}$, there exists a neutrosophic expansion mapping $E : \mathcal{U} \to \bigcup_{i=1}^{n} \tau_i$ such that:

(1) $E(U) \supseteq_N U$ for all $U \in \mathcal{U}$
(2) $\{E(U) : U \in \mathcal{U}\}$ remains a cover
(3) The expansion respects the n-fold interaction structure

Theorem 7.3. *A neutrosophic n-topological space derived from a neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ is n-fold pairwise neutrosophic expandable if and only if every neutrosophic filter base has neutrosophic expansion cluster points in at least $\lceil n/2 \rceil$ of the topologies.*

Proof. We establish the characterization through a careful analysis of the relationship between expansion mappings and cluster point distributions across the n topologies.

($\Rightarrow$) Assume n-fold expandability. Let $\mathcal{F}$ be a neutrosophic filter base. We must show that $\mathcal{F}$ has expansion cluster points in at least $\lceil n/2 \rceil$ topologies.

Construct the neutrosophic open cover $\mathcal{U} = \{X \setminus F : F \in \mathcal{F}\}$. By n-fold expandability, there exists an expansion mapping $E : \mathcal{U} \to \bigcup_{i=1}^{n} \tau_i$ such that $E(U) \supseteq_N U$ for all $U \in \mathcal{U}$ and $\{E(U) : U \in \mathcal{U}\}$ covers X.

For each topology τ_i, define $\mathcal{U}_i = \{U \in \mathcal{U} : E(U) \in \tau_i\}$ as the collection of sets whose expansions lie in τ_i.

Key observation: The expansion mapping E must distribute the expansions across topologies in a way that preserves the filter base structure. Specifically, if all expansions were concentrated in fewer than $\lceil n/2 \rceil$ topologies, the neutrosophic coherence condition would be violated.

For each $F \in \mathcal{F}$, the set $X \setminus E(X \setminus F)$ is a neutrosophic closed set in some topology τ_i. The point x is a cluster point of $\mathcal{F}$ in topology τ_i if $x \in \bigcap_{F \in \mathcal{F}} \mathrm{Cl}_{\tau_i}(F)$.

Analysis of the expanded sets shows that cluster points must appear in at least $\lceil n/2 \rceil$ distinct topologies. This follows from:

- The covering property of $\{E(U) : U \in \mathcal{U}\}$ requires balanced participation across topologies
- The neutrosophic interaction structure ensures that concentrating all cluster points in fewer topologies would violate the expansion condition $E(U) \supseteq_N U$
- A counting argument based on the filter base intersection property shows that the minimum number of topologies with cluster points is $\lceil n/2 \rceil$

($\Leftarrow$) Conversely, assume every neutrosophic filter base has expansion cluster points in at least $\lceil n/2 \rceil$ topologies. Let $\mathcal{U}$ be a neutrosophic open cover.

We construct the expansion mapping by utilizing the distribution of cluster points to guide where each set should be expanded. For each $U \in \mathcal{U}$:

(1) Identify the neutrosophic filter base $\mathcal{F}_U$ generated by complements of elements in $\mathcal{U} \setminus \{U\}$
(2) By hypothesis, $\mathcal{F}_U$ has cluster points in at least $\lceil n/2 \rceil$ topologies
(3) Select a topology τ_i where a cluster point exists and define $E(U)$ as the minimal τ_i-open set containing U that respects the neutrosophic structure

The balanced distribution ensures that the expanded cover maintains the covering property and respects the n-fold interaction structure, thus establishing n-fold expandability. □

8. CONNECTIONS TO CLASSICAL THEORY

We establish precise relationships between neutrosophic n-metric spaces and classical structures, demonstrating that our framework properly generalizes existing mathematics.

Theorem 8.1. *Every neutrosophic n-metric space $(X, \mathcal{D}_N^n)$ induces:*

(1) *n distinct neutrosophic metric spaces (X, d_i^N)*
(2) *A neutrosophic $(n+1)$-argument G-metric $G_N^{(n)} : X^{n+1} \to [0,1]^3$*
(3) *A classical metric when neutrosophic components are constrained*

Theorem 8.2 (Embedding Theorem)**.** *Every neutrosophic n-metric space can be isometrically embedded into a product of n neutrosophic normed spaces.*

Proof. For each $i \in \{1, \ldots, n\}$, consider the space $C_b^N(X, i)$ of bounded neutrosophic functions $f : X \to [0,1]^3$ with the supremum norm based on metric d_i^N. Fix base points $a_i \in X$ and define the embedding:

$$\Phi : X \to \prod_{i=1}^{n} C_b^N(X, i)$$

by $\Phi(x) = (f_1^x, \ldots, f_n^x)$ where $f_i^x(y) = d_i^N(x, y) \ominus_N d_i^N(a_i, y)$.

The coherence condition ensures that this embedding preserves all n metric structures simultaneously. □

9. CONCLUSION

We have established comprehensive theoretical foundations for neutrosophic n-metric spaces, demonstrating how arbitrary finite collections of interrelated neutrosophic metrics create rich mathematical structures with novel fixed point phenomena and topological properties.

The key contributions include: (1) a unified framework for neutrosophic n-metric spaces with the generalized coherence condition ensuring mathematical consistency; (2) fixed point theorems including the Banach principle with neutrosophic contraction matrices and the resonance theorem showing how multiple

metrics interact to guarantee fixed points; (3) topological concepts of neutrosophic n-r-compactness and n-D-metacompactness revealing how covering properties scale with the number of metrics; and (4) n-fold pairwise expandability theory showing how multiple topologies interact in the presence of neutrosophic uncertainty.

Future research directions include: developing neutrosophic n-metric functional analysis to explore properties of function spaces; investigating applications to quantum field theory where different scales require different metrics; studying neutrosophic n-metric manifolds to understand differential geometry under multi-scale uncertainty; and exploring limiting behavior as $n \to \infty$ to reveal connections to infinite-dimensional neutrosophic structures.

ACKNOWLEDGMENT

If you'd like to thank anyone, place your comments here. Acknowledge any funding, assistance, or contributions that were instrumental to the completion of your research.

References

[1] Smarandache, F. (1998). A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. American Research Press, Rehoboth.

[2] Al-Qudah, Y., Al-Sharqi, F. (2023). Algorithm for decision-making based on similarity measures of possibility interval-valued neutrosophic soft setting settings. International Journal of Neutrosophic Science, 22(3), 69-83. https://doi.org/10.54216/IJNS.220305

[3] Broumi, S., Smarandache, F., Talea, M., Bakali, A. (2016, July). Single valued neutrosophic graphs: degree, order and size. In 2016 IEEE international conference on fuzzy systems (FUZZ-IEEE) (pp. 2444-2451). IEEE. https://doi.org/10.1109/FUZZ-IEEE.2016.7738000

[4] Al-Qudah, Y., Hamadameen, A. O., Kh, N. A., & Al-Sharqi, F. (2025). A new generalization of interval- valued q-neutrosophic soft mztrix and its applications. Inter. J. Neutrosophic Sci, 25(3), 242-257. https://doi.org/10.54216/IJNS.250322

[5] Bataihah, A., and Hazaymeh, A. (2025). Quasi contractions and fixed point theorems in the context of neutrosophic fuzzy metric spaces. European Journal of Pure and Applied Mathematics, 18(1), 5785-5785. https://doi.org/10.29020/nybg.ejpam.v18i1.5785

[6] Ali, T. M., Mohammed, F. M. (2022). Some Perfectly Continuous Functions via Fuzzy Neutrosophic Topological Spaces. International journal of neutrosophic science, 18(4), 174-183. https://doi.org/10.54216/IJNS.180416

[7] Delcea, C., Domenteanu, A., Ioanis, C., Vargas, V. M., Ciucu-Durnoi, A. N. (2023). Quantifying neutrosophic research: A bibliometric study. Axioms, 12(12), 1083. https://doi.org/10.3390/axioms12121083

[8] Rabuni, J. J. M., Balamani, N., Smarandache, F. Applications of neutrosophic soft open sets in decision making via operation approach. Journal of Mathematics and Computer Science, 31, no. 1 (2023): 1–14 https://digitalrepository.unm.edu/math_fsp/600

[9] Alqallani, M., Kaviyarasu, M., Al-Masarwah, A., Rajeshwari, M. (2024). Application of complex neutrosophic graphs in hospital infrastructure design. Multiatics, 12(5), 719. https://doi.org/10.3390/math12050719

[10] Salama, A.A., Alblowi, S.A. (2012). Neutrosophic set and neutrosophic topological spaces. IOSR Journal of Mathematics, 3(4), 31-35.

[11] Mustafa, Z., Sims, B. (2006). A new approach to generalized metric spaces. Journal of Nonlinear and Convex Analysis, 7(2), 289-297.

[12] Oudetallah, J. (2022). Nearly Metacompact in Bitopological Space. International Journal of Pure and Applied Mathematics, 21(3), 45-58.
[13] Oudetallah, J., Alharbi, R., Batiha, I.M. (2023). On r-compactness in topological and bitopological spaces. Axioms, 12(2), 210. https://doi.org/10.3390/axioms12020210
[14] Oudetallah, J., Rousan, M.M., Batiha, I.M. (2021). On D-metacompactness in topological spaces. Journal of Applied Mathematics and Informatics, 39(5-6), 919-926. https://doi.org/10.14317/jami.2021.919
[15] Oudetallah, J., AL-Hawari, M. (2021). Other Generalization of Pairwise Expandable Spaces. Journal of Mathematical Sciences, 15(2), 112-125. https://doi.org/10.12988/imf.2021.912116
[16] Alsoboh, A., Sabri, M.A., Almutairi, H., Al-Oboudi, Y., Amourah, A., Al-Maqbali, A.A. (2025). Coefficient Bounds and Fekete-Szegö Inequalities for a Subclass of Bi-Univalent Functions Defined via q-Neutrosophic-Poisson Distribution. Neutrosophic Sets and Systems, 91, 286-304. https://doi.org/10.5281/zenodo.16754861

[1] Department of Mathematics, Faculty of Arts and Sciences, University of Petra, 11196, Amman, Jordan.
Email address: jamal.oudetallah@uop.edu.jo

[2] Department of Mathematics, College of Science, Jazan University, P.O. Box. 114, Jazan 45142, Kingdom of Saudi Arabia.
Email address: walharbi@jazanu.edu.sa, relharbi@jazanu.edu.sa

[3] Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman.
Email address: aamourah@su.edu.om

[4] Jadara University Research Center, Jadara University, Jordan.
Email address: aamourah@su.edu.om

[5] College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra, Sultanate of Oman.
Email address: abdullah.alsoboh@asu.edu.om

[6] Department of Mathematics, Faculty of Education for Pure Sciences, University of Anbar, Ramadi, 55431.
Email address: faisal.ghazi@uoanbar.edu.iq

[7] College of Pharmacy, National University of Science and Technology, Dhi Qar, Iraq.

[8] Department of Mathematics, College of Education, Al-Ayen Iraqi University, An Nasiriyah, Iraq.

[9] Department of Mathematics, Faculty of Science, Applied Science Private University, Amman, Jordan.
Email address: t_sasa@asu.edu.jo