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TIME METRIC SPACES A FRAMEWORK FOR TIME SENSITIVE DISTANCE

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Abstract. Distance is no longer measured only in traditional units such as kilometers or miles, but is better understood as a time-sensitive concept. We introduce time-metric spaces where distance reflects a dynamic time-dependent cost influenced by absolute time, history of paths taken, or environmental factors. When this cost (distance) is independent of time, the structure becomes a classical metric space. Time-metric spaces extend metric theory to describe systems where distance is measured in minutes and not in meters, and provide a natural way to model evolving time-dependent phenomena.

Keywords. Time metric, distance spaces, Banach, fixed point, Kannan
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1. Introduction

Classical metric spaces $(\overline{\Xi}, d)$ with their static distance functions $d : \overline{\Xi} \times \overline{\Xi} \to \mathbb{R}$ have long served as the foundation for modern analysis. However, many contemporary applications demand more flexible, dynamic distance measures that evolve over time. In transportation networks, for instance, travel times between locations fluctuate with changing congestion patterns.

Time-metric framework addresses these real-world complexities by introducing time parameterized distance functions $d_t : T \times \overline{\Xi} \times \overline{\Xi} \to \mathbb{R}$, where $t \in T \subseteq \mathbb{R}$ reflects the dynamic context of the measurement. This generalization preserves the essential structure of metric spaces while allowing the for the time-related changes that are common in contemporary applications.

2. Preliminary

The classical concept of metric spaces was first introduced by Maurice Fréchet [4] in 1906. This formal definition provided the groundwork for modern topology and analysis.

To study more complex structures, mathematicians have developed generalizations of metric spaces. In pseudometric spaces, different points can have zero

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distance, while quasi-metric spaces, introduced by Kopperman and others, do not require distances to be symmetric [8].

In the 1990s, Czerwik [3] introduced b-metric spaces, which generalize the usual triangle inequality by allowing a constant factor $s \geq 1$, so that the distance between two points satisfies

$$d(x,z) \leq s\big(d(x,y) + d(y,z)\big).$$

This concept has been widely used in fixed point theory and nonlinear analysis.

Partial metric spaces, introduced by Matthews [10], allow a nonzero self-distance $d(x,x) \geq 0$, which makes them useful for modeling incomplete information systems and computations. Probabilistic metric spaces, first proposed by Menger and later developed by Schweizer and Sklar, incorporate randomness or uncertainty into the metric structure [12].

Fuzzy metric spaces were introduced by Kramosil and Michalek [9] and further developed by George and Veeramani [5]. These spaces are important in situations where precision is either unattainable or undesirable. A more recent development is the neutrosophic metric space, proposed by Kirişçi [7], which incorporates degrees of truth, indeterminacy, and falsity to handle more complex uncertain environments.

Other generalizations, such as cone metric spaces [6] and G-metric spaces [11] extend metric theory to cover a wider range of topological behaviors while retaining essential analytical tools. Together, these developments show the steady evolution of metric theory from static, symmetric structures to more flexible, context-aware formulations that better model the complexities of modern mathematical and applied problems.

Recent advances in generalized metric spaces have broadened the scope of fixed point theory. In particular, [1] established new fixed point results in novel distance spaces, with direct applications to fractional differential equations. Building on this work, [2] introduced a new class of distance spaces that further generalizes the framework for fixed point analysis.

3. Time-Metric Space Axioms

We propose a concept called a time-metric space, which extends classical metric spaces by including a time component and causal relationships. This approach is useful for modeling dynamic systems where the "distance" or "cost" between points varies over time.

Definition 3.1. Let $\overline{\Xi}$ be a nonempty set and $T \subseteq \mathbb{R}$ be a time domain. A function $d_t \colon T \times \overline{\Xi} \times \overline{\Xi} \to [0,\infty)$ is called a time-metric if for all $x, y, z \in \overline{\Xi}$ and $t_1, t_2 \in T$ it satisfies

(TM1) $d_t(x,y) \geq 0$ for all $x, y \in \overline{\Xi}$.
(TM2) $d_t(x,y) = 0 \iff x = y$.
(TM3) $d_t(x,y) = d_t(y,x)$ for all $x, y \in \overline{\Xi}$.
(TM4) $d_{t_1}(x,y) \leq d_{t_1}(x,u) + d_{t_2}(u,y)$ for all $x, u, y \in \overline{\Xi}$, where $t_2 = t_1 + \Delta t(t,x,u)$, where $\Delta t(t,x,u)$ represents the time increment necessary to transition from x to u beginning at time t.

The triple $(\overline{\Xi}, T, d_t)$ is called a time metric spaces.

Next, we address some examples on time metric space.

Example 3.2. Let $\overline{\Xi} = \mathbb{R}$, $T = [0, 1]$, and define a function $d_t : T \times \overline{\Xi} \times \overline{\Xi} \to [0, \infty)$ by

$$d_t(x, y) = (1 + t)|x - y|, \quad \text{for all } x, y \in \overline{\Xi},\ t \in T.$$

Then $(\overline{\Xi}, T, d_t)$ is a time metric space.
To show this, we verify the Time-Metric Axioms

(TM1) Clearly, $d_t(x, y) \geq 0$ for all $x, y \in \overline{\Xi}$.
(TM2) $d_t(x, y) = 0 \iff |x - y| = 0 \iff x = y$.
(TM3) Symmetry holds since $|x - y| = |y - x|$.
(TM4) Let $t_1 \in T$, and define $\Delta t(t_1, x, y) = |x - y|$, so that $t_2 = t_1 + |x - y|$. Then:

$$d_{t_1}(x, z) = (1 + t_1)|x - z| \leq (1 + t_1)|x - y| + (1 + t_2)|y - z| = d_{t_1}(x, y) + d_{t_2}(y, z),$$

which satisfies the generalized triangle inequality.

Thus, $(\overline{\Xi}, T, d_t)$ is a time-metric space.

Example 3.3. Let $\overline{\Xi} = [0, 1] \subset \mathbb{R}$, $T = [0, \infty)$, and define the function $d_t : T \times \overline{\Xi} \times \overline{\Xi} \to [0, \infty)$ by

$$d_t(x, y) = \frac{|x - y|}{1 + t}, \quad \text{for all } x, y \in \overline{\Xi},\ t \in T.$$

Note that (TM1)–(TM3) are easily verified as in previous example.
To verify (TM4) Let $t_1 \in T$, and define $\Delta t(t_1, x, y) = |x - y|$, so $t_2 = t_1 + |x - y|$. Then

$$d_{t_1}(x, z) \leq d_{t_1}(x, y) + d_{t_2}(y, z),$$

as shown in the previous examples.

Hence, $(\overline{\Xi}, T, d_t)$ is a time-metric space.

Remark 3.4. The axiom (TM4) is called the Time-Dependent Triangle Inequality. It generalizes the classical triangle inequality by accounting for temporal costs of movement (Δt term), and allowing the second segment's metric (d_{t_2}) to evolve with time, when $\Delta t \equiv 0$ and d_t is constant in t, this reduces to standard metric axioms.

In real-world applications (e.g., traffic networks), the symmetry condition $d_t(x, y) = d_t(y, x)$ often fails. We thus define the following relaxed structure:

Definition 3.5. A triple $(\overline{\Xi}, T, q_t)$ is called a time quasi-metric space if $q_t \colon T \times \overline{\Xi} \times \overline{\Xi} \to [0, \infty)$ satisfies (TM1), (TM2), and (TM4), but not necessarily (TM3) (symmetry may fail).

Example 3.6 (Urban Traffic)**.** Let $\overline{\Xi}$ be city intersections with

$$d_t(A, B) = \begin{cases} \text{Travel time } A \to B \text{ at } t & \text{(one-way)} \\ \infty & \text{if } B \to A \text{ forbidden.} \end{cases}$$

Then, $(\overline{\Xi}, T, d_t)$ is a time-quasi-metric.

Remark 3.7. The class of time-metric spaces is strictly larger than the class of classical metric spaces. Indeed, every classical metric space $(\overline{\Xi}, d)$ can be embedded into the time-metric framework by defining a constant-in-time distance function:

$$d_t(x, y) = d(x, y), \quad \forall t \in T.$$

In this case, the time-dependent triangle inequality and causal consistency arise trivially satisfied, and the structure reduced to the classical setting.

However, time-metric spaces allow the distance between two points to evolve over time and may depend on traversal duration or historical data, which classical metrics cannot capture. Therefore, time-metric spaces form a proper generalization.

Example 3.8 (Weather-Dependent Routing)**.** Let $\overline{\Xi} = \{A, B, C\}$ represent delivery locations, and let $T = [0, 24]$ be time in hours. Define

$$d_t(A, B) = 1{+}0.5{\cdot}\sin\left(\frac{\pi t}{12}\right), \quad d_t(B, C) = 2, \quad d_t(A, C) = d_t(A, B){+}d_{t+\Delta t(t,x,y)}(B, C).$$

where $\Delta t(t, x, y) = d_t(A, B)$

This time-metric depends on periodic fluctuations (e.g., due to heatwaves or storms), and sequential time increments based on travel duration.
This models a dynamic routing problem where transit cost varies with the hour of the day.

4. Topologies of time metric spaces

Definition 4.1 (Topology Induced by a Time Metric Spaces)**.** Let $(\overline{\Xi}, T, d_t)$ be a time metric space, where $d_t \colon T \times \overline{\Xi} \times \overline{\Xi} \to [0, \infty)$. For each fixed $t \in T$, define the open ball centered at $x \in \overline{\Xi}$ with radius $r > 0$ as

$$B(x, t, r) = \{y \in \overline{\Xi} \mid d_t(x, y) < r\}.$$

The collection

$$\mathcal{B} = \{B(x, t, r) \mid x \in \overline{\Xi},\ t \in T,\ r > 0\}$$

forms a basis for a topology τ_t on $\overline{\Xi}$, called the time-t topology.

The family $\{\tau_t\}_{t\in T}$ represents a time-evolving topological structure on $\overline{\Xi}$.

Definition 4.2. Let $(\overline{\Xi}, T, d_t)$ and (Y, T, p_t) be time metric spaces, and let $f \colon \overline{\Xi} \to Y$ be a function. Fix $t \in T$, and let $\tau_t^{\overline{\Xi}}$, τ_t^Y be the topologies on $\overline{\Xi}$ and Y, respectively, induced by the metrics d_t and p_t.

We say that f is continuous at time t if for every open set $V \in \tau_t^Y$, the preimage $f^{-1}(V) \in \tau_t^{\overline{\Xi}}$; that is,

$$\forall V \in \tau_t^Y, \quad f^{-1}(V) \in \tau_t^{\overline{\Xi}}.$$

Equivalently, f is continuous at time t if for every $x \in \overline{\Xi}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_t(x, x') < \delta \quad \Rightarrow \quad p_t(f(x), f(x')) < \varepsilon.$$

Definition 4.3 (Time-convergence sequence)**.** Let $(\overline{\Xi}, T, d_t)$ be a time metric space. A sequence (x_n) in $\overline{\Xi}$ converges to $x \in \overline{\Xi}$ if

$$\forall t \in T \ \forall \varepsilon > 0, \exists N = N(\epsilon, t) \in \mathbb{N}, \text{ such that } \forall n \geq N, \quad d_t(x_n, x) < \varepsilon.$$

i.e.,

$$\forall t \in T \lim_{n\to\infty} d_t(x_n, x) = 0.$$

Definition 4.4 (Time-Cauchy sequence)**.** Let $(\overline{\Xi}, T, d_t)$ be a time metric space. A sequence $(x_n) \subseteq \overline{\Xi}$ is called a pointwise time-Cauchy sequence if

$$\forall t \in T \ \forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall m, n \geq N, \quad d_t(x_n, x_m) < \varepsilon.$$

i.e.,

$$\forall t \in T \lim_{n\to\infty} d_t(x_n, x_m) = 0.$$

Definition 4.5. A time-metric space $(\overline{\Xi}, T, d_t)$ is said to be time-complete if every uniform time-Cauchy sequence converges at every time $t \in T$.

5. Fixed Point Theory

In this section, we extend classical fixed point results to the setting of time-metric spaces. We introduce time-dependent contraction mappings and prove the existence and uniqueness of fixed points under suitable completeness conditions. These results provide a theoretical foundation for studying iterative processes in dynamic environments where distances vary over time. Before we address our main results, we introduce the following definition.

5.1. **Time-Banach Contractions.**

Definition 5.1. We say that $(\overline{\Xi}, T, d_t)$ is T-bounded if there is some $\mathcal{M} > 0$ such that for any $a, b \in \overline{\Xi}$, $\sup_t d_t(a, b) \leq \mathcal{M}$.

In the next two results, we assume that $(\overline{\Xi}, T, d_t)$ is T-bounded.

Definition 5.2. A self-map $f : \overline{\Xi} \to \overline{\Xi}$ is a time-Banach contraction if there exists an $\alpha \in [0, 1)$ such that for all $x, y \in \overline{\Xi}$ and $t \in T$,

$$d_t(f(x), f(y)) \leq \alpha \, d_t(x, y).$$

Theorem 5.3. *Let $(\overline{\Xi}, T, d_t)$ be a time-complete space. If $f : \overline{\Xi} \to \overline{\Xi}$ is a time-Banach contraction, then there is a unique $x^* \in \overline{\Xi}$ such that $f(x^*) = x^*$, and for any $x_0 \in \overline{\Xi}$, the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*.*

Proof. Let $x_0 \in \overline{\Xi}$ be arbitrary and consider the sequence (x_n) by $x_{n+1} = f(x_n)$ for all $n \geq 0$.
For any $t \in T$, and for all $n \geq 1$ we have

$$\begin{aligned} d_t(x_n, x_{n+1}) &\leq \alpha \cdot d_t(x_{n-1}, x_n), \\ &\vdots \\ &\leq \alpha^n \cdot d_t(x_0, x_1). \end{aligned}$$

Using the time-aware triangle inequality iteratively, we obtain

$$\begin{aligned} d_t(x_n, x_m) &\leq d_t(x_n, x_{n+1}) + d_{t_1}(x_{n+1}, x_{n+2}) + \cdots + d_{t_{m-n-1}}(x_{m-1}, x_m) \\ &\leq \alpha^n d_t(x_0, x_1) + \alpha^{n+1} d_{t_1}(x_0, x_1) + \cdots + \alpha^{m-1} d_{t_{m-n-1}}(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} \sup_{1 \leq i} d_{t_i}(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} \mathcal{M}. \end{aligned}$$

So (x_n) is a time-Cauchy sequence.
Since $(\overline{\Xi}, T, d_t)$ is time-complete, there exists $x^* \in \overline{\Xi}$ such that for all $t \in T$

$$\lim_{n \to \infty} d_t(x_n, x^*) = 0.$$

Now, we show that $f(x^*) = x^*$.
We have for all $t \in T$

$$d_t(f(x^*), f(x_n)) \leq \alpha \cdot d_t(x^*, x_n) \to 0.$$

By (TM4), we have

$$d_t(f(x^*), x^*) \leq d_t(f(x^*), f(x_n)) + d_{t'}(f(x_n), x^*),$$

Both terms on the right tend to zero as $n \to \infty$. Hence

$$d_t(f(x^*), x^*) = 0 \Rightarrow f(x^*) = x^*.$$

Suppose there exists another fixed point $y^* \neq x^*$. Then

$$d_t(x^*, y^*) = d_t(f(x^*), f(y^*)) \leq \alpha \cdot d_t(x^*, y^*).$$

Since $0 \leq \alpha < 1$, this implies $d_t(x^*, y^*) = 0$, hence $x^* = y^*$.

□

5.2. **Time-Kannan Contractions.**

Definition 5.4. A map $f : \overline{\Xi} \to \overline{\Xi}$ is called a time-Kannan contraction if there exists $\lambda \in [0, \frac{1}{2})$ such that for all $x, y \in \overline{\Xi}$, and $t \in T$,

$$d_t(f(x), f(y)) \leq \lambda \left[d_t(f(x), x) + d_t(f(y), y,) \right].$$

Theorem 5.5. *Let $(\overline{\Xi}, T, d_t)$ be a time-complete space. If f is a time-Kannan contraction, then there is a unique $x^* \in \overline{\Xi}$ such that $f(x^*) = x^*$, and for any $x_0 \in \overline{\Xi}$, the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*.*

Proof. Let $x_0 \in \overline{\Xi}$ be arbitrary and consider the sequence (x_n) by $x_{n+1} = f(x_n)$ for all $n \geq 0$.
For any $t \in T$, and for all $n \geq 1$ we have

$$\begin{aligned} d_t(x_{n+1}, x_n) &= d_t(f(x_n), f(x_{n-1})) \\ &\leq \lambda \left[d_t(f(x_n), x_n) + d_t(f(x_{n-1}), x_{n-1}) \right] \\ &= \lambda \left[d_t(x_{n+1}, x_n) + d_t(x_n, x_{n-1}) \right]. \end{aligned}$$

Rearranging gives,

$$d_t(x_{n+1}, x_n) \leq \frac{\lambda}{1-\lambda} d_t(x_n, x_{n-1}).$$

Let $R = \frac{\lambda}{1-\lambda}$. Since $\lambda \in [0, \frac{1}{2})$, then $R \in [0, 1)$.
By induction, we get

$$d_t(x_{n+1}, x_n) \leq R^n d_t(x_1, x_0). \tag{5.1}$$

Next, for any $m > n$, using the time-dependent triangle inequality (TM4) repeatedly, with appropriate time shifts t_k due to the traversal durations, we get

$$d_t(x_m, x_n) \leq \sum_{k=n}^{m-1} d_{t_k}(x_{k+1}, x_k).$$

Using the above bound (Equation5.1),

$$d_t(x_m, x_n) \leq \sum_{k=n}^{m-1} R^k d_{t_k}(x_1, x_0) \leq d_{\sup} \sum_{k=n}^{\infty} R^k = d_{\sup} \frac{R^n}{1-R},$$

where

$$d_{\sup} = \sup_{1 \leq k} d_{t_k}(x_1, x_0) < \infty,$$

Hence, for all $t \in T$

$$\lim_{n,m \to \infty} d_t(x_m, x_n) = 0,$$

showing (x_n) is time–Cauchy.

By time-completeness, there exists $x^* \in \overline{\Xi}$ such that

$$\lim_{n \to \infty} d_t(x_n, x^*) = 0.$$

Now, to show x^* is a fixed point of f.
Using the Kannan contraction with $x = x^*$, $y = x_n$,

$$\begin{aligned} d_t(f(x^*), f(x_n)) &\leq \lambda \left[d_t(x^*, f(x^*)) + d_t(x_n, f(x_n)) \right] \\ &= \lambda \left[d_t(x^*, f(x^*)) + d_t(x_n, x_{n+1}) \right]. \end{aligned}$$

By taking the limit, we get

$$d_t(f(x^*), x^*) \leq \lambda d_t(f(x^*), x^*).$$

Since $\lambda < 1$, it implies

$$d_t(f(x^*), x^*) = 0 \implies f(x^*) = x^*.$$

To prove the uniqueness, suppose $y^* \neq x^*$ is also a fixed point. Then for all $t \in T$

$$d_t(x^*, y^*) = d_t(f(x^*), f(y^*)) \leq \lambda \left[d_t(f(x^*), x^*) + d_t(f(y^*), y^*) \right] = 0,$$

and so $x^* = y^*$.

□

6. Consequences and Examples

Corollary 6.1 (Classical Banach Fixed Point Theorem)**.** *Let $(\overline{\Xi}, d)$ be a complete metric space, and let $f : \overline{\Xi} \to \overline{\Xi}$ be a contraction, i.e., there exists $\alpha \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in \overline{\Xi}.$$

Then there is a unique $x^* \in \overline{\Xi}$ such that $f(x^*) = x^*$, and for any $x_0 \in \overline{\Xi}$, the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*.

Proof. This follows directly from the time-Banach fixed point theorem by choosing the time-metric

$$d_t(x, y) = d(x, y), \quad \text{for all } t \in T,$$

so that the time-dependence disappears and $(\overline{\Xi}, T, d_t)$ reduces to a classical metric space. The contraction condition becomes time-independent, and the result reduces to the classical Banach contraction principle. □

Corollary 6.2 (Classical Kannan's Fixed Point Theorem)**.** *Let $(\overline{\Xi}, d)$ be a complete metric space. Suppose $f : \overline{\Xi} \to \overline{\Xi}$ satisfies*

$$d(f(x), f(y)) \leq \lambda\, [d(f(x), x) + d(f(y), y)]$$

for all $x, y \in \overline{\Xi}$, where $\lambda \in [0, \frac{1}{2})$. Then there is a unique $x^* \in \overline{\Xi}$ such that $f(x^*) = x^*$, and for any $x_0 \in \overline{\Xi}$, the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*.

Proof. Define a time-metric $d_t(x, y) = d(x, y)$ for all $t \in T$. Then the time-Kannan condition reduces to the classical Kannan condition. Since $(\overline{\Xi}, d)$ is complete, the induced time-metric space is time-complete. Thus, by the time-Kannan fixed point theorem, the result follows.

□

Example 6.3. Let $\overline{\Xi} = \mathbb{R}$, $T = [0, 1]$, and define a function $d_t : T \times \overline{\Xi} \times \overline{\Xi} \to [0, \infty)$ by

$$d_t(x, y) = (1 + t)|x - y|, \quad \text{for all } x, y \in \overline{\Xi},\ t \in T.$$

Then $(\overline{\Xi}, T, d_t)$ is a complete time-metric space, and T-bounded (see Example 3.2).

Define a self-map $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2}x.$$

Then for all $x, y \in \mathbb{R}$ and $t \in T$, we have

$$d_t(f(x), f(y)) = (1 + t)\left|\frac{1}{2}x - \frac{1}{2}y\right| = \frac{1}{2}(1 + t)|x - y| = \frac{1}{2}d_t(x, y).$$

Hence, f is a time-Banach contraction with contraction constant $\alpha = \frac{1}{2} \in [0, 1)$.

For any $x_0 \in \mathbb{R}$, the iteration $x_{n+1} = f(x_n)$ yields

$$x_n = \left(\frac{1}{2}\right)^n x_0 \to 0,$$

so the unique fixed point is $x^* = 0$. Moreover, $x_n \to x^*$ in d_t for all $t \in [0, 1]$.

Example 6.4. Let $\overline{\Xi} = [0,1] \subset \mathbb{R}$, $T = [0,\infty)$, and define the function $d_t : T \times \overline{\Xi} \times \overline{\Xi} \to [0,\infty)$ by

$$d_t(x,y) = \frac{|x-y|}{1+t}, \quad \text{for all } x,y \in \overline{\Xi},\ t \in T.$$

then $(\overline{\Xi}, T, d_t)$ is a complete time-metric space and T-bounded (see Example 3.3).

Let $f : \overline{\Xi} \to \overline{\Xi}$ be given by

$$f(x) = \frac{1}{4}(1+x).$$

Then for any $x,y \in [0,1]$ and $t \in T$, we have

$$|f(x) - f(y)| = \left|\frac{1}{4}(1+x) - \frac{1}{4}(1+y)\right| = \frac{1}{4}|x-y|,$$

$$d_t(f(x),f(y)) = \frac{1}{1+t} \cdot \frac{1}{4}|x-y|,$$

$$d_t(f(x),x) = \frac{1}{1+t}\left|\frac{1}{4}(1+x) - x\right| = \frac{1}{1+t}\left|\frac{1-3x}{4}\right|,$$

$$d_t(f(y),y) = \frac{1}{1+t}\left|\frac{1-3y}{4}\right|.$$

Now consider the sum

$$d_t(f(x),x) + d_t(f(y),y) = \frac{1}{1+t}\left(\frac{|1-3x| + |1-3y|}{4}\right).$$

To show that f is Time-Kannan contraction it is suffices to show that $\max\limits_{x,y\in[0,1]} h(x,y) \leq \frac{1}{2}$, where

$$h(x,y) = \frac{d_t(f(x),f(y))}{d_t(f(x),x) + d_t(f(y),y)} = \frac{|x-y|}{|1-3x| + |1-3y|}, \quad \text{for } x,y \in [0,1].$$

Using MATLAB, we get that $h(x,y) \leq \frac{1}{3}$, see Figure (1)

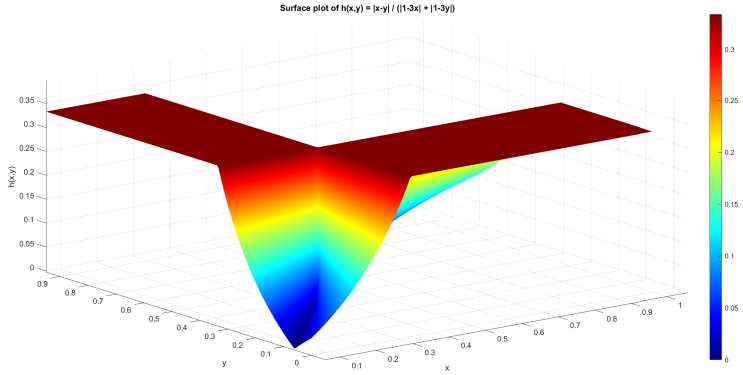


FIGURE 1. Plot of $h(x,y) = \dfrac{|x-y|}{|1-3x| + |1-3y|}$ on $[0,1]^2$.

In fact by solving $f(x) = x$, we get

$$\frac{1}{4}(1+x) = x \quad \Rightarrow \quad x = \frac{1}{3}.$$

Hence, the unique fixed point is $x^* = \frac{1}{3}$.

Conclusion

In this work, we introduced the concept of time-metric spaces, a natural generalization of classical metric spaces where distances evolve with time. By incorporating temporal dynamics, these spaces provide a rigorous framework to model real-world phenomena in which travel cost, effort, or process duration depends on time, history, or environmental conditions.

We established foundational properties, and extended classical fixed point results like Banach and Kannan contractions to time-dependent setting. Illustrative examples on fixed point results were introduced, highlighting their potential in applications ranging from traffic networks to iterative computational processes.

Time-metric spaces provide opportunities for additional investigation, including probabilistic, fuzzy, and stochastic extensions, along with algorithmic applications for dynamic optimization and control.

In summary, this framework presents a useful tool for the analysis of evolving systems.

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