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CONFORMABLE FRACTIONAL DERIVATIVE APPLIED TO FRACTIONAL BLOCH MODEL

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Abstract. This paper explores the fractional Bloch equation, a widely applied model across diverse scientific fields, including physics, chemistry, and magnetic resonance imaging (MRI). By employing the conformable fractional derivative, analytic solutions for nuclear magnetization are derived. An investigation of the effect of the fractional order on system dynamics for different parameter values is conducted. The results unveil novel characteristics of the Bloch equation within the conformable fractional framework. This contributes to offering valuable insights for both scientific exploration and technological innovation.

Keywords. Fractional order Bloch model, Conformable fractional derivatives, magnetization, conformable transform
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1. Introduction

Fractional calculus extends the traditional concepts of calculus by allowing derivatives and integrals of arbitrary (non-integer) order. The origins of this field date back to the late 1600s, with Leibniz (1695) first raising the idea. Over the centuries, mathematicians such as Liouville (1834) and Riemann (1892) contributed to its development. The subject gained significant momentum after the publication of the first dedicated book by Oldham and Spanier in 1974 [21]. In recent decades, fractional calculus has become increasingly popular due to its effectiveness in modeling complex phenomena in areas such as nuclear magnetic resonance [25], bioengineering [26, 17, 3], viscoelastic materials [16, 20], plasma physics [19], robotics [18], and many other disciplines.

The Bloch equations, introduced in 1946, provide a macroscopic description of nuclear magnetic resonance (NMR) through a system of first-order differential equations [14, 1]. These equations characterize the time evolution of the magnetization vector $\mathcal{M} = (\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z)$ during the relaxation processes governed by ρ_1 and ρ_2, which denote the longitudinal and transverse relaxation

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times, respectively. The equilibrium magnetization is denoted by $\mathcal{M}_0$ [15]. In many cases, magnetic resonance phenomena are better described by replacing the integer-order derivatives in the Bloch equations with fractional-order derivatives. Physically, the replacement of integer-order time derivatives by a conformable fractional derivative can be interpreted as introducing an effective time-scaling that models anomalous relaxation and transport processes often observed in complex or heterogeneous samples. In NMR experiments, relaxation and dephasing in systems with restricted or anomalous diffusion frequently deviate from single-exponential behavior and may display power-law or stretched-exponential decay [34, 32, 33]. Fractional-order models have been used to provide deeper insight into the dynamics of magnetic resonance systems. Several studies have explored fractional versions of the Bloch equation [2]. For example, Magin et al. [13] investigated spin dynamics and magnetization relaxation for a single spin in a static magnetic field using time-fractional Bloch equations (TFBE) with Caputo derivatives. Similarly, Velasco et al. [12] applied fractional Bloch equations (FBE) with Riemann–Liouville derivatives to fit experimental spin–spin relaxation data. In another study, Singh et al. [25] formulated a fractional Bloch equation using the Liouville-Caputo derivative.

In this paper, we introduce a new variant of the fractional Bloch equation based on a local operator of the conformable fractional derivative [31], which leads to a system equivalent to a classical ODEs system with a time-scaling factor.

The structure of this paper is as follows: Section 2 introduces the mathematical background. Section 3 presents the formulation of the fractional Bloch system based on the conformable derivative, where the conformable fractional Laplace transform is applied to derive analytical and numerical results. Finally, Section 4 discusses the physical implications of the findings and outlines possible directions for future work.

2. Preliminaries

2.1. The conformable fractional derivative.

Definition 2.1. The conformable fractional derivative of order $0 < \alpha \leq 1$ for a function $f : [0, \infty) \to \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\zeta \to 0} \frac{f(t + \zeta t^{1-\alpha}) - f(t)}{\zeta} \tag{2.1}$$

for all $t > 0$.

This formulation extends the classical derivative to fractional orders in a direct manner, without relying on integral representations, which makes it more straightforward to apply in many contexts.

Remark 2.2.

(1) If $D^\alpha f$ exists, the function f is called α-differentiable.
(2) If f is α-differentiable on an interval $(0, a)$ with $a > 0$, and $\lim_{t \to 0^+} D^\alpha f(t)$ exists, then $D^\alpha f(0)$ is defined as this limit.

(3) For polynomials, the conformable derivative produces results of the same form as the Riemann–Liouville and Caputo fractional derivatives, differing only by a multiplicative constant.

Theorem 2.3. [31] *Suppose $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$ for some $0 < \alpha \leq 1$. Then f is continuous at t_0.*

Proof. Let $t_0 > 0$, for $h > 0$, there exists $\zeta > 0$ such that $h = \zeta t_0^{1-\alpha}$. Assuming that $D^\alpha f(t_0)$ exists and equals l, yields

$$\lim_{\zeta \to 0} \frac{f(t_0 + \zeta t_0^{1-\alpha}) - f(t_0)}{\zeta} = l \tag{2.2}$$

hence,

$$\begin{aligned} \lim_{h \to 0} f(t_0 + h) - f(t_0) &= \lim_{\zeta \to 0} \frac{f\left(t_0 + \zeta t_0^{1-\alpha}\right) - f(t_0)}{\zeta} . \zeta \\ &= \lim_{\zeta \to 0} \frac{f\left(t_0 + \zeta t_0^{1-\alpha}\right) - f(t_0)}{\zeta} . \lim_{\zeta \to 0} \zeta \\ &= l.0 \\ &= 0 \end{aligned}$$

Therefore f is continuous at t_0. □

Theorem 2.4. [31] *Let $0 < \alpha \leq 1$ and suppose f and g are α-differentiable at $t > 0$. Then:*

(1) *Linearity:* $(af + bg)^{(\alpha)} = a(f)^{(\alpha)} + b(g)^{(\alpha)}$ *for all* $a, b \in \mathbb{R}$.
(2) *Product Rule:* $(fg)^{(\alpha)} = f(g)^{(\alpha)} + g(f)^{(\alpha)}$.
(3) *Quotient Rule:* $\left(\frac{f}{g}\right)^{(\alpha)} = \frac{g(f)^{(\alpha)} - f(g)^{(\alpha)}}{g^2}$.
(4) *Power Rule:* $(t^p)^{(\alpha)} = pt^{p-\alpha}$ *for any* $p \in \mathbb{R}$.
(5) *Constant Rule:* $(\lambda)^{(\alpha)} = 0$ *for any constant* λ.

Proof. These properties follow directly from the definition of the conformable fractional derivative. □

Theorem 2.5. *The conformable fractional derivative is applicable to various types of functions, as illustrated in the following cases:*

(1) $D^\alpha(1) = 0$
(2) *For any* $a \in \mathbb{R}$: $D^\alpha(\sin(at)) = at^{1-\alpha}\cos(at)$
(3) *For any* $a \in \mathbb{R}$: $D^\alpha(\cos(at)) = -at^{1-\alpha}\sin(at)$
(4) *For any* $a \in \mathbb{R}$: $D^\alpha(e^{at}) = t^{(1-\alpha)}ae^{at}$
(5) $D^\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$
(6) $D^\alpha\left(\sin\left(\frac{1}{\alpha}t^\alpha\right)\right) = \cos(\frac{1}{\alpha}t^\alpha)$
(7) $D^\alpha\left(\cos\left(\frac{1}{\alpha}t^\alpha\right)\right) = -\sin(\frac{1}{\alpha}t^\alpha)$
(8) $D^\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$

Theorem 2.6. *Assume that $f : [a, \infty) \to \mathbb{R}$ is α-differentiable for $0 < \alpha \leq 1$. Then, for all $t > a$ we have $(f)^{(\alpha)}(t) = t^{1-\alpha}\frac{df}{dt}(t)$.*

Theorem 2.7 (Rolle's theorem)**.** *Let $a > 0$, $b > a$, and $0 < \alpha \leq 1$. Assume that $f : [a, +\infty) \to \mathbb{R}$ is continuous and α-differentiable, and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that*

$$D^{\alpha} f(c) = 0. \tag{2.3}$$

Proof. Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there must exist $c \in (a, b)$ where f attains a local extremum. At such a point,

$$D^{(\alpha)} f(c) = \lim_{\rho \to 0^+} \frac{f(c + \rho c^{1-\alpha}) - f(c)}{\rho} = \lim_{\rho \to 0^-} \frac{f(c + \rho c^{1-\alpha}) - f(c)}{\rho} \tag{2.4}$$

The first limit is nonnegative and the second is nonpositive, so $D^{(\alpha)} f(c) = 0$. □

Theorem 2.8. *Let $a > 0$, $b > a$, and $0 < \alpha \leq 1$. Assume that $f : [a, +\infty) \to \mathbb{R}$ is continuous and α-differentiable. Then there exists $c \in (a, b)$ such that*

$$D^{\alpha} f(c) = \frac{f(b) - f(a)}{\frac{b^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}}. \tag{2.5}$$

This is the mean value theorem for conformable fractional derivatives [31].

Theorem 2.9 (Chain rule)**.** *f and $g : [0, +\infty) \to \mathbb{R}$ are α-differentiable functions, where $0 < \alpha \leq 1$. Then, the composite function $h(t) = f(g(t))$ is α-differentiable, and for all $t \neq 0$ and $g(t) \neq 0$, we write*

$$D^{\alpha} h(t) = D^{\alpha} f[g(t)] \cdot D^{\alpha} g(t) \cdot (g(t))^{\alpha - 1} \tag{2.6}$$

if $t = 0$, we get

$$D^{\alpha} h(0) = \lim_{t \to 0^+} D^{\alpha} f[g(t)] \cdot D^{\alpha} g(t) \cdot (g(t))^{\alpha - 1} \tag{2.7}$$

Proof. Setting $u = t + \varepsilon(t)^{1-\alpha}$ in the definition and applying the continuity of g, we obtain

$$\begin{aligned}
D^{\alpha} h(t) &= \lim_{u \to t} \frac{f(g(u)) - f(g(t))}{u - t} t^{1-\alpha} \\
&= \lim_{u \to t} \frac{f(g(u)) - f(g(t))}{g(u) - g(t)} \cdot \lim_{u \to t} \frac{g(u) - g(t)}{u - t} t^{1-\alpha} \\
&= g(t)^{\alpha - 1} \cdot g(t)^{1-\alpha} \lim_{g(u) \to g(t)} \frac{f(g(u)) - f(g(t))}{g(u) - g(t)} \cdot t^{1-\alpha} \lim_{u \to t} \frac{g(u) - g(t)}{u - t} \\
&= g(t)^{\alpha - 1} \cdot g(t)^{1-\alpha} f'(g(u)) \cdot t^{1-\alpha} g'(t) \\
&= g(t)^{\alpha - 1} \cdot (D^{\alpha} f)(g(t)) \cdot (D^{\alpha} g)(t)
\end{aligned}$$

□

Definition 2.10. Let $\alpha \in (n, n + 1]$ and suppose f is n-times differentiable at $t > 0$. The conformable fractional derivative of f of order α is defined as

$$D^{\alpha} f(t) = \lim_{\zeta \to 0} \frac{f^{\lceil \alpha \rceil - 1}(t + \zeta t^{\lceil \alpha \rceil - \alpha}) - f^{\lceil \alpha \rceil - 1}(t)}{\zeta}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

Remark 2.11. A useful result from Definition 2.10 is

$$D^{\alpha}f(t) = t^{(\lceil\alpha\rceil-\alpha)}f^{\lceil\alpha\rceil}(t)$$

where n is a non-negative integer, $\alpha \in (n, n+1]$, f is $(n+1)$-times differentiable at $t > 0$, and $\lceil\alpha\rceil$ is the ceiling of α.

Corollary 2.12. *Suppose* $0 < \alpha \leq 1$, $a > 0$, *and* $f, g : [a, +\infty) \to \mathbb{R}$ *satisfy* $D^{\alpha}f(t) = D^{\alpha}g(t)$. *Then there exists* $c \in \mathbb{R}$ *such that* $g(t) = f(t) + c$.

2.2. **Fractional integral.**

The α-fractional integral generalizes the classical integral to non-integer orders. As described by Khalil et al. [31], the α-fractional integral of a function f from $a \geq 0$ is defined as following

Definition 2.13. for a suitable real valued function f, we define

$$I_a^{\alpha}(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx \tag{2.8}$$

where $0 < \alpha \leq 1$ and the integral is taken in the usual (improper) Riemann sense.

Remark 2.14. It is easy to see that

$$I_a^{\alpha}(f)(t) = I_a^1(t^{\alpha-1}f)$$

Theorem 2.15. *For any continuous function* f *where* $I_{\alpha}(f)$ *exists and for all* $t \geq a$,

$$D^{\alpha}\left[I_a^{\alpha}(f)\right](t) = f(t) \tag{2.9}$$

Proof. If f is continuous, then $I_a^{\alpha}(f)(t)$ is differentiable and

$$\begin{aligned} D^{\alpha}[I_a^{\alpha}(f)](t) &= t^{1-\alpha}\frac{d}{dt}I_a^{\alpha}(f)(t) \\ &= t^{1-\alpha}\frac{d}{dt}\int_a^t \frac{f(x)}{x^{1-\alpha}}dx \\ &= t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}} \\ &= f(t) \end{aligned}$$

□

Lemma 2.16. [29] *Let* $0 < \alpha \leq 1$ *and* f *be* α*-differentiable. Then, for all* $a > 0$,

$$I_a^{\alpha}\left[D^{\alpha}(f)\right](t) = f(t) - f(a) \tag{2.10}$$

2.3. **Conformable Laplace transform.**

Definition 2.17. Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function and $0 < \alpha \leq 1$. The conformable Laplace transform of f is defined as:

$$F_{\alpha}(s) = \mathcal{L}_{\alpha}[f(t)](s) = \int_0^{\infty} e^{-s\frac{t^{\alpha}}{\alpha}}t^{\alpha-1}f(t)dt. \tag{2.11}$$

provided the integral exists.

Let us illustrate the conformable fractional Laplace transform with some elementary functions, as presented in the following theorem.

Theorem 2.18. *Let $a, p, c \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then:*

(1) $\mathcal{L}_\alpha\{c\}(s) = \dfrac{c}{s}, \quad s > 0.$

(2) $\mathcal{L}_\alpha\{t^p\}(s) = \dfrac{\Gamma(1+p)}{\alpha^{p/\alpha} s^{1+p/\alpha}}, \quad s > 0.$

(3) $\mathcal{L}_\alpha\{e^{at}\}(s) = \dfrac{1}{s-a}, \quad s > a.$

(4) $\mathcal{L}_\alpha\{\sin(at)\}(s) = \dfrac{a}{s^2+a^2}, \quad s > 0.$

(5) $\mathcal{L}_\alpha\{\cos(at)\}(s) = \dfrac{s}{s^2+a^2}, \quad s > 0.$

(6) $\mathcal{L}_\alpha\{\sinh(at)\}(s) = \dfrac{a}{s^2-a^2}, \quad s > |a|.$

(7) $\mathcal{L}_\alpha\{\cosh(at)\}(s) = \dfrac{s}{s^2-a^2}, \quad s > |a|.$

Proof. The results follow directly from applying Definition 2.1 and evaluating the corresponding integrals. □

The relationship between the classical Laplace transform and its conformable fractional counterpart is established in the following theorem.

Theorem 2.19. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $\mathcal{L}_\alpha\{f\}(s) = F_\alpha(s)$ exists. Then*

$$\mathcal{L}_\alpha\{f(t)\}(s) = \int_0^\infty e^{-su} f\big((\alpha u)^{1/\alpha}\big)\, du = \mathcal{L}\big\{ f\big((\alpha t)^{1/\alpha}\big) \big\}(s), \quad 0 < \alpha \leq 1. \tag{2.12}$$

where $\mathcal{L}$ represents the classical Laplace transform, given by

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad s > 0.$$

Proof. Recall the definition of the conformable fractional Laplace transform:

$$\mathcal{L}_\alpha\{f\}(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\, t^{\alpha-1}\, dt, \qquad 0 < \alpha \leq 1, \tag{2.13}$$

which is assumed to converge for the values of s under consideration.

Let us perform the change of variable

$$u = \frac{t^\alpha}{\alpha} \quad \Longleftrightarrow \quad t = (\alpha u)^{1/\alpha}.$$

The corresponding differential is

$$du = \frac{d}{dt}\left(\frac{t^\alpha}{\alpha}\right) dt = t^{\alpha-1}\, dt.$$

Substituting $t^{\alpha-1}\, dt = du$ into (2.13) yields

$$\mathcal{L}_\alpha\{f\}(s) = \int_0^\infty e^{-su} f\big((\alpha u)^{1/\alpha}\big)\, du.$$

The right-hand side corresponds to the classical Laplace transform of the function

$$g(t) = f\big((\alpha t)^{1/\alpha}\big)$$

Hence, we obtain

$$\mathcal{L}_\alpha\{f\}(s) = \mathcal{L}\big\{\, f\big((\alpha t)^{1/\alpha}\big)\,\big\}(s),$$

□

Remark 2.20. The conformable Laplace transform constitutes a generalization of the classical Laplace transform. For $\alpha = 1$, it coincides with the classical case:

$$\mathcal{L}_1[f(t)](s) = \mathcal{L}[f(t)](s) \tag{2.14}$$

Remark 2.21. The relation established in theorem 2.19 implies that all properties of the classical Laplace transform have their conformable counterparts. In particular:

Lemma 2.22. *Let $f : [0, \infty) \to \mathbb{R}$ be an α-differentiable function. Then:*

$$\mathcal{L}_\alpha\{D^\alpha f(t)\}(s) = s\mathcal{L}_\alpha\{f(t)\}(s) - f(0).$$

Proof. By the definition of the conformable derivative, $D^\alpha f(t) = t^{1-\alpha} f'(t)$. Applying the conformable Laplace transform:

$$\mathcal{L}_\alpha\{D^\alpha f(t)\}(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} \left[t^{1-\alpha} f'(t)\right] t^{\alpha-1} dt = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f'(t) dt.$$

Using integration by parts with $u = e^{-s\frac{t^\alpha}{\alpha}}$ and $dv = f'(t)dt$, we obtain $du = -st^{\alpha-1}e^{-s\frac{t^\alpha}{\alpha}}dt$ and $v = f(t)$. Thus:

$$\int_0^\infty u dv = [uv]_0^\infty - \int_0^\infty v du.$$

Substituting:

$$\mathcal{L}_\alpha\{D^\alpha f(t)\}(s) = \left[e^{-s\frac{t^\alpha}{\alpha}} f(t)\right]_0^\infty + s\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t) t^{\alpha-1} dt.$$

Assuming $f(t)$ is of suitable exponential order, the boundary term evaluates to $-f(0)$. The remaining integral is exactly $\mathcal{L}_\alpha\{f(t)\}(s)$. Therefore:

$$\mathcal{L}_\alpha\{D^\alpha f(t)\}(s) = s\mathcal{L}_\alpha\{f(t)\}(s) - f(0),$$

□

3. Main results

3.1. **Model formulation.** The Bloch equations, as introduced by Felix Bloch, can be reformulated as follows [11]

$$\begin{cases} \frac{d\mathcal{M}_x(t)}{dt} &= \gamma(\mathbf{M}(t) \times \mathbf{B}(t))_x - \frac{\mathcal{M}_x(t)}{\tau_2}, \\ \frac{d\mathcal{M}_y(t)}{dt} &= \gamma(\mathbf{M}(t) \times \mathbf{B}(t))_y - \frac{\mathcal{M}_y(t)}{\tau_2}, \\ \frac{d\mathcal{M}_z(t)}{dt} &= \gamma(\mathbf{M}(t) \times \mathbf{B}(t))_z - \frac{\mathcal{M}_z(t) - \mathcal{M}_0}{\tau_1}, \end{cases} \tag{3.1}$$

where $\mathbf{M} = (\mathcal{M}_x(t), \mathcal{M}_y(t), \mathcal{M}_z(t))$ is the nuclear magnetization, $\mathbf{B} = (B_x(t), B_y(t), B_0 + \Delta B_z(t))$ is the magnetic field; $\mathcal{M}_0$ is the equilibrium magnetization. τ_1 (spin–lattice) and τ_2 (spin–spin) are the relaxation times, and $\gamma/2\pi$ represents the gyromagnetic ratio. The relaxation terms characterize the process by which the system returns to its equilibrium state.

When the magnetic field is oriented along the z-axis, the Bloch equations 3.1 corresponding to a constant static magnetic field B_0(z-component) simplify to the following expressions

$$\begin{cases} \frac{d\mathcal{M}_x(t)}{dt} &= \omega_0 \mathcal{M}_y(t) - \frac{\mathcal{M}_x(t)}{\tau_2}, \\ \frac{d\mathcal{M}_y(t)}{dt} &= -\omega_0 \mathcal{M}_x(t) - \frac{\mathcal{M}_y(t)}{\tau_2}, \\ \frac{d\mathcal{M}_z(t)}{dt} &= \frac{\mathcal{M}_0 - \mathcal{M}_z(t)}{\tau_1}, \end{cases} \tag{3.2}$$

where $\omega_0 = \gamma B_0$.

Variable	Physical signification
$\mathbf{M}(t)$	Nuclear magnetization vector
$\mathcal{M}_x(t)$	x-component of the nuclear magnetization
$\mathcal{M}_y(t)$	y-component of the nuclear magnetization
$\mathcal{M}_z(t)$	z-component of the nuclear magnetization
$\mathbf{B}(t)$	Total magnetic field vector
$B_x(t)$	x-component of the magnetic field
$B_y(t)$	y-component of the magnetic field
B_0	Constant static magnetic field (z-component)
$\Delta B_z(t)$	Time-varying component of the magnetic field in the z-direction
γ	Gyromagnetic ratio
ω_0	Angular frequency ($\omega_0 = \gamma B_0$)
τ_1	Spin-lattice relaxation time
τ_2	Spin-spin relaxation time
$\mathcal{M}_0$	Equilibrium magnetization

TABLE 1. Variables and their physical significance in the Bloch equations.

Now, we focus on the fractional-order Bloch equations, characterized by the replacement of integer-order derivatives with conformable fractional derivatives. The mathematical formulation of this fractional-order system can be represented as follows

$$\begin{cases} D^\alpha \mathcal{M}_x(t) &= \omega_0' \mathcal{M}_y(t) - \frac{\mathcal{M}_x(t)}{\tau_2'}, \\ D^\alpha \mathcal{M}_y(t) &= -\omega_0' \mathcal{M}_x(t) - \frac{\mathcal{M}_y(t)}{\tau_2'}, \\ D^\alpha \mathcal{M}_z(t) &= \frac{\mathcal{M}_0 - \mathcal{M}_z(t)}{\tau_1'}, \end{cases} \tag{3.3}$$

where α is the conformable fractional derivative order. For the magnetization to possess consistent units, the parameters ω_0', τ_1' and τ_2' must be expressed in units of $s^{-\alpha}$.

3.2. **Explicit solution.** This section presents the explicit solution of the Bloch equations using the mathematical framework developed earlier. For notational clarity, in (3.3) the parameters ω_0', τ_1' and τ_2' have been replaced by μ_0, ρ_1 and ρ_2, respectively. By applying the conformable Laplace transform, we derive the following:

$$\begin{cases} sL_\alpha[\mathcal{M}_x(t)] - \mathcal{M}_x(0) &= \mu_0 L_\alpha[\mathcal{M}_y(t)] - \frac{1}{\rho_2} L_\alpha[\mathcal{M}_x(t)], \\ sL_\alpha[\mathcal{M}_y(t)] - \mathcal{M}_y(0) &= -\mu_0 L_\alpha[\mathcal{M}_x(t)] - \frac{1}{\rho_2} L_\alpha[\mathcal{M}_y(t)], \\ sL_\alpha[\mathcal{M}_z(t)] - \mathcal{M}_z(0) &= L_\alpha\left[\frac{\mathcal{M}_0}{\rho_1}\right] - L_\alpha\left[\frac{\mathcal{M}_z(t)}{\rho_1}\right], \end{cases} \tag{3.4}$$

which leads to

$$\begin{cases} s\mathcal{M}_x(s) - \mathcal{M}_x(0) &= \mu_0 \mathcal{M}_y(s) - \frac{1}{\rho_2} \mathcal{M}_x(s), \\ s\mathcal{M}_y(s) - \mathcal{M}_y(0) &= -\mu_0 \mathcal{M}_x(s) - \frac{1}{\rho_2} \mathcal{M}_y(s), \\ s\mathcal{M}_z(s) - \mathcal{M}_z(0) &= \frac{\mathcal{M}_0}{\rho_1 s} - \frac{1}{\rho_1} \mathcal{M}_z(s), \end{cases} \tag{3.5}$$

and thus

$$\begin{cases} \left(s + \frac{1}{\rho_2}\right) \mathcal{M}_x(s) - \mu_0 \mathcal{M}_y(s) &= \mathcal{M}_x(0), \\ \mu_0 \mathcal{M}_x(s) + \left(s + \frac{1}{\rho_2}\right) \mathcal{M}_y(s) &= \mathcal{M}_y(0), \\ \left(s + \frac{1}{\rho_1}\right) \mathcal{M}_z(s) &= \frac{\mathcal{M}_0}{\rho_1 s} + \mathcal{M}_z(0), \end{cases} \tag{3.6}$$

Let $M_\perp(t) = \mathcal{M}_x(t) + i\mathcal{M}_y(t)$. The system can be combined and solved as follows:

$$\begin{aligned} D_0^\alpha M_\perp(t) &= D_0^\alpha \mathcal{M}_x(t) + iD_0^\alpha \mathcal{M}_y(t) \\ &= \mu_0 \mathcal{M}_y(t) - \frac{\mathcal{M}_x(t)}{\rho_2} + i\left(-\mu_0 \mathcal{M}_x(t) - \frac{\mathcal{M}_y(t)}{\rho_2}\right) \\ &= \left(-\frac{1}{\rho_2} - i\mu_0\right) M_\perp(t) \end{aligned} \tag{3.7}$$

where $\lambda = -\frac{1}{\rho_2} - i\mu_0$. Thus,

$$D_0^\alpha M_\perp(t) = \lambda M_\perp(t) \tag{3.8}$$

Given $M_\perp(0) = \mathcal{M}_x(0) + i\mathcal{M}_y(0)$, applying the α-fractional integral operator I_0^α to both sides yields

$$M_\perp(t) = M_\perp(0) + \lambda I_0^\alpha M_\perp(t) \tag{3.9}$$

By discretizing time as $t_n = n\Delta t$ and setting $\mathcal{U}_n = M_\perp(t_n)$, the equation becomes recursively

$$\mathcal{U}_{n+1}(t) = \mathcal{U}_0 + \lambda I_0^\alpha \mathcal{U}_n(t), \qquad n = 1, 2, ... \tag{3.10}$$

which leads to

$$\begin{cases} \mathcal{U}_1 &= \mathcal{U}_0 + \lambda \mathcal{U}_0 \frac{t^\alpha}{\alpha}, \\ \mathcal{U}_2 &= \mathcal{U}_0 \left(1 + \lambda \frac{t^\alpha}{\alpha} + \frac{\lambda^2 t^{2\alpha}}{2!\alpha^2}\right), \\ \vdots & \\ \mathcal{U}_n &= \mathcal{U}_0 \sum_{k=0}^{n} \frac{\lambda^k t^{k\alpha}}{\alpha^k k!}, \end{cases} \tag{3.11}$$

Taking the limit as $n \to \infty$ gives

$$M_{\perp}(t) = \mathcal{U}_0 \sum_{k=0}^{\infty} \frac{\lambda^k t^{k\alpha}}{\alpha^k k!} = \mathcal{U}_0 e^{\lambda \frac{t^\alpha}{\alpha}} \tag{3.12}$$

With $\lambda = -\frac{1}{\rho_2} - i\mu_0$, since $M_{\perp}(t) = \mathcal{M}_x(t) + i\mathcal{M}_y(t)$, we have

$$\mathcal{M}_x(t) + i\mathcal{M}_y(t) = \mathcal{U}_0 e^{\left(-\frac{1}{\rho_2} - i\mu_0\right)\frac{t^\alpha}{\alpha}}$$

or equivalently,

$$\mathcal{M}_x(t) + i\mathcal{M}_y(t) = \left(\mathcal{M}_x(0) + i\mathcal{M}_y(0)\right) e^{-\frac{1}{\rho_2}\frac{t^\alpha}{\alpha}} \left[\cos\left(\mu_0 \frac{t^\alpha}{\alpha}\right) + i \sin\left(\mu_0 \frac{t^\alpha}{\alpha}\right)\right]$$

which gives the explicit solutions

$$\begin{cases} \mathcal{M}_x(t) &= \mathcal{M}_x(0) e^{-\frac{1}{\rho_2}\frac{t^\alpha}{\alpha}} \cos\left(\mu_0 \frac{t^\alpha}{\alpha}\right) - \mathcal{M}_y(0) e^{-\frac{1}{\rho_2}\frac{t^\alpha}{\alpha}} \sin\left(\mu_0 \frac{t^\alpha}{\alpha}\right), \\ \mathcal{M}_y(t) &= \mathcal{M}_y(0) e^{-\frac{1}{\rho_2}\frac{t^\alpha}{\alpha}} \cos\left(\mu_0 \frac{t^\alpha}{\alpha}\right) + \mathcal{M}_x(0) e^{-\frac{1}{\rho_2}\frac{t^\alpha}{\alpha}} \sin\left(\mu_0 \frac{t^\alpha}{\alpha}\right) \end{cases} \tag{3.13}$$

To find $\mathcal{M}_z$, we use the last equation of (3.6):

$$\mathcal{M}_z(s) = \frac{\frac{\mathcal{M}_0}{\rho_1 s} + \mathcal{M}_z(0)}{s + \frac{1}{\rho_1}}$$

Assuming $\mathcal{M}_z(0) = 0$, this simplifies to

$$\mathcal{M}_z(t) = \frac{\mathcal{M}_0}{\rho_1}\left(1 - e^{-\frac{1}{\rho_1}\frac{t^\alpha}{\alpha}}\right) \tag{3.14}$$

3.3. **Numerical simulation.** This subsection presents numerical results for the proposed model, which is based on the conformable fractional derivative. The analytical solutions were obtained using a modified Laplace transform adapted to the new derivative, allowing each component of the nuclear magnetization $(\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z)$ to be expressed as a function of time.

Figures 1–7 illustrate the time evolution and phase-space behavior of the magnetization components under the fractional derivative model. Figures 1 and 2 show the transverse magnetization components $\mathcal{M}_x$ and $\mathcal{M}_y$ for various values of the fractional order α, with parameters $\rho_1 = \rho = 1$ and $\mathcal{M}_0 = 100$. Figure 3 displays the longitudinal component $\mathcal{M}_z$ for different α values, using $\rho_2 = \rho = 1$ and $\mathcal{M}_0 = 100$. Figures 4–6 present 2D phase planes to demonstrate the dynamic relationships between the components, while Figure 7 provides a 3D phase-space visualization of the magnetization trajectory for $\rho_1 = 1$ and $\rho_2 = 20$.

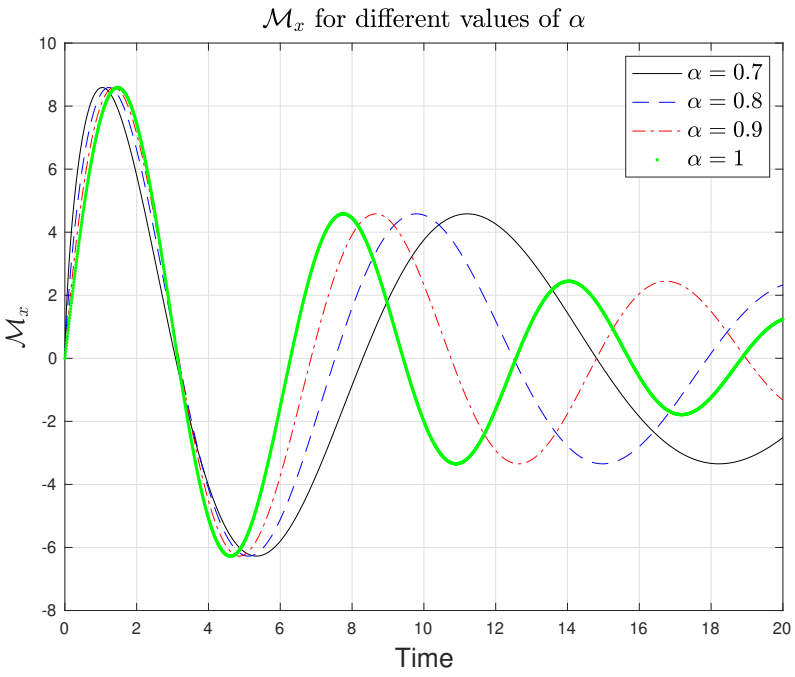


FIGURE 1. Time evolution of the longitudinal magnetization component $\mathcal{M}_x$ for $\rho = 1$, $\mathcal{M}_x(0) = 100$, and fractional orders $\alpha = \{0.7, 0.8, 0.9, 1\}$.

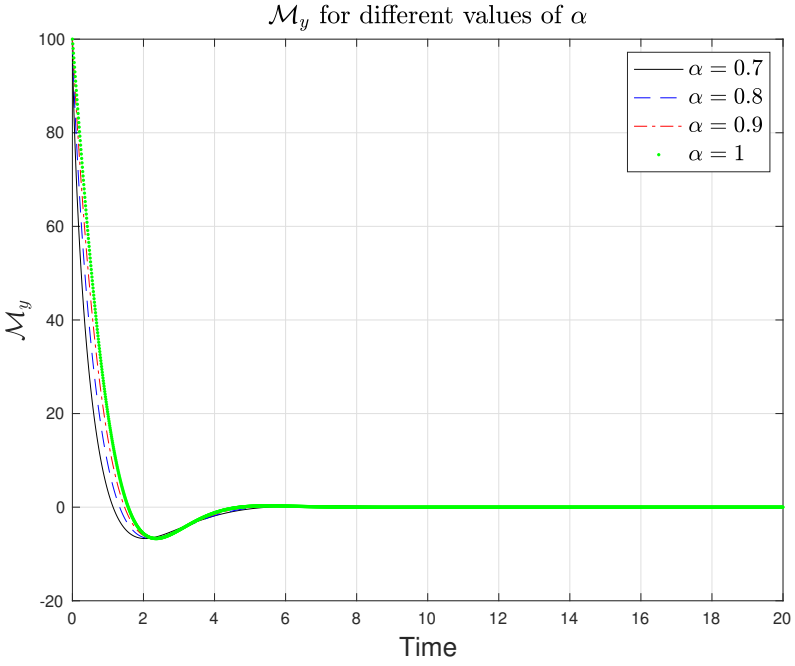


FIGURE 2. Time evolution of the transverse magnetization component $\mathcal{M}_y$ for $\rho = 1$, $\mathcal{M}_y(0) = 100$, and fractional orders $\alpha = \{0.7, 0.8, 0.9, 1\}$.

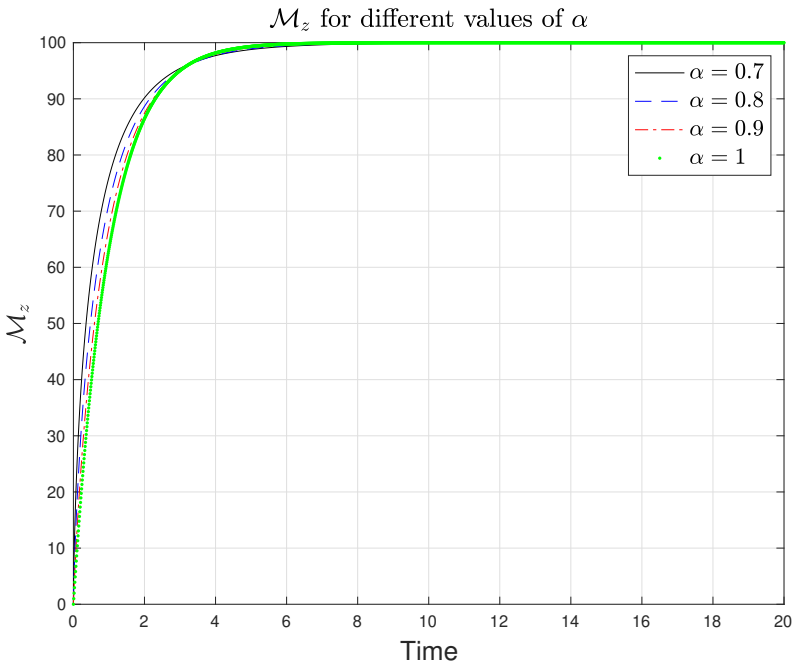


FIGURE 3. Time evolution of the transverse magnetization component $\mathcal{M}_z$ for $\rho = 1$, $\mathcal{M}_0 = 100$, and fractional orders $\alpha = \{0.7, 0.8, 0.9, 1\}$.

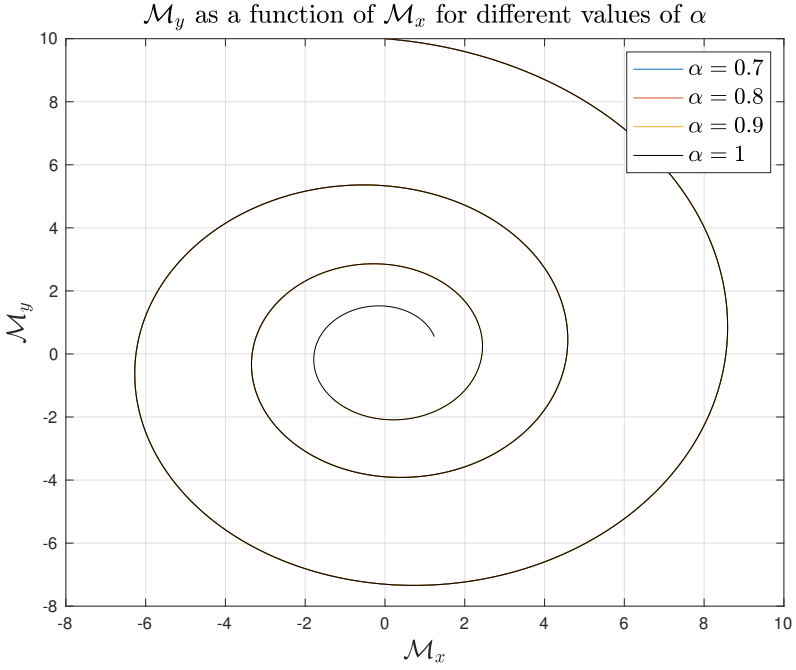


FIGURE 4. 2D phase plane of the components $\mathcal{M}_x$ and $\mathcal{M}_y$ for $\rho = 1$, $\mathcal{M}_0 = 100$, and fractional orders $\alpha = \{0.7, 0.8, 0.9, 1\}$.

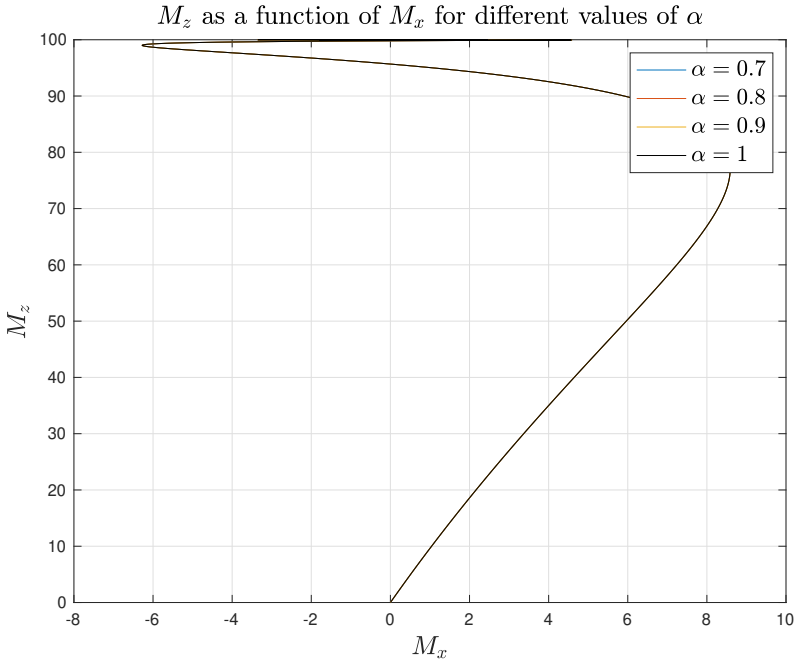


FIGURE 5. 2D phase plane of the components $\mathcal{M}_x$ and $\mathcal{M}_z$, $\rho = 1$, $\mathcal{M}_0 = 100$, and fractional orders $\alpha = \{0.7, 0.8, 0.9, 1\}$.

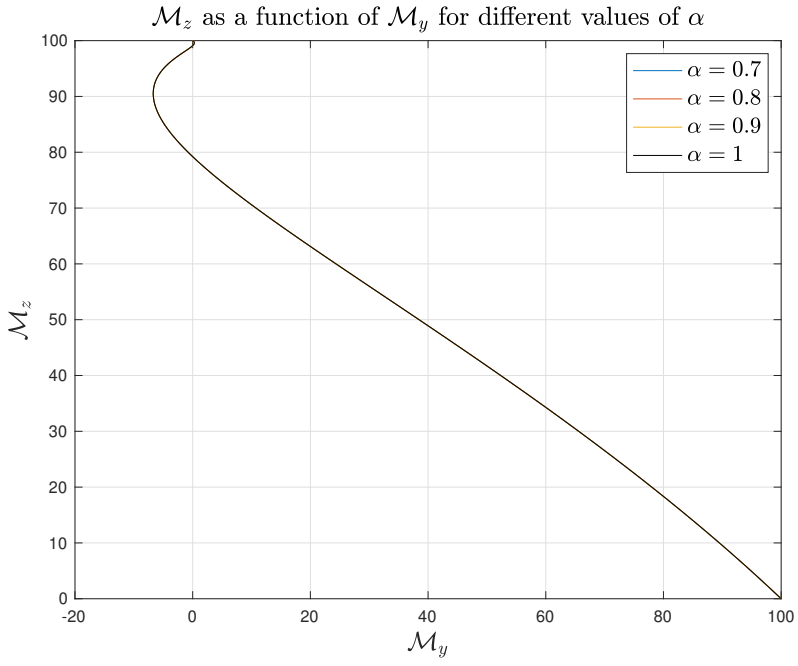


FIGURE 6. 2D phase plane of the components $\mathcal{M}_y$ and $\mathcal{M}_z$, $\rho = 1$, $\mathcal{M}_0 = 100$, and fractional orders $\alpha = \{0.7, 0.8, 0.9, 1\}$.

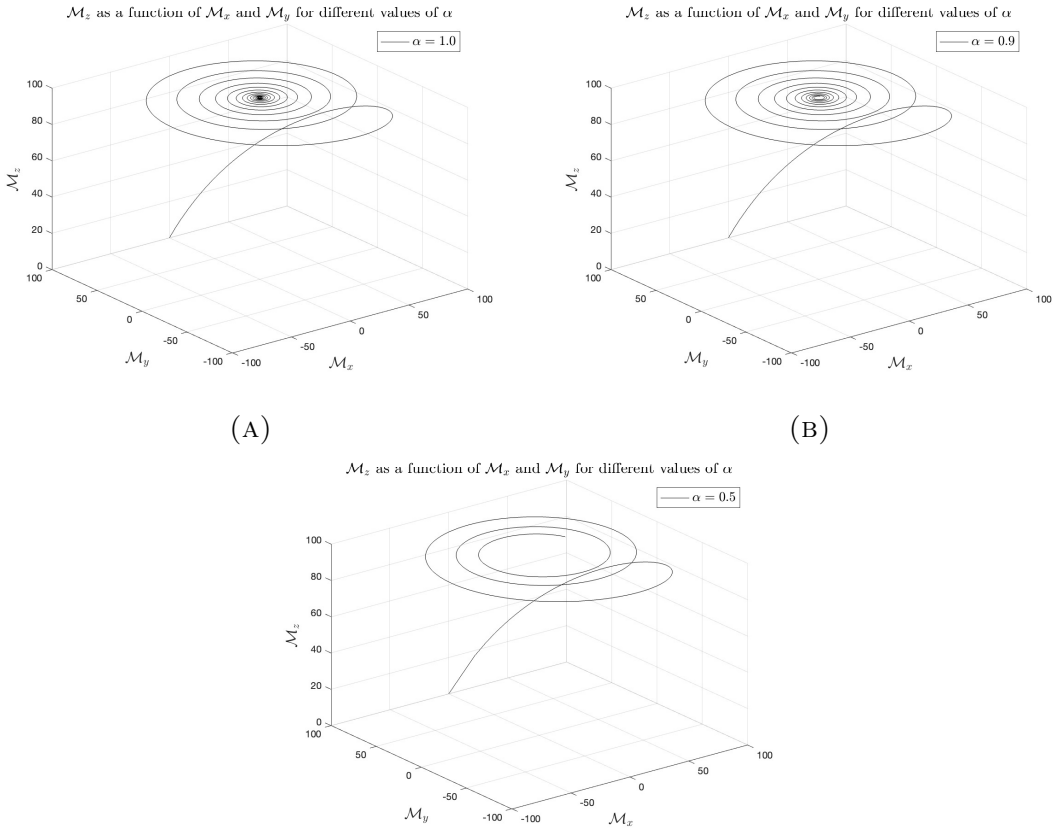


FIGURE 7. 3D Phase space of magnetization components $(\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z)$ for distinct values of the order of derivative α, (A) : $\alpha = 1, (B) : \alpha = 0.9, (C) : \alpha = 0.5$, with: $\mathcal{M}_x(0) = \mathcal{M}_y(0) = 0$, $\mathcal{M}_z(0) = 100$, $\rho_1 = 1$ and $\rho_2 = 20$.

4. Conclusions

This paper investigates fractional Bloch equations using a local operator in the sense of the conformable derivative. The mathematical study is essentially based on the fractional Laplace transform and its key properties to obtain the analytic solution of the problem. Furthermore, this research integrates a rigorous theoretical framework associated with numerical validation, implemented in MATLAB, to show the computational efficiency of the proposed methodology through several graphical illustrations.

The results highlight the conformable derivative as a more intuitive and tractable foundation for fractional-order Bloch models compared to the classical Caputo and Riemann-Liouville definitions. Its straightforward calculus, which preserves the familiar rules of differentiation, has proven advantageous for both the analytical derivation of the model and the development of stable, efficient numerical simulations.

In future work, we intend to extend this work by employing a nonlinear version of the Bloch equations to better explain the underlying phenomena. Moreover, we aim to generalize the fractional model by introducing a variable-order derivative.

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