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A CAUCHY PROBLEM FOR THE SUB-DIFFUSION EQUATION WITH THE PRABHAKAR FRACTIONAL DERIVATIVE

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Abstract. In this work, we have investigated the Cauchy problem for the second order partial differential equation involving the Prabhakar fractional derivative of the Riemann-Liouville type in the time variable. This equation has been solved by converting it into a system of two first order differential equations. Therefore, the boundary and Cauchy problems for the first order equations have first been solved as auxiliary problems. We have presented the explicit solution of the Cauchy problem for the second order differential equation using these auxiliary problems. Then, it has been proven that the obtained solution satisfies the problem.

Keywords. Generalized Mittag-Leffler function, Prabhakar fractional integral operator, Prabhakar fractional derivative of Riemann-Liouville type, Bivariate Mittag-Leffler type function, Cauchy problem.
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1. Introduction and preliminaries

In recent years, fractional calculus has attracted growing interest due to its applications in science and engineering [8], [14], [16], [22], [26], [27]. It is increasingly used to model complex processes such as anomalous diffusion, viscoelasticity, and nonlocal phenomena. For example, [8] provides an introduction for physicists and a collection of review articles on its applications, while [22] highlights key concepts and major applications from leading researchers in the field.
Significant progress has been made in Cauchy problems for fractional-order partial differential equations. The author [17] constructs a fundamental solution for a diffusion-wave equation with the Dzhrbashyan–Nersesyan derivative, proving uniqueness under a Tychonoff-type condition. The work [28] studies the Caputo derivative case, expressing the solution via Mittag-Leffler and H-functions using Laplace and Fourier transforms, illustrated with Mathematica graphs.
In addition, the work [10] studies the Cauchy problem for equations with the Riemann–Liouville fractional derivative, providing a fundamental solution and

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an explicit solution for the nonhomogeneous case using Duhamel's principle and the three-parameter Mittag-Leffler function. In [4], a linear ordinary differential equation with a fractional derivative and an involution operator in the subordinate term is considered, with existence and uniqueness results established and an explicit solution given via the fundamental solution. [3] studies an evolution equation with a regularized fractional time derivative and a uniformly elliptic operator with variable spatial coefficients, arising in diffusion on inhomogeneous fractals. The fundamental solution to the Cauchy problem is constructed and analyzed.

Recent attention has focused on Prabhakar type fractional operators, which generalize classical fractional operators and capture a wide range of memory effects, attracting significant interest for their mathematical structure and applications in modeling real-world phenomena [7], [19], [21], [5], [11], [12], [24], [23]. The author [7] reviews key results and applications of the three-parameter Mittag-Leffler (Prabhakar) function, while [19] studies properties and extensions of Prabhakar integrals and derivatives, including the regularized and Hilfer-Prabhakar derivatives. In [21], the structure and operators of Prabhakar fractional calculus are analyzed, focusing on Caputo-type operators, while in [5], linear differential equations with variable coefficients and Prabhakar-type kernels are studied. In [11], the unique solvability of a boundary value problem involving the Caputo-type Prabhakar derivative is established, with the Cauchy problem solution expressed via a bivariate Mittag-Leffler function and uniform convergence of the solution series proven. In [12], direct and inverse problems for mixed equations with the regularized Prabhakar operator are briefly presented, and the Cauchy problem for an ordinary differential equation is solved using successive iteration. The work [1] studies a generalized space-time fractional reaction-diffusion equation with the Hilfer-Prabhakar time derivative and space fractional Laplacian, obtaining solutions via Laplace and Fourier transforms in terms of the three-parameter Mittag-Leffler function. The Formable transform for Hilfer-Prabhakar derivatives has been derived in [9], where Cauchy-type equations are also solved using Fourier methods. The method is further extended to applications involving the three-parameter Mittag-Leffler function. In [29], the Cauchy problem with singular coefficients and the Prabhakar derivative is analyzed, proving uniqueness and obtaining solutions via Hankel and Fourier transforms, with properties of bivariate Mittag-Leffler and Bessel functions discussed. In the reference [24], unique solvability of the diffusion-wave equation with dynamical boundary conditions and the regularized Caputo-type Prabhakar derivative is addressed. In [23], unique solvability of the initial-boundary problem for mixed equations with the regularized Prabhakar derivative is studied. In [13], the wave-diffusion equation under initial, boundary and transmission conditions modeling gas flow in a channel is analyzed, applying separation of variables and establishing uniform convergence of the resulting series using a bivariate Mittag-Leffler-type function, with the method extendable to more general cases.

Motivated by recent developments, our work focuses on fractional diffusion processes governed by generalized time-fractional operators. We study the Cauchy problem for a sub-diffusion equation with the regularized Prabhakar fractional derivative in time. This derivative generalizes several classical operators, allowing more realistic modeling of memory effects. The main novelty of our study is the explicit construction of the solution, which facilitates analysis of its properties and will be a basis for future developments, including the construction of Green's functions for related initial-boundary value problems.

Definition 1.1. Let $k \in \mathbb{N}, \alpha, \beta, \gamma \in \mathbb{R}, \alpha > 0, \beta > 0$. The generalized Mittag-Leffler function [20] is defined as

$$E_{\alpha,\beta}^{\gamma}[z] = \sum_{k=0}^{+\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!},$$

where $(\gamma)_k = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+(k-1))$ is the Pochhammer symbol.

Definition 1.2. For $\alpha, \gamma, \delta, \gamma \in \mathbb{R}$ with $\alpha > 0, \beta > 0$ and $y \in L_1[0,T]$, $0 < t < T < \infty$ the Prabhakar fractional integral operator [20] is defined as follows

$${}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left[\delta(t-s)^{\alpha}\right] y(s) ds.$$

Definition 1.3. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha > 0, \beta > 0, m = [\beta] + 1 \in \mathbb{N}$, and $y \in AC^m(0,T)$, then the Prabhakar fractional derivative of Riemann-Liouville type is presented by [6]

$${}^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = \frac{d^m}{dt^m} {}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta} y(t), \quad 0 < t < T < \infty$$

Further, the Caputo-Prabhakar fractional derivative for $y \in AC^m[0,T]$ is defined as follows [6]:

$${}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = {}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta} \frac{d^m}{dt^m} y(t),$$

where $AC^m[0,T] = \left\{ y : I \to \mathbb{R} : \frac{d^{m-1}}{dt^{m-1}} y(t) \in AC[0,T] \right\}$ and $AC[0,T]$ means the space of absolutely continuous functions on $[0,T]$[14].

Definition 1.4. $E_{12}(x,y)$ is the bivariate Mittag-Leffler type function defined as [25]:

$$E_{12}\left(\begin{matrix} \alpha_1, \beta_1, \delta_1; \\ \alpha_2, \beta_2, \delta_2; \alpha_3, \delta_3; \alpha_4, \delta_4; \beta_3, \delta_5 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right) =$$

$$= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{\Gamma(\alpha_1 n + \beta_1 m + \delta_1) x^n y^m}{\Gamma(\alpha_2 n + \beta_2 m + \delta_2) \Gamma(\alpha_3 n + \delta_3) \Gamma(\alpha_4 n + \delta_4) \Gamma(\beta_3 m + \delta_5)},$$

$(x, y, \alpha_l, \beta_i, \delta_j \in \mathbb{R}; \min\{\alpha_l, \beta_i, \delta_j\} > 0; (l = \{1, ..., 4\}, i = \{1, 2, 3\}, j = \{1, ..., 5\}))$, in which the double series converges for $x, y \in \mathbb{R}$, if $\Delta_1 > 0$, and $\Delta_2 > 0$. Here $\Delta_1 = \alpha_2 + \alpha_3 + \alpha_4 - \alpha_1$, $\Delta_2 = \beta_2 + \beta_3 - \beta_1$.

Lemma 1.1. *The following equality*

$$\sum_{n=0}^{+\infty} \frac{(-1)^n x^n t^{-\beta n}}{n!} \sum_{k=0}^{\infty} \frac{(-\gamma n)_k t^{\alpha k} \delta^k}{k! \Gamma(\alpha k - \beta n)} = \frac{1}{2\pi i} \int_H e^{s - x t^{-\beta} s^{\beta} \left(1 - t^{\alpha} \delta s^{-\alpha}\right)^{\gamma}} ds \qquad (1.1)$$

holds for $\alpha > 0$, $0 < \beta < 1$, $\gamma > 0$, $t > 0$, $x \in \mathbb{R}$.

Proof. First, we consider the integral on the right-hand side and rewrite it as follows:

$$I = \frac{1}{2\pi i}\int\limits_{H} e^{s}\cdot e^{-xt^{-\beta}s^{\beta}\left(1-t^{\alpha}\delta s^{-\alpha}\right)^{\gamma}}ds.$$

Using the formula

$$e^{y} = \sum_{n=0}^{+\infty}\frac{y^{n}}{n!},$$

we rewrite I as

$$I = \frac{1}{2\pi i}\int\limits_{H} e^{s}\cdot\sum_{n=0}^{+\infty}\frac{\left(-xt^{-\beta}s^{\beta}\right)^{n}(1-t^{\alpha}\delta s^{-\alpha})^{\gamma n}}{n!}\,ds.$$

Considering

$$\left(1-t^{\alpha}\delta s^{-\alpha}\right)^{\gamma n} = \sum_{k=0}^{+\infty}\frac{(-\gamma n)_{k}}{k!}t^{\alpha k}\delta^{k}s^{-\alpha k},$$

we rewrite I in the following form:

$$I = \frac{1}{2\pi i}\int\limits_{H} e^{s}\cdot\sum_{n=0}^{+\infty}\frac{\left(-xt^{-\beta}s^{\beta}\right)^{n}}{n!}\sum_{k=0}^{+\infty}\frac{(-\gamma n)_{k}}{k!}t^{\alpha k}\delta^{k}s^{-\alpha k}\,ds =$$

$$= \sum_{n=0}^{+\infty}\frac{\left(-xt^{-\beta}\right)^{n}}{n!}\sum_{k=0}^{+\infty}\frac{(-\gamma n)_{k}t^{\alpha k}\delta^{k}}{k!}\frac{1}{2\pi i}\int\limits_{H} e^{s}\cdot s^{\beta n-\alpha k}\,ds.$$

Now, using

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int\limits_{H} e^{s}\cdot s^{-z}\,ds,$$

we have the following:

$$I = \sum_{n=0}^{+\infty}\frac{(-1)^{n}x^{n}t^{-\beta n}}{n!}\sum_{k=0}^{+\infty}\frac{(-\gamma n)_{k}t^{\alpha k}\delta^{k}}{k!\Gamma(\alpha k-\beta n)}.$$

Therefore, (1.1) is established.

Lemma 1.1 is proven. □

Lemma 1.2. *If $\delta \le 0$, then the following limit is valid:*

$$\lim_{x\to+\infty}\sum_{n=0}^{+\infty}\frac{(-1)^{n}x^{n}t^{-\beta n}}{n!}\sum_{k=0}^{+\infty}\frac{(-\gamma n)_{k}t^{\alpha k}\delta^{k}}{k!\Gamma(\alpha k-\beta n)} = 0. \qquad (1.2)$$

Proof. By using Lemma 1.1, we write the limit as follows:

$$\lim_{x\to+\infty}\frac{1}{2\pi i}\int\limits_{H} e^{s-xt^{-\beta}s^{\beta}\left(1-t^{\alpha}\delta s^{-\alpha}\right)^{\gamma}}ds.$$

Since $\delta \le 0$, $(1-t^{\alpha}\delta s^{-\alpha}) > 0$. When x is sufficiently large, $e^{-x\cdot(\ldots)}$ rapidly tends to zero.

We use saddle-point method [2]:

1. Find the phase function. We write the integral as the following form:

$$I(\lambda) = \int\limits_{C} e^{\lambda f(z)} g(z)\, dz,$$

where λ is the large parameter, $f(z)$ is the phase function, $g(z)$ is the slowly varying function.

2. Determining the saddle points. We find the critical points of $f(z)$:

$$f'(z_0) = 0.$$

These points may be maxima, minima, or saddle points of the function.

3. Contour deformation. We deform the path of integration (the contour) so that it passes through the neighborhood of the saddle point, where the main contribution to the integral arises.

4. Asymptotic estimation. We expand $f(z)$ into a Taylor series around the saddle point and evaluate the integral approximately:

$$I(\lambda) \approx e^{\lambda f(z_0)} \sqrt{\frac{2\pi}{\lambda \left| f''(z_0) \right|}}\, g(z_0), \quad f''(z_0) \neq 0.$$

In our limit, $f(s) = s - xt^{-\beta} s^{\beta} (1 - t^{\alpha} \delta s^{-\alpha})^{\gamma}$. The saddle point is found from the condition $f'(s_*) = 0$:

$$1 - xt^{-\beta} \beta s_*^{\beta-1} \left(1 - t^{\alpha} \delta s_*^{-\alpha}\right)^{\gamma} + O\left(x s_*^{\beta-\alpha-1}\right) = 0.$$

As $x \to +\infty$, s_* becomes large, therefore $s_*^{-\alpha} \to 0$. So $s_* \approx \left(x\beta t^{-\beta}\right)^{1/(1-\beta)}$.

Asymptotic conclusion: The integral is exponentially decreasing at $x \to +\infty$. That means,

$$\lim_{x \to +\infty} I(x,t) = 0.$$

Lemma 1.2 is proven. □

2. Main results

We now examine the following time-fractional diffusion equation

$$ {}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta} u(t,x) - u_{xx}(t,x) = f(t,x) \tag{2.1}$$

in a domain $\Omega = \{(t,x):\ 0 < t < T,\ -\infty < x < +\infty\}$. Here $f(t,x)$ is a given function. The analysis is carried out for (2.1) in the specific case $0 < \beta < 1$.

Definition 2.1. A regular solution of the equation (2.1) in the domain Ω is called a function $u(t,x)$ with the regularity ${}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} u(t,x) \in C\left(\overline{\Omega}\right)$, $u_{xx}(t,x)$, ${}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta} u(t,x) \in C(\Omega)$ that satisfies the equation (2.1) at all points $(t,x) \in \Omega$.

Problem 2.1. (Cauchy problem). Determine a regular function $u(t,x)$ that satisfies the equation (2.1) in the domain Ω and fulfills the following condition:

$$\lim_{t \to 0} {}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} u(t,x) = \tau(x), \quad x \in \mathbb{R}, \tag{2.2}$$

where $\tau(x)$ is a given function.

Before solving this Cauchy problem, we need to consider the following auxiliary problems.

Problem 2.2. Find the regular solution $u(t,x)$ of the equation

$$^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x)-u_x(t,x)=g_1(t,x) \tag{2.3}$$

that satisfies the following boundary conditions:

$$\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u(t,x)=\psi(x)\,,\quad 0<x<a, \tag{2.4}$$

$$u(t,0)=\varphi(t)\,,\quad 0<t<T \tag{2.5}$$

in a domain $\Omega_1=\{(t,x):\ 0<t<T,\ 0<x<a\}\,, 0<a,T<\infty$, where $g_1(t,x)\,,$ $\psi(x)$ and $\varphi(t)$ are given functions.

Lemma 2.1. [15] *Let* $\psi(x)\in C[0;a]\,,$ $t^{1-\beta}\varphi(t)\in C[0,T]\,,$ $t^{1-\beta}g_1(t,x)\in C\left(\overline{\Omega_1}\right),$ $g_1(t,x)$ *satisfies the Hölder condition with respect to at least one of its variables and the following compatibility condition is satisfied:*

$$\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\varphi(t)=\psi(0)\,.$$

Then there exists a unique regular solution of the equation(2.3) *in the domain* Ω_1, *satisfying the boundary conditions* (2.4) *and* (2.5). *The solution has the form*

$$u(t,x)=\int\limits_0^t\varphi(\eta)(t-\eta)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}x(t-\eta)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right)d\eta+$$

$$+\int\limits_0^x\psi(\xi)\,t^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}(x-\xi)\,t^{-\beta}\\\delta t^{\alpha}\end{matrix}\right)d\xi+$$

$$+\int\limits_0^t\int\limits_0^x g_1(\eta,\xi)(t-\eta)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}(x-\xi)(t-\eta)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right)d\xi d\eta.$$

Problem 2.3. Find the regular solution $u(t,x)$ of the equation (2.3) in a domain $\Omega_2=\{(t,x):\ 0<t<T,\quad x>a\}\,,\ a>-\infty$, that satisfies the following initial condition:

$$\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u(t,x)=\tau_1(x)\,,\quad x>a, \tag{2.6}$$

where $\tau_1(x)$ is a given function.

Lemma 2.2. *Let* $t^{1-\beta}g_1(t,x)\in C\left(\overline{\Omega_2}\right),$ $g_1(t,x)$ *satisfies the Hölder condition with respect to one of its variables,* $\tau_1(x)\in C[a,\infty)\,,$ *and the conditions*

$$\lim_{x\to\infty}\tau_1(x)\,e^{-\rho|x|^{1/(1-\beta)}}=0,\quad \lim_{x\to\infty}t^{1-\beta}g_1(t,x)\,e^{-\rho|x|^{1/(1-\beta)}}=0$$

are satisfied, where

$$\rho<(1-\beta)\left(\frac{\beta}{T}\right)^{\beta/(1-\beta)}, \tag{2.7}$$

and the convergence is uniform on the set $\{t \in (0;T)\}$. *In the domain* Ω_2, *the regular solution of the equation* (2.3) *satisfying the condition* (2.6) *is expressed as follows:*

$$u(t,x) = -\int_x^\infty \tau_1(\xi)\, t^{-1} E_{12}\left(\begin{matrix} -\gamma, 1, 0; \\ -\beta, \alpha, 0; -\gamma, 0; 1, 1; 1, 1 \end{matrix} \middle| \begin{matrix} (x-\xi)\, t^{-\beta} \\ \delta t^{\alpha} \end{matrix} \right) d\xi -$$

$$-\int_0^t \int_x^\infty g_1(\eta,\xi)\,(t-\eta)^{-1} E_{12}\left(\begin{matrix} -\gamma, 1, 0; \\ -\beta, \alpha, 0; -\gamma, 0; 1, 1; 1, 1 \end{matrix} \middle| \begin{matrix} (x-\xi)\,(t-\eta)^{-\beta} \\ \delta(t-\eta)^{\alpha} \end{matrix} \right) d\xi d\eta.$$

Proof. In Problem 2.3, if either δ or γ is equal to zero, the problem reduces to the case considered in [18]. Therefore, the proof of this lemma 2.2 follows the same approach as Theorem 2.5.1 in [18]. $\square$

Lemma 2.3. *There exists at most one regular solution to the problem* (2.3)-(2.6) *in the class of functions satisfying the condition*

$$\lim_{x\to\infty} t^{1-\beta} u(t,x)\, e^{-k|x|^{1/(1-\beta)}} = 0 \tag{2.8}$$

for some positive k. *The convergence in* (2.8) *is uniform on the set* $\{t \in (0;T)\}$.

Proof. In Problem 2.3, if either δ or γ is equal to zero, the problem reduces to the case considered in [18]. Therefore, the proof of this lemma 2.3 follows the same approach as Theorem 2.5.2 in [18]. $\square$

Problem 2.4. Find the regular solution $u(t,x)$ of the equation

$$^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta} u(t,x) + u_x(t,x) = g_2(t,x) \tag{2.9}$$

that satisfies the boundary conditions (2.4) and (2.5) in a domain Ω_1, where $g_2(t,x)$ is a given function.

Lemma 2.4. *Let* $\psi(x) \in C[0;a]$, $t^{1-\beta}\varphi(t) \in C[0,T]$, $t^{1-\beta} g_2(t,x) \in C\left(\overline{\Omega_1}\right)$, $g_2(t,x)$ *satisfies the Hölder condition with respect to at least one of its variables and the following compatibility condition is satisfied:*

$$\lim_{t\to 0} {}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta} \varphi(t) = \psi(0).$$

Then there exists a unique regular solution of the equation (2.9) *in the domain* Ω_1, *satisfying the boundary conditions* (2.4) *and* (2.5). *The solution has the form*

$$u(t,x) = -\int_0^t \varphi(\eta)\,(t-\eta)^{-1} E_{12}\left(\begin{matrix} -\gamma, 1, 0; \\ -\beta, \alpha, 0; -\gamma, 0; 1, 1; 1, 1 \end{matrix} \middle| \begin{matrix} (-x)\,(t-\eta)^{-\beta} \\ \delta(t-\eta)^{\alpha} \end{matrix} \right) d\eta -$$

$$-\int_0^x \psi(\xi)\, t^{-1} E_{12}\left(\begin{matrix} -\gamma, 1, 0; \\ -\beta, \alpha, 0; -\gamma, 0; 1, 1; 1, 1 \end{matrix} \middle| \begin{matrix} (\xi-x)\, t^{-\beta} \\ \delta t^{\alpha} \end{matrix} \right) d\xi -$$

$$-\int_0^t \int_0^x g_2(\eta,\xi)\,(t-\eta)^{-1} E_{12}\left(\begin{matrix} -\gamma, 1, 0; \\ -\beta, \alpha, 0; -\gamma, 0; 1, 1; 1, 1 \end{matrix} \middle| \begin{matrix} (\xi-x)\,(t-\eta)^{-\beta} \\ \delta(t-\eta)^{\alpha} \end{matrix} \right) d\xi d\eta.$$

The equation (2.9) differs from the equation (2.3) only by the sign in front of $u_x(t,x)$. Therefore, Lemma 2.4 can also be proved in the same way as in [15].

Problem 2.5. Find the regular solution $u(t,x)$ of the equation (2.9) in a domain $\Omega_3 = \{(t,x):\ 0<t<T,\ \ x<a\}$, $a<\infty$, that satisfies the following initial condition:

$$\lim_{t\to 0} {}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} u(t,x) = \tau_2(x), \quad x<a, \tag{2.10}$$

where $\tau_2(x)$ is a given function.

Lemma 2.5. *Let $t^{1-\beta} g_2(t,x) \in C\left(\overline{\Omega_3}\right)$, $g_2(t,x)$ satisfies the Hölder condition with respect to one of its variables, $\tau_2(x)\in C(-\infty;a]$, and the conditions*

$$\lim_{x\to-\infty} \tau_2(x)\, e^{-\rho|x|^{1/(1-\beta)}} = 0, \quad \lim_{x\to-\infty} t^{1-\beta} g_2(t,x)\, e^{-\rho|x|^{1/(1-\beta)}} = 0$$

are satisfied, where ρ satisfies (2.7), and the convergence is uniform on the set $\{t\in(0;T)\}$. Then the function $u(t,x)$, defined by this formula

$$u(t,x) = -\int_{-\infty}^{x} \tau_2(\xi)\, t^{-1} E_{12}\left(\begin{matrix} -\gamma,1,0; \\ -\beta,\alpha,0;-\gamma,0;1,1;1,1 \end{matrix}\middle|\, \begin{matrix} (\xi-x)\,t^{-\beta} \\ \delta t^{\alpha} \end{matrix}\right) d\xi -$$

$$-\int_{0}^{t}\int_{-\infty}^{x} g_2(\eta,\xi)\,(t-\eta)^{-1} E_{12}\left(\begin{matrix} -\gamma,1,0; \\ -\beta,\alpha,0;-\gamma,0;1,1;1,1 \end{matrix}\middle|\, \begin{matrix} (\xi-x)\,(t-\eta)^{-\beta} \\ \delta (t-\eta)^{\alpha} \end{matrix}\right) d\xi d\eta$$

is the regular solution of the problem (2.9)-(2.10).

Proof. Lemma 2.5 is proven in the same way as Lemma 2.2. □

Lemma 2.6. *There exists at most one regular solution $u(t,x)$ to the problem (2.9)-(2.10) in the domain Ω_3, for which, for some $k>0$, there exists a uniform limit on the set $\{t\in(0;T)\}$ and*

$$\lim_{x\to-\infty} t^{1-\beta} u(t,x)\, e^{-k|x|^{1/(1-\beta)}} = 0.$$

Proof. Lemma 2.6 is proven in the same way as Lemma 2.3. □

Now, let us proceed to solve the Cauchy problem (2.1)-(2.2). Let $u(t,x)$ is the solution of the equation (2.1) and satisfies the condition (2.2), since $t^{1-\beta} u(t,x) \in C\left(\overline{\Omega}\right)$, it is obvious that

$$\lim_{t\to 0} I_{0t}^{\alpha,\,1-\frac{\beta}{2},\,-\frac{\gamma}{2},\,\delta} u(t,x) = 0. \tag{2.11}$$

Since ${}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta} u(t,x) = {}^{PRL}D_{0t}^{\alpha,\,\frac{\beta}{2},\,\frac{\gamma}{2},\,\delta}({}^{PRL}D_{0t}^{\alpha,\,\frac{\beta}{2},\,\frac{\gamma}{2},\,\delta} u(t,x))$, we can rewrite the equation (2.1) as follows:

$$\left({}^{PRL}D_{0t}^{\alpha,\,\frac{\beta}{2},\,\frac{\gamma}{2},\,\delta} - \frac{\partial}{\partial x}\right)\left({}^{PRL}D_{0t}^{\alpha,\,\frac{\beta}{2},\,\frac{\gamma}{2},\,\delta} + \frac{\partial}{\partial x}\right) u(t,x) = f(t,x).$$

By denoting

$$v(t,x) = {}^{PRL}D_{0t}^{\alpha,\,\beta_1,\,\gamma_1,\,\delta} u(t,x) + u_x(t,x), \tag{2.12}$$

$\beta_1 = \frac{\beta}{2},\ \ \gamma_1 = \frac{\gamma}{2}$, we obtain that the function $u\,(t,x)$ is a solution of the system

$$\begin{cases} {}^{PRL}D_{0t}^{\alpha,\,\beta_1,\,\gamma_1,\,\delta} u\,(t,x) + u_x\,(t,x) = v\,(t,x)\,, \\ {}^{PRL}D_{0t}^{\alpha,\,\beta_1,\,\gamma_1,\,\delta} v\,(t,x) - v_x\,(t,x) = f\,(t,x)\,. \end{cases} \tag{2.13}$$

Henceforth, we shall denote by

$$\omega\,(t,x) = -\sum_{n=0}^{+\infty} \frac{(-x)^n}{n!} t^{-\beta_1 n-1} E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\,[\delta t^{\alpha}] =$$

$$= -t^{-1} E_{12}\left(\begin{matrix} -\gamma_1, 1, 0; \\ -\beta_1, \alpha, 0; -\gamma_1, 0; 1, 1; 1, 1 \end{matrix} \middle| \begin{matrix} (-x)\,t^{-\beta_1} \\ \delta t^{\alpha} \end{matrix} \right). \tag{2.14}$$

It follows from the Lemma 2.5 and the equality (2.11) that

$$u\,(t,x) = \int\limits_0^t \int\limits_{-\infty}^x v\,(y,s)\,\omega\,(t-y,x-s)\,dsdy. \tag{2.15}$$

We apply the operator $I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}$ to both sides of the equality (2.12):

$$I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta} v(t,x) =$$

$$= I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta}\left({}^{PRL}D_{0t}^{\alpha,\beta_1,\gamma_1,\delta} u(t,x)\right) + I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta} u_x(t,x). \tag{2.16}$$

First, we will calculate the first term on the right-hand side of this expression. We write the expression using the form of the operator, and then integrate it with respect to s by parts:

$$I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta}\left({}^{PRL}D_{0t}^{\alpha,\beta_1,\gamma_1,\delta} u(t,x)\right) =$$

$$= \int_0^t (t-s)^{-\beta_1} E_{\alpha,1-\beta_1}^{-\gamma_1}\left[\delta(t-s)^{\alpha}\right] \frac{\partial}{\partial s} I_{0s}^{\alpha,1-\beta_1,-\gamma_1,\delta} u(s,x)ds =$$

$$= (t-s)^{-\beta_1} E_{\alpha,1-\beta_1}^{-\gamma_1}\left[\delta(t-s)^{\alpha}\right] I_{0s}^{\alpha,1-\beta_1,-\gamma_1,\delta} u(s,x)\Big|_0^t +$$

$$+ \int_0^t (t-s)^{-\beta_1-1} E_{\alpha,-\beta_1}^{-\gamma_1}\left[\delta(t-s)^{\alpha}\right] I_{0s}^{\alpha,1-\beta_1,-\gamma_1,\delta} u(s,x)ds =$$

$$= \int_0^t (t-s)^{-\beta_1-1} E_{\alpha,-\beta_1}^{-\gamma_1}\left[\delta(t-s)^{\alpha}\right] ds \int_0^s (s-z)^{-\beta_1} E_{\alpha,1-\beta_1}^{-\gamma_1}\left[\delta(s-z)^{\alpha}\right] u(z,x)dz.$$

In the resulting equality, we change the order of integration and apply the formula $\sum\limits_{k=0}^{+\infty} a_k \sum\limits_{m=0}^{+\infty} b_m = \sum\limits_{k=0}^{+\infty}\sum\limits_{m=0}^{k} a_m b_{k-m}$:

$$I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta}\left({}^{PRL}D_{0t}^{\alpha,\beta_1,\gamma_1,\delta} u(t,x)\right) =$$

$$= \int_0^t u(z,x)dz \int_z^t (t-s)^{-\beta_1-1}(s-z)^{-\beta_1} E_{\alpha,1-\beta_1}^{-\gamma_1}\left[\delta(s-z)^{\alpha}\right] E_{\alpha,-\beta_1}^{-\gamma_1}\left[\delta(t-s)^{\alpha}\right] ds =$$

$$= \int_0^t u(z,x)dz \int_z^t (t-s)^{-\beta_1-1}(s-z)^{-\beta_1} \times$$

$$\times \sum_{k=0}^{\infty} \frac{(-\gamma_1)_k\,\delta^k (s-z)^{\alpha k}}{\Gamma\left(\alpha k + 1 - \beta_1\right) k!} \sum_{m=0}^{\infty} \frac{(-\gamma_1)_m\,\delta^m (t-s)^{\alpha m}}{\Gamma\left(\alpha m - \beta_1\right) m!} ds =$$

$$= \int_0^t u(z,x)dz \int_z^t (t-s)^{-\beta_1-1}(s-z)^{-\beta_1} \times$$

$$\times \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-\gamma_1)_m (-\gamma_1)_{k-m} \delta^k (s-z)^{\alpha m}(t-s)^{\alpha(k-m)}}{\Gamma(\alpha m+1-\beta_1)\Gamma(\alpha k-\alpha m-\beta_1) m!(k-m)!} ds =$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-\gamma_1)_m (-\gamma_1)_{k-m} \delta^k}{\Gamma(\alpha m+1-\beta_1)\Gamma(\alpha k-\alpha m-\beta_1) m!(k-m)!} \times$$

$$\times \int_0^t u(z,x)dz \int_z^t (t-s)^{\alpha k-\alpha m-\beta_1-1}(s-z)^{\alpha m-\beta_1} ds.$$

Next, we perform the substitution $s = (t-z)y+z$ and utilize $\sum\limits_{m=0}^{k} \frac{(\delta)_m (\gamma)_{k-m}}{m!(k-m)!} = \frac{(\delta+\gamma)_k}{k!}$:

$$I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta}\left({}^{PRL}D_{0t}^{\alpha,\beta_1,\gamma_1,\delta} u(t,x)\right) =$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-\gamma_1)_m (-\gamma_1)_{k-m} \delta^k}{\Gamma(\alpha m+1-\beta_1)\Gamma(\alpha k-\alpha m-\beta_1) m!(k-m)!} \times$$

$$\times \int_0^t (t-z)^{\alpha k-2\beta_1} u(z,x)dz \int_0^1 y^{\alpha m-\beta_1}(1-y)^{\alpha k-\alpha m-\beta_1-1} dy =$$

$$= \sum_{k=0}^{\infty} \frac{\delta^k}{\Gamma(\alpha k-2\beta_1+1)} \sum_{m=0}^{k} \frac{(-\gamma_1)_m (-\gamma_1)_{k-m}}{m!(k-m)!} \int_0^t (t-z)^{\alpha k-2\beta_1} u(z,x)dz =$$

$$= \sum_{k=0}^{\infty} \frac{\delta^k}{\Gamma(\alpha k-2\beta_1+1)} \sum_{m=0}^{k} \frac{(-\gamma_1)_m (-\gamma_1)_{k-m}}{m!(k-m)!} \int_0^t (t-z)^{\alpha k-2\beta_1} u(z,x)dz =$$

$$= \int_0^t (t-z)^{-2\beta_1} \sum_{k=0}^{\infty} \frac{(-2\gamma_1)_k \delta^k (t-z)^{\alpha k}}{\Gamma(\alpha k-2\beta_1+1) k!} u(z,x)dz =$$

$$= \int_0^t (t-z)^{-2\beta_1} E_{\alpha,1-2\beta_1}^{-2\gamma_1}\left[\delta(t-z)^{\alpha}\right] u(z,x)dz = I_{0t}^{\alpha,1-2\beta_1,-2\gamma_1,\delta} u(t,x).$$

The second term can be written as follows:

$$I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta} u_x(t,x) = \frac{\partial}{\partial x} I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta} u(t,x)$$

Let us take the limit as $t \to 0$ in (2.16) and according to conditions (2.2) and (2.11), we get

$$\lim_{t\to 0} I_{0t}^{\alpha,1-\beta_1,-\gamma_1,\delta} v(t,x) = \lim_{t\to 0} I_{0t}^{\alpha,1-\beta,-\gamma,\delta} u(t,x) = \tau(x).$$

Therefore, in accordance with Lemma 2.2, the solution of the second equation of the system (2.13) has the following form:

$$v(t,x) = \int_x^{+\infty} \tau(s)\,\omega(t,s-x)\,ds + \int_x^{+\infty}\int_0^t f(s,y)\,\omega(t-y,s-x)\,dyds. \qquad (2.17)$$

After substituting (2.17) into (2.15) and changing the order of integration, we obtain the following:

$$u(t,x)=\left(\int_{-\infty}^{x}d\xi\int_{-\infty}^{\xi}ds+\int_{x}^{+\infty}d\xi\int_{-\infty}^{x}ds\right)\tau(\xi)\int_{0}^{t}\omega(t-y,x-s)\,\omega(y,\xi-s)\,dy+$$

$$+\left(\int_{-\infty}^{x}d\xi\int_{-\infty}^{\xi}ds+\int_{x}^{+\infty}d\xi\int_{-\infty}^{x}ds\right)\times$$

$$\times\int_{0}^{t}f(\eta,\xi)\,d\eta\int_{\eta}^{t}\omega(t-y,x-s)\,\omega(y-\eta,\xi-s)\,dy. \tag{2.18}$$

Now we will use this formula:

$$\int_{0}^{y}\omega(y-t,x_1)\,\omega(t,x_2)\,dt=-\omega(y,x_1+x_2)\,. \tag{2.19}$$

We will prove this statement. Firstly, we substitute the appropriate expression for the function $\omega(t,x)$ into the left-hand side of (2.19):

$$\int_{0}^{y}\omega(y-t,x_1)\,\omega(t,x_2)\,dt=$$

$$=\int_{0}^{y}\sum_{n=0}^{+\infty}\frac{(-1)^n{x_1}^n}{n!}(y-t)^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta(y-t)^{\alpha}\right]\times$$

$$\times\sum_{m=0}^{+\infty}\frac{(-1)^m{x_2}^m}{m!}t^{-\beta_1 m-1}E_{\alpha,-\beta_1 m}^{-\gamma_1 m}\left[\delta t^{\alpha}\right]dt.$$

Then we apply the following formula

$$\sum_{k=0}^{+\infty}a_k\sum_{m=0}^{+\infty}b_m=\sum_{k=0}^{+\infty}\sum_{m=0}^{k}a_m b_{k-m}$$

twice in the subsequent calculations:

$$\int_{0}^{y}\omega(y-t,x_1)\,\omega(t,x_2)\,dt=$$

$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\frac{(-1)^n x_1^m x_2^{n-m}}{m!\,(n-m)!}\int_{0}^{y}(y-t)^{-\beta_1 m-1}t^{-\beta_1 n+\beta_1 m-1}\times$$

$$\times\sum_{k=0}^{+\infty}\frac{(-\gamma_1 m)_k\delta^k(y-t)^{\alpha k}}{\Gamma(\alpha k-\beta_1 m)\,k!}\sum_{z=0}^{+\infty}\frac{(-\gamma_1 n+\gamma_1 m)_z\delta^z t^{\alpha z}}{\Gamma(\alpha k-\beta_1 n+\beta_1 m)\,z!}dt=$$

$$= \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{(-1)^n x_1^m x_2^{n-m}}{m!\,(n-m)!} \sum_{k=0}^{+\infty} \sum_{z=0}^{k} \frac{(-\gamma_1 m)_z (-\gamma_1 n + \gamma_1 m)_{k-z} \delta^k}{\Gamma(\alpha z - \beta_1 m)\,\Gamma(\alpha k - \alpha z - \beta_1 n + \beta_1 m)\, z!\,(k-z)!} \times$$

$$\times \int_0^y (y-t)^{\alpha z - \beta_1 m - 1} t^{\alpha k - \alpha z - \beta_1 n + \beta_1 m - 1} dt.$$

After using the formulas

$$\sum_{m=0}^{n} \frac{a^m b^{n-m}}{m!\,(n-m)!} = \frac{(a+b)^n}{n!}$$

and

$$\sum_{m=0}^{k} \frac{(\delta)_m (\gamma)_{k-m}}{m!\,(k-m)!} = \frac{(\delta+\gamma)_k}{k!},$$

we express the resulting expression in terms of the function $\omega(t,x)$:

$$\int_0^y \omega(y-t, x_1)\, \omega(t, x_2)\, dt =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n (x_1+x_2)^n}{n!} \sum_{k=0}^{+\infty} \frac{(-\gamma_1 n)_k \delta^k y^{\alpha k - \beta_1 n - 1}}{k!\,\Gamma(\alpha k - \beta_1 n)} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n (x_1+x_2)^n}{n!} y^{-\beta_1 n - 1} E_{\alpha, -\beta_1 n}^{-\gamma_1 n} [\delta y^\alpha] =$$

$$= -\omega(y, x_1 + x_2).$$

As a result, the right-hand side of (2.19) is obtained. The formula (2.19) is proven.

We apply the formula (2.19) to the equality (2.18):

$$\int_0^t \omega(t-y, x-s)\, \omega(y, \xi - s)\, dy = -\omega(t, x + \xi - 2s).$$

We replace the function $\omega(t, x+\xi-2s)$ with its expression and compute the integral with respect to s:

$$-\left(\int_{-\infty}^{\xi} + \int_{-\infty}^{x} \right) \omega(t, x+\xi-2s)\, ds =$$

$$= \left(\int_{-\infty}^{\xi} + \int_{-\infty}^{x} \right) \sum_{n=0}^{+\infty} \frac{(-1)^n (x+\xi-2s)^n}{n!} t^{-\beta_1 n - 1} E_{\alpha, -\beta_1 n}^{-\gamma_1 n} [\delta t^\alpha]\, ds =$$

$$= -\frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n (x+\xi-2s)^{n+1}}{(n+1)!} t^{-\beta_1 n - 1} E_{\alpha, -\beta_1 n}^{-\gamma_1 n} [\delta t^\alpha] \Bigg|_{s=-\infty}^{s=\xi} -$$

$$-\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(x+\xi-2s)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]\Bigg|_{s=-\infty}^{s=x}=$$

$$=-\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(x-\xi)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]+$$

$$+\frac{1}{2}\lim_{s\to-\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n(x+\xi-2s)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]-$$

$$-\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(\xi-x)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]+$$

$$+\frac{1}{2}\lim_{s\to-\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n(x+\xi-2s)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]=$$

$$=-\frac{1}{2}\sum_{n=-1}^{+\infty}\frac{(-1)^n(x-\xi)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]-\frac{1}{2}t^{\beta_1-1}E_{\alpha,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]-$$

$$-\frac{1}{2}\sum_{n=-1}^{+\infty}\frac{(-1)^n(\xi-x)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]-\frac{1}{2}t^{\beta_1-1}E_{\alpha,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]+$$

$$+\lim_{s\to-\infty}\sum_{n=-1}^{+\infty}\frac{(-1)^n(x+\xi-2s)^{n+1}}{(n+1)!}t^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]+t^{\beta_1-1}E_{\alpha,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]=$$

$$=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(x-\xi)^{n}}{n!}t^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right]+$$

$$+\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(\xi-x)^{n}}{n!}t^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right]-$$

$$-\lim_{s\to-\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n(x+\xi-2s)^{n}}{n!}t^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right].$$

According to Lemma 1.2,

$$\lim_{s\to-\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n(x+\xi-2s)^{n}}{n!}t^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right]=0,$$

so

$$-\left(\int_{-\infty}^{\xi}+\int_{-\infty}^{x}\right)\omega\left(t,x+\xi-2s\right)ds=$$

$$=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(x-\xi)^{n}}{n!}t^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right]+$$

$$+\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(\xi-x)^{n}}{n!}t^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right].$$

The above procedure is repeated once more as follows:

$$\int\limits_{\eta}^{t} \omega\left(t-y, x-s\right)\omega\left(y-\eta, \xi-s\right)dy = \left\{y-\eta=z\right\} =$$

$$= \int\limits_{0}^{t-\eta} \omega\left(\left(t-\eta\right)-z, x-s\right)\omega\left(z, \xi-s\right)dz = -\omega\left(x+\xi-2s, t-\eta\right);$$

$$-\left(\int\limits_{-\infty}^{\xi} + \int\limits_{-\infty}^{x}\right)\omega\left(x+\xi-2s, t-\eta\right)ds =$$

$$= \left(\int\limits_{-\infty}^{\xi} + \int\limits_{-\infty}^{x}\right)\sum_{n=0}^{+\infty}\frac{(-1)^n(x+\xi-2s)^n}{n!}(t-\eta)^{-\beta_1 n-1}E_{\alpha,-\beta_1 n}^{-\gamma_1 n}\left[\delta(t-\eta)^{\alpha}\right]ds =$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(x-\xi)^n}{n!}(t-\eta)^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta(t-\eta)^{\alpha}\right]+$$

$$+\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n(\xi-x)^n}{n!}(t-\eta)^{-\beta_1 n+\beta_1-1}E_{\alpha,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta(t-\eta)^{\alpha}\right].$$

In view of the most recent calculations, the equality (2.18) is rewritten as follows:

$$u\left(t,x\right) = \int\limits_{-\infty}^{\infty}\tau\left(\xi\right)v\left(t,x;0,\xi\right)d\xi + \int\limits_{0}^{t}\int\limits_{-\infty}^{+\infty}f\left(\eta,\xi\right)v\left(t,x;\eta,\xi\right)d\xi d\eta, \tag{2.20}$$

where

$$v\left(t,x;\eta,\xi\right) = \frac{(t-\eta)^{\beta_1-1}}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{|x-\xi|}{(t-\eta)^{\beta_1}}\right)^n E_{\alpha,\,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta(t-\eta)^{\alpha}\right].$$

Based on the obtained results, we formulate the following theorem.

Theorem 2.2. *Let $\tau\left(x\right)\in C\left(\mathbb{R}\right)$ and $t^{1-\beta}f\left(t,x\right)\in C\left(\overline{\Omega}\right)$, $f\left(t,x\right)$ satisfies the Hölder condition with respect to x and the conditions*

$$\lim_{|x|\to\infty}\tau\left(x\right)e^{-\rho|x|^{2/(2-\beta)}} = 0, \quad \lim_{|x|\to\infty}t^{1-\beta}f\left(t,x\right)e^{-\rho|x|^{2/(2-\beta)}} = 0$$

are satisfied, where

$$\rho < (1-\beta_1)\left(\frac{\beta_1}{T}\right)^{\beta_1/(1-\beta_1)},$$

and the convergence is uniform on the set $\left\{t\in(0;T)\right\}$. Then the unique regular solution to the Cauchy problem (2.1)-(2.2) exists and has a form:

$$u\left(t,x\right) = \int\limits_{-\infty}^{\infty}\tau\left(\xi\right)v\left(t,x;0,\xi\right)d\xi + \int\limits_{0}^{t}\int\limits_{-\infty}^{+\infty}f\left(\eta,\xi\right)v\left(t,x;\eta,\xi\right)d\xi d\eta,$$

where

$$v(t,x;\eta,\xi)=\frac{(t-\eta)^{\beta_1-1}}{2}E_{12}\left(\begin{matrix}-\gamma_1,1,\gamma_1;\\ -\beta_1,\alpha,\beta_1;-\gamma_1,\gamma_1;1,1;1,1;\end{matrix}\middle|\begin{matrix}(-|x-\xi|)(t-\eta)^{-\beta_1}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right).$$

Remark 2.3. If $\delta=0$ or $\gamma=0$ in the considered problem, these cases have been studied in work [18]. In reference [18], the following Cauchy problem has been studied:

Find a solution of the following equation

$$D_{0t}^{\alpha}u(t,x)-u_{xx}(t,x)=f(t,x)$$

in a domain $D=\{(t,x):0<t<T,x\in\mathbb{R}\}$, satisfying the following condition:

$$\lim_{t\to 0}D_{0t}^{\alpha-1}u(t,x)=\tau(x),\quad x\in\mathbb{R},$$

where D_{0t}^{α} is the Riemann-Liouville operator, $0<\alpha\le 1$.

The solution of this Cauchy problem has the following form:

$$u(t,x)=\int\limits_{-\infty}^{\infty}\tau(\xi)v(t,x;0,\xi)d\xi+\int\limits_{0}^{t}\int\limits_{-\infty}^{+\infty}f(\eta,\xi)v(t,x;\eta,\xi)d\xi d\eta,$$

here

$$v(t,x;\eta,\xi)=\frac{1}{2}(t-\eta)^{\frac{\alpha}{2}-1}e_{1,\frac{\alpha}{2}}^{1,\frac{\alpha}{2}}\left(-\frac{|x-\xi|}{(t-\eta)^{\frac{\alpha}{2}}}\right),$$

$e_{1,\frac{\alpha}{2},\frac{\alpha}{2}}^{1,\frac{\alpha}{2}}\left(-\frac{|x-\xi|}{(t-\eta)^{\frac{\alpha}{2}}}\right)$ is the Wright function.

Appendix. We prove that the function defined by the formula (2.20) is the solution of the problem (2.1)-(2.2) Let us denote by

$$u_0(t,x)=\int\limits_{-\infty}^{\infty}\tau(\xi)v(t,x;0,\xi)d\xi,\tag{2.21}$$

$$u_1(t,x)=\int\limits_{0}^{t}\int\limits_{-\infty}^{+\infty}f(\eta,\xi)v(t,x;\eta,\xi)d\xi d\eta.\tag{2.22}$$

$u_0(t,x)$ satisfies the homogeneous equation

$$^{PRL}D_{0t}^{\alpha,\beta,\gamma,\delta}u_0(t,x)-\frac{\partial^2}{\partial x^2}u_0(t,x)=0\tag{2.23}$$

and the non-homogeneous initial condition

$$\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u_0(t,x)=\tau(x).\tag{2.24}$$

Let us start to check the condition (2.24):

$$\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}u_0(t,x)=$$

$$=\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,1-\beta,-\gamma,\delta}\left(\int\limits_{-\infty}^{x}+\int\limits_{x}^{+\infty}\right)\tau(\xi)v(t,x;0,\xi)d\xi.\tag{2.25}$$

We see the first integral:

$$ {}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} \int\limits_{-\infty}^{x} \tau\left(\xi\right) v\left(t,x;0,\xi\right) d\xi = $$

$$ = \int\limits_{0}^{t} (t-s)^{-\beta} E_{\alpha,\,1-\beta}^{-\gamma} \left[\delta(t-s)^{\alpha}\right] ds \int\limits_{-\infty}^{x} \tau\left(\xi\right) v\left(s,x;0,\xi\right) d\xi = $$

$$ = \int\limits_{-\infty}^{x} \tau\left(\xi\right) d\xi \int\limits_{0}^{t} (t-s)^{-\beta} E_{\alpha,\,1-\beta}^{-\gamma} \left[\delta(t-s)^{\alpha}\right] v\left(s,x;0,\xi\right) ds = $$

$$ = \int\limits_{-\infty}^{x} \tau\left(\xi\right) d\xi \int\limits_{0}^{t} (t-s)^{-\beta} E_{\alpha,\,1-\beta}^{-\gamma} \left[\delta(t-s)^{\alpha}\right] \times $$

$$ \times \frac{s^{\beta_1-1}}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{|x-\xi|}{s^{\beta_1}}\right)^{n} E_{\alpha,\,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1} \left[\delta s^{\alpha}\right] ds = $$

$$ = \int\limits_{-\infty}^{x} \tau\left(\xi\right) d\xi \int\limits_{0}^{t} (t-s)^{-\beta} \sum_{k=0}^{+\infty} \frac{(-\gamma)_k \delta^k (t-s)^{\alpha k}}{\Gamma\left(\alpha k+1-\beta\right) k!} \times $$

$$ \times \frac{s^{\beta_1-1}}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{|x-\xi|}{s^{\beta_1}}\right)^{n} \sum_{m=0}^{+\infty} \frac{(-\gamma_1 n+\gamma_1)_m \delta^m s^{\alpha m}}{\Gamma\left(\alpha m-\beta_1 n+\beta_1\right) m!} ds. $$

We use this formula

$$ \sum_{k=0}^{+\infty} a_k \sum_{m=0}^{+\infty} b_m = \sum_{k=0}^{+\infty} \sum_{m=0}^{k} a_m b_{k-m}, $$

then we get

$$ {}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} \int\limits_{-\infty}^{x} \tau\left(\xi\right) v\left(t,x;0,\xi\right) d\xi = $$

$$ = \frac{1}{2} \sum_{\substack{n=0\\k=0}}^{+\infty} \sum_{m=0}^{k} \frac{(-\gamma)_m (-\gamma_1 n+\gamma_1)_{k-m} \delta^k}{\Gamma\left(\alpha m+1-\beta\right) \Gamma\left(\alpha k-\alpha m-\beta_1 n+\beta_1\right) m!\,(k-m)!n!} \times $$

$$ \times \int\limits_{-\infty}^{x} (\xi-x)^n \tau\left(\xi\right) d\xi \int\limits_{0}^{t} (t-s)^{-\beta+\alpha m} s^{\beta_1-1-\beta_1 n+\alpha k-\alpha m} ds = $$

$$ = \frac{1}{2} \sum_{\substack{n=0\\k=0}}^{+\infty} \sum_{m=0}^{k} \frac{(-\gamma)_m (-\gamma_1 n+\gamma_1)_{k-m} \delta^k t^{\alpha k-\beta_1 n-\beta_1}}{\Gamma\left(\alpha m-\beta+1\right) \Gamma\left(\alpha k-\alpha m-\beta_1 n+\beta_1\right) m!\,(k-m)!n!} \times $$

$$ \times \int\limits_{-\infty}^{x} (\xi-x)^n \tau\left(\xi\right) d\xi \int\limits_{0}^{1} z^{\alpha k-\alpha m-\beta_1 n+\beta_1-1} (1-z)^{\alpha m-\beta} dz = $$

$$= \frac{1}{2}\sum_{\substack{n=0\\k=0}}^{+\infty} \frac{\delta^k t^{\alpha k-\beta_1 n-\beta_1}}{n!\Gamma\left(\alpha k-\beta_1 n-\beta_1+1\right)} \sum_{m=0}^{k} \frac{(-\gamma)_m(-\gamma_1 n+\gamma_1)_{k-m}}{m!\,(k-m)!}\times$$

$$\times\int\limits_{-\infty}^{x} (\xi-x)^n \tau\left(\xi\right)d\xi =$$

$$= \frac{1}{2}\sum_{\substack{n=0\\k=0}}^{+\infty} \frac{\delta^k t^{\alpha k-\beta_1 n-\beta_1}(-\gamma_1 n+\gamma_1-\gamma)_k}{n!\Gamma\left(\alpha k-\beta_1 n-\beta_1+1\right)k!} \int\limits_{-\infty}^{x} (\xi-x)^n \tau\left(\xi\right)d\xi =$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty} \frac{t^{-\beta_1}}{n!} E_{\alpha,\,-\beta_1-\beta_1 n+1}^{-\gamma_1 n-\gamma_1}\left[\delta t^\alpha\right] \int\limits_{-\infty}^{x} \left(\frac{\xi-x}{t^{\beta_1}}\right)^n \tau\left(\xi\right)d\xi =$$

$$= \left\{\frac{\xi-x}{t^{\beta_1}}=y\right\} = \frac{1}{2}\sum_{n=0}^{+\infty} \frac{E_{\alpha,\,-\beta_1-\beta_1 n+1}^{-\gamma_1 n-\gamma_1}\left[\delta t^\alpha\right]}{n!} \int\limits_{-\infty}^{0} y^n \tau\left(x+yt^{\beta_1}\right)dy.$$

Now we calculate the limit of the last equality that $t\to 0$:

$$\lim_{t\to 0} {}^P I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} \int\limits_{-\infty}^{x} \tau\left(\xi\right) v\left(t,x;0,\xi\right)d\xi =$$

$$= \lim_{t\to 0}\frac{1}{2}\sum_{n=0}^{+\infty} \frac{E_{\alpha,\,-\beta_1 n-\beta_1+1}^{-\gamma_1 n-\gamma_1}\left[\delta t^\alpha\right]}{n!} \int\limits_{-\infty}^{0} y^n \tau\left(x+yt^{\beta_1}\right)dy =$$

$$= \frac{\tau\left(x\right)}{2}\lim_{t\to 0}\sum_{n=0}^{+\infty} \frac{E_{\alpha,\,-\beta_1 n-\beta_1+1}^{-\gamma_1 n-\gamma_1}\left[\delta t^\alpha\right]}{n!} \int\limits_{-\infty}^{0} y^n dy =$$

$$= \frac{\tau\left(x\right)}{2}\sum_{n=0}^{+\infty} \frac{y^{n+1}}{\Gamma\left(n+2\right)\Gamma\left(1-\beta_1-\beta_1 n\right)}\Bigg|_{-\infty}^{0} =$$

$$= \frac{\tau\left(x\right)}{2}\, y e_{1,\,\beta_1}^{2,\,1-\beta_1}\left(y\right)\Big|_{-\infty}^{0} = \frac{\tau\left(x\right)}{2}.$$

After doing the same calculations as above for the second integral of (2.25), we get the following result:

$$\lim_{t\to 0} {}^P I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} \int\limits_{x}^{+\infty} \tau\left(\xi\right) v\left(t,x;0,\xi\right)d\xi = \frac{\tau\left(x\right)}{2}.$$

Therefore, $u_0\left(t,x\right)$ satisfies the initial condition (2.24).
Now we see that $u_0\left(t,x\right)$ satisfies the equation (2.23). First, we apply the operator:

$${}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta} u_0\left(t,x\right) = \frac{\partial}{\partial t} {}^P I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} u_0\left(t,x\right) =$$

$$= \frac{\partial}{\partial t} \int_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}^{-\gamma} \left[\delta (t-s)^{\alpha} \right] ds \int_{-\infty}^{\infty} \tau (\xi) v (s, x; 0, \xi) d\xi =$$

$$= \frac{\partial}{\partial t} \int_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}^{-\gamma} \left[\delta (t-s)^{\alpha} \right] ds \left(\int_{-\infty}^{x} + \int_{x}^{+\infty} \right) \tau (\xi) \; v (s, x; 0, \xi) d\xi =$$

$$= \frac{1}{2} \frac{\partial}{\partial t} \sum_{\substack{n=0\\k=0}}^{+\infty} \frac{\delta^k t^{\alpha k - \beta_1 n - \beta_1} (-\gamma_1 n - \gamma_1)_k}{n! \Gamma (\alpha k - \beta_1 n - \beta_1 + 1) k!} \int_{-\infty}^{+\infty} (-|x-\xi|)^n \tau (\xi) d\xi =$$

$$= \frac{1}{2} \sum_{\substack{n=0\\k=0}}^{+\infty} \frac{\delta^k t^{\alpha k - \beta_1 n - \beta_1 - 1} (-\gamma_1 n - \gamma_1)_k}{n! \Gamma (\alpha k - \beta_1 n - \beta_1) k!} \int_{-\infty}^{+\infty} (-|x-\xi|)^n \tau (\xi) d\xi =$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{t^{-\beta_1 - 1}}{n!} E_{\alpha, -\beta_1 n - \beta_1}^{-\gamma_1 n - \gamma_1} \left[\delta t^{\alpha} \right] \int_{-\infty}^{+\infty} \left(- \frac{|x-\xi|}{t^{\beta_1}} \right)^n \tau (\xi) d\xi.$$

Secondly, we calculate the partial derivative. We calculate the derivative separately for the intervals $(-\infty, x]$ and $[x, +\infty)$. For the interval $(-\infty, x]$:

$$\frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \tau (\xi) v (t, x; 0, \xi) d\xi =$$

$$= \frac{1}{2} \frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} \frac{(-1)^n t^{\beta_1 - \beta_1 n - 1}}{n!} E_{\alpha, -\beta_1 n + \beta_1}^{-\gamma_1 n + \gamma_1} \left[\delta t^{\alpha} \right] \int_{-\infty}^{x} \tau (\xi) (x-\xi)^n d\xi =$$

$$= \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(t^{\beta_1 - 1} E_{\alpha, \beta_1}^{\gamma_1} \left[\delta t^{\alpha} \right] \int_{-\infty}^{x} \tau (\xi) d\xi + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{(-1)^n t^{\beta_1 - \beta_1 n - 1}}{n!} E_{\alpha, -\beta_1 n + \beta_1}^{-\gamma_1 n + \gamma_1} \left[\delta t^{\alpha} \right] \int_{-\infty}^{x} \tau (\xi) (x-\xi)^n d\xi \right) =$$

$$= \frac{1}{2} \frac{\partial}{\partial x} \left(t^{\beta_1 - 1} E_{\alpha, \beta_1}^{\gamma_1} \left[\delta t^{\alpha} \right] \tau (x) - \sum_{n=0}^{\infty} \frac{(-1)^n t^{-\beta_1 n - 1}}{n!} E_{\alpha, -\beta_1 n}^{-\gamma_1 n} \left[\delta t^{\alpha} \right] \int_{-\infty}^{x} \tau (\xi) (x-\xi)^n d\xi \right) =$$

$$= \frac{1}{2} \frac{\partial}{\partial x} \left(t^{\beta_1 - 1} E_{\alpha, \beta_1}^{\gamma_1} \left[\delta t^{\alpha} \right] \tau (x) - t^{-1} E_{\alpha, 0}^{0} \left[\delta t^{\alpha} \right] \int_{-\infty}^{x} \tau (\xi) d\xi - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{(-1)^n t^{-\beta_1 n - 1}}{n!} E_{\alpha, -\beta_1 n}^{-\gamma_1 n} + \left[\delta t^{\alpha} \right] \int_{-\infty}^{x} \tau (\xi) (x-\xi)^n d\xi \right) =$$

$$= \frac{1}{2} \left(t^{\beta_1 - 1} E_{\alpha, \beta_1}^{\gamma_1} \left[\delta t^{\alpha} \right] \tau' (x) - t^{-1} E_{\alpha, 0}^{0} \left[\delta t^{\alpha} \right] \tau (x) + \right.$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n t^{-\beta_1 n-\beta_1-1}}{n!}E_{\alpha,\,-\beta_1 n-\beta_1}^{-\gamma_1 n-\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{-\infty}^{x}\tau(\xi)(x-\xi)^n d\xi\Bigg).$$

For the interval $[x,+\infty)$:

$$\frac{\partial^2}{\partial x^2}\int\limits_{x}^{+\infty}\tau(\xi)\,v(t,x;0,\xi)\,d\xi=$$

$$=\frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}\frac{(-1)^n t^{\beta_1-\beta_1 n-1}}{n!}E_{\alpha,\,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)(\xi-x)^n d\xi=$$

$$=\frac{1}{2}\frac{\partial^2}{\partial x^2}\Bigg(t^{\beta_1-1}E_{\alpha,\,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)\,d\xi+$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n t^{\beta_1-\beta_1 n-1}}{n!}E_{\alpha,\,-\beta_1 n+\beta_1}^{-\gamma_1 n+\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)(\xi-x)^n d\xi\Bigg)=$$

$$=\frac{1}{2}\frac{\partial}{\partial x}\left(-t^{\beta_1-1}E_{\alpha,\,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]\tau(x)-\sum_{n=0}^{\infty}\frac{(-1)^n t^{-\beta_1 n-1}}{n!}E_{\alpha,\,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)(\xi-x)^n d\xi\right)=$$

$$=\frac{1}{2}\frac{\partial}{\partial x}\Bigg(-t^{\beta_1-1}E_{\alpha,\,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]\tau(x)-t^{-1}E_{\alpha,\,0}^{0}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)\,d\xi-$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n t^{-\beta_1 n-1}}{n!}E_{\alpha,\,-\beta_1 n}^{-\gamma_1 n}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)(\xi-x)^n d\xi\Bigg)=$$

$$=\frac{1}{2}\Big(-t^{\beta_1-1}E_{\alpha,\,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]\tau'(x)+t^{-1}E_{\alpha,\,0}^{0}\left[\delta t^{\alpha}\right]\tau(x)+$$

$$+\sum_{n=0}^{\infty}\frac{(-1)^n t^{-\beta_1 n-\beta_1-1}}{n!}E_{\alpha,\,-\beta_1 n-\beta_1}^{-\gamma_1 n-\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau(\xi)(\xi-x)^n d\xi\Bigg).$$

Now, we add the results obtained from both intervals:

$$\frac{\partial^2}{\partial x^2}\int\limits_{-\infty}^{x}\tau(\xi)\,v(t,x;0,\xi)\,d\xi+\frac{\partial^2}{\partial x^2}\int\limits_{x}^{+\infty}\tau(\xi)\,v(t,x;0,\xi)\,d\xi=$$

$$=\frac{1}{2}\Big(t^{\beta_1-1}E_{\alpha,\,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]\tau'(x)-t^{-1}E_{\alpha,\,0}^{0}\left[\delta t^{\alpha}\right]\tau(x)+$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n t^{-\beta_1 n-\beta_1-1}}{n!}E_{\alpha,\,-\beta_1 n-\beta_1}^{-\gamma_1 n-\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{-\infty}^{x}\tau(\xi)(x-\xi)^n d\xi-$$

$$-t^{\beta_1-1}E_{\alpha,\,\beta_1}^{\gamma_1}\left[\delta t^{\alpha}\right]\tau'(x)+t^{-1}E_{\alpha,\,0}^{0}\left[\delta t^{\alpha}\right]\tau(x)+$$

$$+\sum_{n=0}^{\infty}\frac{(-1)^n t^{-\beta_1 n-\beta_1-1}}{n!}E_{\alpha,\,-\beta_1 n-\beta_1}^{-\gamma_1 n-\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{x}^{+\infty}\tau\left(\xi\right)\left(\xi-x\right)^n d\xi\Bigg)=$$

$$=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{t^{-\beta_1-1}}{n!}E_{\alpha,\,-\beta_1 n-\beta_1}^{-\gamma_1 n-\gamma_1}\left[\delta t^{\alpha}\right]\int\limits_{-\infty}^{+\infty}\left(-\frac{|x-\xi|}{t^{\beta_1}}\right)^n\tau\left(\xi\right)d\xi.$$

According to the results of the operator and derivative, the equation (2.23) is valid for $u_0\left(t,x\right)$.

$u_1\left(t,x\right)$ satisfies the non-homogeneous equation

$$^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}u_1\left(t,x\right)-\frac{\partial^2}{\partial x^2}u_1\left(t,x\right)=f\left(t,x\right) \tag{2.26}$$

and the homogeneous initial condition

$$\lim_{t\to 0}{}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}u_1\left(t,x\right)=0. \tag{2.27}$$

We use the Duhamel's principle in order to solve the problem (2.26)-(2.27), so we write the function $u_1\left(t,x\right)$ in the following form:

$$u_1\left(t,x\right)=\int\limits_0^t V\left(t,\eta,x\right)d\eta, \tag{2.28}$$

where

$$\begin{cases} ^{PRL}D_{\eta t}^{\alpha,\,\beta,\,\gamma,\,\delta}V\left(t,\eta,x\right)-V_{xx}\left(t,\eta,x\right)=0,\\ ^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}V\left(t,\eta,x\right)\Big|_{\eta=t}=f\left(t,x\right). \end{cases} \tag{2.29}$$

Let us substitute the equality (2.28) into the equation (2.26):

$$1)\ {}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}u_1\left(t,x\right)=\frac{\partial}{\partial t}{}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}\int\limits_0^t V\left(t,\eta,x\right)d\eta=$$

$$=\frac{\partial}{\partial t}\int\limits_0^t\left(t-s\right)^{-\beta}E_{\alpha,\beta}^{-\gamma}\left[\delta(t-s)^{\alpha}\right]ds\int\limits_0^s V\left(s,\eta,x\right)d\eta=$$

$$=\frac{\partial}{\partial t}\int\limits_0^t d\eta\int\limits_{\eta}^t\left(t-s\right)^{-\beta}E_{\alpha,\beta}^{-\gamma}\left[\delta(t-s)^{\alpha}\right]V\left(s,\eta,x\right)ds=$$

$$=\frac{\partial}{\partial t}\int\limits_0^t {}^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}V\left(t,\eta,x\right)d\eta=$$

$$={}^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}V\left(t,\eta,x\right)\Big|_{\eta=t}+\int\limits_0^t\frac{\partial}{\partial t}{}^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}V\left(t,\eta,x\right)d\eta=$$

$$= {}^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} V\left(t,\eta,x\right)\Big|_{\eta=t} + \int\limits_{0}^{t} {}^{PRL}D_{\eta t}^{\alpha,\,\beta,\,\gamma,\,\delta} V\left(t,\eta,x\right) d\eta;$$

$$2)\,\frac{\partial^2}{\partial x^2} u_1\left(t,x\right) = \int\limits_{0}^{t} V_{xx}\left(t,\eta,x\right) d\eta;$$

$$3)\,{}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta} u_1\left(t,x\right) - \frac{\partial^2}{\partial x^2} u_1\left(t,x\right) =$$

$$= {}^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} V\left(t,\eta,x\right)\Big|_{\eta=t} + \int\limits_{0}^{t} {}^{PRL}D_{\eta t}^{\alpha,\,\beta,\,\gamma,\,\delta} V\left(t,\eta,x\right) d\eta - \int\limits_{0}^{t} V_{xx}\left(t,\eta,x\right) d\eta =$$

$$= {}^{P}I_{\eta t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta} V\left(t,\eta,x\right)\Big|_{\eta=t} + \int\limits_{0}^{t} \left[{}^{PRL}D_{\eta t}^{\alpha,\,\beta,\,\gamma,\,\delta} V\left(t,\eta,x\right) - V_{xx}\left(t,\eta,x\right)\right] d\eta = f\left(t,x\right).$$

According to the system (2.29) and calculations 1)-3), one can state that $u_1\left(t,x\right)$ satisfies the equation (2.26). Therefore, we can write the following equalities:

$$V\left(t,\eta,x\right) = \int\limits_{-\infty}^{+\infty} f\left(\eta,\xi\right) v\left(t,x;\eta,\xi\right) d\xi,$$

$$u_1(t,x) = \int_0^t d\eta \int_{-\infty}^{+\infty} f(\eta,\xi) v(t,x;\eta,\xi) d\xi.$$

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