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# ANALYTICAL METHODS FOR SOLVING TIME-FRACTIONAL STEFAN PROBLEM FOR DIFFUSION-CONVECTION-REACTION EQUATION

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Abstract. This study investigates a time-fractional Stefan problem that models phase-change processes in media exhibiting anomalous diffusion and memory effects. A fractional Stefan model with an advective–diffusive flux is developed by deriving the Stefan condition from the time-fractional diffusion–convection–reaction equation. A rescaling technique reduces the original fractional PDE to a self-similar form that enables deeper analytical exploration. The interface function is constructed to satisfy boundary and Stefan conditions. Approximate solutions are obtained using fractional derivatives in the Riemann–Liouville and Caputo formulations. The model is applicable to heat transport in porous media, cryosurgery, groundwater contamination, and thermal energy storage.

*Keywords.* Fractional Stefan problem, Anomalous diffusion, Advective-diffusive flux, Self-similar solutions, Fractional calculus.
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## 1. Introduction

A partial differential equation that characterizes heat transfer in a material undergoing a phase change is referred to as a Stefan Problem (SP). Since the evolving interface is not known beforehand, determining the boundary becomes an essential part of the solution [12]. A key variation of this problem is the tracking of the front during pressure-driven liquid infiltration into a HeleShaw cell initially filled with gas [4], or into a porous medium [14, 8]. For such SPs, two primary modeling approaches are used. The traditional approach, known as the sharp-interface model [15], exhibits a discontinuous jump in the liquid fraction across the melting front. The alternative, the diffuse-interface method, separates the solid and liquid phases with a finitethickness interface. The enthalpy method [21] and the phase-field models are examples of this approach. The enthalpy technique assumes that the liquid fraction transitions over a range of temperatures, implicitly defining the thickness of the interface. Our study focuses on how incorporating memory

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effects alters the formulation of the problem. Specifically, we investigate whether the convergence of the diffuse-interface model to the sharp-interface model still holds as the interface thickness approaches zero. We demonstrate that introducing memory into the flux formulation disrupts this convergence by comparing both modeling approaches in the context of a Stefan melting problem. In particular, we show that the way latent heat accumulates under memory effects is significantly different. The problem of tracking the solidliquid interface during the melting of a pure solid at the phase transition temperature is therefore the quintessential example of a SP. An initially separated domain $D$ is placed in a

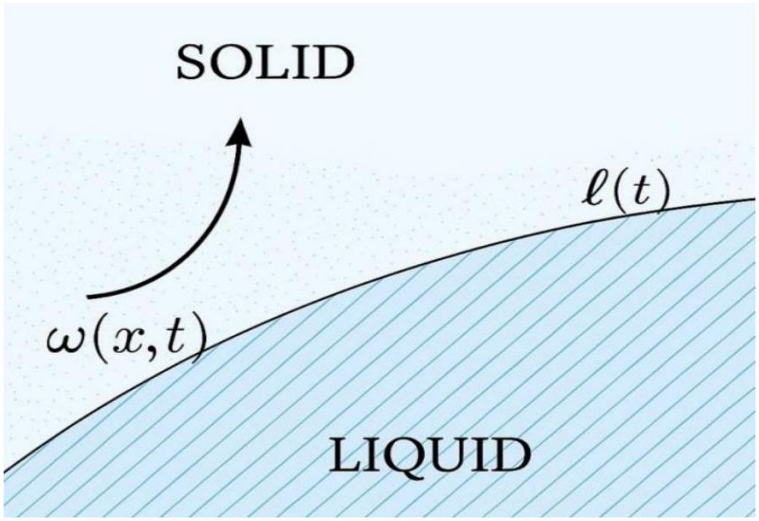



FIGURE 1. Schematic of the melting problem with sharp-and diffuse-interface model.

suitable geometric setup. As this process continues in a physical environment, the liquid phase $D_L$ and the solid phase $D_S$ become divided by a clearly defined moving boundary $\ell$(t). The speed at which the solid can take in the latent heat L required for melting controls how this boundary moves. In the sharp-interface model, the location and movement of the contact are directly related to the memory of latent heat, which is treated as a single combined value needed for melting determines how this boundary shifts. In the sharp-interface model, the position and motion of the contact are directly connected to the stored effect of latent heat, which is considered as a single overall quantity and the advection-diffusion flux

$$\mathcal{F}(x,t) = -\partial^{1-\lambda}\omega_x(x,t) + \alpha\partial^{1-\lambda}\omega(x,t) + \mu\omega(x,t) \tag{1.1}$$

where $\alpha$ is the plasma velocity. The authors developed the fractional Stefan model with a diffusive interface and the sharp-interface model based on this idea. The fractional Caputo derivative replaces the standard time derivative to describe the sharp-interface model introduced in [9]. This problem has been studied and tackled several times over the past few years. In [3, 2, 9], the authors showed the existence of weak solutions in non-cylindrical regions with fixed boundaries. In [10], the Hopf lemma was proven under suitable smoothness conditions. It's

also worth mentioning the work in [16], where a unique solution to the problem was found. However, the diffusive-interface model that was obtained does not match the sharp-interface results, as shown in [17]. This has led researchers to take a closer look at the time-fractal version of Stefan's model. Other different ways of formulating the time and space fractional Stefan problem have also been considered in in [17, 19, 18] and [7]. Using the minimal smoothness assumptions, we present in this work the sharp results involving the non-local flux given in Eq.(1.1). We reach a model that resembles the one shown in [20], but with additional boundary conditions. In the end, we find self-similar solutions to the problem that were not considered carefully in [2].

Let us consider the domain $D = (0, \ell)$ in $\mathbb{R}$. Let $t = 0$, and $D$ can be divided into $\mathrm{D}_L = (0, x_0)$ which is a liquid domain and $\mathrm{D}_S = (x_0, \ell)$ is a solid domain. The enthalpy function $E = \sigma c\omega + \sigma H\varphi$ is defined from [2], where $\omega(x, t)$ is the temperature at point $x \in D$ and time $t, \varphi$ represents the latent heat, $\sigma$ is the density, $c$ is the volumetric specific heat and $H$ is the latent heat of fusion. To assume that $\varphi$ has the following formula

$$\varphi = 1 \text{ in } D_L, \quad \varphi = 0 \text{ in } D_S \tag{1.2}$$

$\mathcal{F}(x, t)$ is defined as the flux at $x \in D$ and time $t$. To define the energy conservation has the formula: for every $S = (a, b) \subseteq D$

$$\frac{\partial}{\partial t} \int_S E(x, t)dx = Q(a, t) - Q(b, t). \forall t > 0, \text{ and } S \subset (0, \infty) \tag{1.3}$$

The total enthalpy is given by the initial enthalpy combined with the time-averaged changes in the local fluxes at the boundaries of $S$.

By applying the diffusive flux defined in Eq.(1.1), we derive the fractional SP starting from the steady-state case Eq.(1.3). We need to apply certain smoothness conditions on the temperature function $\omega(x, t)$ and the phase interface $\ell(t)$ to carry this out accurately. At the beginning, we consider $t^*$ to be positive

$$\begin{gathered} \ell(t) \in AC\left[0, t^*\right], \quad t \in (0, t^*) \\ \omega_t(\cdot, t) \in L^1(0, \ell(t)), \omega_x(x, \cdot) \in L^\infty\left(U^x\right), \ x \in D, t \in (0, t^*) \\ \omega_x(\cdot, t) \in AC[0, \ell(t) - \varepsilon] \text{ for every } \varepsilon > 0 \text{ and every } t \in (0, t^*) \end{gathered} \tag{H.1}$$

$$\begin{gathered} U^x = \left\{t : (x, t) \in \mathrm{D}_{\ell, \mathrm{t}^*}\right\}, \text{where} \\ \mathrm{D}_{\ell, \mathrm{t}^*} = \left\{(x, t) : 0 < x < \ell(t), 0 < t < t^*\right\}, \end{gathered}$$

From now on, the space of absolutely continuous functions will be denoted by AC. A function $\omega : [a, b] \to IR$ is said to be absolutely continuous on [a, b]. The initial-boundary conditions for the SP in standard setting are considered as follows condition $\omega(x, 0) = \omega_0(x)$, and the boundary condition $\omega(0, t) = \omega_D(t)$ are non-negative or $\omega_x(0, t) = \omega_N(t)$ is non-positive. In particular, if $\omega_D = 0$, or $\omega_N = 0$ then $\omega = 0$. Otherwise, for expecting

$$\ell'(t) > 0. \tag{H.2}$$

This represents the melting of the solid phase. The temperature within the solid is assumed to vanish when considering the one-phase Stefan problem. Therefore,

the flux is given by the equation and is only active in the liquid part of the domain $(D_{\ell,\mathrm{t}^*})$.

$$\mathcal{F}(x,t) = \begin{cases} -\partial^{1-\lambda}\omega_x(x,t) + \alpha\partial^{1-\lambda}\omega(x,t) + \mu\omega(x,t) \text{ for } (\mathrm{x},\mathrm{t}) & \in \mathrm{D}_{\ell,\mathrm{t}^*} \\ 0, & \text{o.w.} \end{cases} \tag{1.4}$$

where

$$\partial_\tau^{1-\lambda}\omega(x,t) = \begin{cases} \frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_0^t (t-\tau)^{\lambda-1}\omega(x,\tau)d\tau, & \text{for } x \le \ell(0) \\ \frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_\tau^t (t-\tau)^{\lambda-1}\omega(x,\tau)d\tau, & \text{for } x > \ell(0) \end{cases} \tag{1.5}$$

$$\partial_\tau^{1-\lambda}\omega_x(x,t) = \begin{cases} \frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_0^t (t-\tau)^{\lambda-1}\omega_x(x,\tau)d\tau, & \text{for } x \le \ell(0) \\ \frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_\tau^t (t-\tau)^{\lambda-1}\omega_x(x,\tau)d\tau & \text{for } x > \ell(0) \end{cases} \tag{1.6}$$

Combining this with Eq. (1.2) results in the following expression of equality, shown in Eq.(1.3), and let $W = S \cap \mathrm{D}_{\ell,\mathrm{t}^*}$, then

$$\frac{d}{dt}\left(\int_W (\omega(x,t)+1)dx\right) = \mathcal{F}(a,t) - \mathcal{F}(b,t) \tag{1.7}$$

The final regularity condition, which we will be as follows

$$\begin{gathered} \ell'(t) \in L^\infty_{\mathrm{loc}}\ ([0,t^*]) \\ \partial^\lambda_{\ell(x)}\omega(\cdot,t) \in L^1(0,\ell(t)) \forall t \in (0,t^*) \end{gathered} \tag{H.3}$$

Next, we develop the Stefan problem for Diffusion-Convection-Reaction equation (DCRE), which presents the first main result of this work in the upcoming section.

## 2. Formulating the time-fractional Stefan problem for DCRE

The Fractional Stefan Problem for the DCRE is a mathematical model that builds on the classical Stefan problem by including fractional calculus. The classical Stefan problem describes phase transitions, such as ice turning into water, and is controlled by a diffusion equation with a boundary that moves as the phase changes, see [11]. The main ideas of the Stefan problem involve the classical heat equation a form of diffusion equation that represents how heat moves through a material. The moving interface, called the Stefan boundary, separates two phases (like solid and liquid), and its position must be determined as part of the solution. This problem is usually described by partial differential equations and boundary conditions that include a term for latent heat. Another important idea is fractional calculus, which expands the traditional concept of derivatives and integrals from whole-number (integer) orders to non-integer (fractional) orders. This extension allows for more precise modeling of systems that show memory effects or depend on past behavior. In diffusion processes, using a fractional derivative helps describe anomalous diffusion, where the diffusion rate behaves differently than in the standard (integer-order) case. In this setting, the Fractional Stefan Problem for DCRE merges the classical Stefan model with fractional derivatives, resulting in a fractional form of the diffusion equation. The time derivative used in this model is often the Caputo derivative or the Riemann-Liouville derivative, both of which generalize how the system evolves over time. This fractional framework helps capture more complicated phase-change behaviors that do not follow

regular diffusion rules. It includes effects such as non-local behavior and long-term memory, making the model more realistic in some physical situations. The boundary conditions and the equation for the moving interface (Stefan condition) are adjusted to reflect the fractional characteristics of the diffusion process. The fractional Stefan problem has important applications in fields such as modeling the freezing and thawing of soil, the solidification of magma, phase changes in materials with complex thermal behavior, and in medical fields like modeling the freezing and thawing of biological tissues.

Now, let us look at the sharp Stefan problem for DCRE with the boundary condition $\omega(\ell(t),t)=0$, and under the assumptions (H.1)-( H.2), the flux is given by Eq.(1.1) and Eq.(1.3) which points to the next equation

$$\partial_\tau^\lambda \omega(x,t)-\omega_{xx}(x,t)+\alpha\omega_x(x,t)+\mu\omega(x,t)=0, 0<x<\ell(t) \tag{2.1}$$

For almost every $(x,t)\in D_{\ell,t^*}$, where

$$\partial_\tau^\lambda \omega(x,t)=\begin{cases} \frac{1}{\Gamma(1-\lambda)}\int_0^t (t-\tau)^{-\lambda}\frac{d}{d\tau}\omega(x,\tau)d\tau, & \forall x\le \ell(0)\\ \frac{1}{\Gamma(1-\lambda)}\int_\tau^t (t-\tau)^{-\lambda}\frac{d}{dt}\omega(x,\tau)d\tau, & \forall x>\ell(0)\end{cases} \tag{2.2}$$

Moreover, the functions $\omega$ and $\ell$ are connected through the formula

$$\ell'(t)=\lim_{a\to\ell(t)}\left(\alpha\frac{d}{dt}\int_A^t\frac{\omega(a,\tau)}{(t-\tau)^{1-\lambda}}d\tau-\frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_A^t\frac{\omega_x(a,\tau)}{(t-\tau)^{1-\lambda}}d\tau+\mu\omega(a,t)\right) \tag{2.3}$$

Where $A=\ell^{-1}(a)$, and if (H.3)is satisfied, the boundary condition is given as follows

$$\lim_{\varepsilon\to 0^+}\left(\alpha\omega(\ell(t)-\varepsilon,t)-\omega_x(\ell(t)-\varepsilon,t)+\mu I_A^{1-\lambda}\omega(\ell(t)-\varepsilon,\mathrm{t})\right)=0. \tag{2.4}$$

It is worth mentioning that [2, 6] studied the fractional Stefan problem using the flux defined by the Riemann-Liouville (RL) derivative. However, the researchers managed to derive the following system of equations.

$$\partial^\lambda \omega(x,t)-\omega_{xx}(x,t)+\alpha\omega_x(x,t)+\mu\omega(x,t)+\frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}=0 \tag{2.5}$$

$$\partial^\lambda \ell(t)=\alpha\omega(\ell(t),t)-\omega_x(\ell(t),t) \tag{2.6}$$

The second main outcome of this study will be presented by identifying self-similar solutions to the Stefan problem within the domain.

$$D=\{(x,t):0<x<\ell(t),0<t<\infty\}. \tag{2.7}$$

The left boundary condition will be applied under the assumption $\ell(0)=0$. We will present the problem in the following theorem.

**Theorem 2.1.** *Let us suppose that (H.1)-(H.2), and Eq.(1.3) with the flux given by Eq.(1.1), then the SP is satisfied for Eq.(2.5) in D with the initial, boundary conditions*

$$\omega(0,\mathrm{t})=\varpi,\quad \omega(\ell(t),\mathrm{t})=0,\quad \omega_x(\ell(t),\mathrm{t})=0, \tag{2.8}$$

*and the Stefan condition*

$$\ell'(t) = \lim_{a\to\ell(t)} \left( \alpha \frac{d}{dt} \int_A^t \frac{\omega(a,\tau)}{(t-\tau)^{1-\lambda}} d\tau - \frac{1}{\Gamma(\lambda)} \frac{d}{dt} \int_A^t \frac{\omega_x(a,\tau)}{(t-\tau)^{1-\lambda}} d\tau + \mu\omega(a,t) \right) \tag{2.9}$$

*Proof.* To use the principle of energy conservation on the subset $S$ and time $t \in (0, t^*)$ and to derive the system of equations from Eq.(1.7), by looking at two cases:
i. If $S = (a, b) \subseteq (0, \ell(0))$, then from (H.2) we own $S \subseteq (0, \ell(t))$ for each $t \in (0, t^*)$ and Eq.(1.7) gives

$$\frac{d}{dt}\left[\int_S (\omega(x,t)+1)dx\right] = \mathcal{F}(a,t) - \mathcal{F}(b,t)$$

Hence

$$\begin{aligned}\int_W \frac{d}{dt}\omega(x,t)dx = \alpha\partial^{1-\lambda}(\omega(a,t)-\omega(b,t)) - \partial^{1-\lambda}\left(\omega_x(a,t)-\omega_x(b,t)\right)\\ + \mu(\omega(a,t)-\omega(a,t))\end{aligned}$$

Let use $I^{1-\lambda}$ which is The fractional integral w.r.t. the time under assumption (H.1), therefore

$$\begin{aligned}\int_S \partial^\lambda \omega(x,t)dx =& \alpha(\omega(a,t)-\omega(b,t)) - (\omega_x(a,t)-\omega_x(b,t))\\ &+ \mu I^{1-\lambda}(\omega(a,t)-\omega(a,t))\end{aligned}$$

Using Fundamental Theorem, to get and since S $\subseteq (0, \ell(0))$ is for an arbitrary case, then

$$\partial^\lambda \omega(x,t) - \omega_{xx}(x,t) + \alpha\omega_x(x,t) + \mu\omega(x,t) = 0. \forall (x,t) \in (0,\ell(0)) \times (0,t^*) \tag{2.10}$$

ii. If $S = (a, b)$, where $\ell(0) < a < \ell(t) < b$, and Let $A = \ell^{-1}(a)$ then Eq (2.1) has the form

$$\frac{d}{dt}\left(\int_a^{\ell(t)} \omega(x,t) + 1 dx\right) = Q(a,t)$$

and since $\mathcal{F}(a,t) = -\partial^{1-\lambda}\omega_x(a,t) + \alpha\partial^{1-\lambda}\omega(a,t) + \mu\omega(a,t)$

$$\mathcal{F}(a,t) = \alpha \frac{d}{dt} \int_A^t \frac{\omega(a,\tau)}{(t-\tau)^{1-\lambda}} d\tau - \frac{1}{\Gamma(\lambda)} \frac{d}{dt} \int_A^t \frac{\omega_x(a,\tau)}{(t-\tau)^{1-\lambda}} d\tau + \mu\omega(a,\tau)$$

Differentiating the integral on the left-hand side

$$\begin{aligned}\int_a^{\ell(t)} \frac{d}{dt}\omega(x,t)dx + \ell'(t)[\omega(\ell(t),t)+1]\\ = \alpha \frac{d}{dt} \int_A^t \frac{\omega(a,\tau)}{(t-\tau)^{1-\lambda}} d\tau - \frac{1}{\Gamma(\lambda)} \frac{d}{dt} \int_A^t \frac{\omega_x(a,\tau)}{(t-\tau)^{1-\lambda}} d\tau + \mu\omega(a,\tau)\end{aligned}$$

Using $\omega(\ell(t),t)=0$, we get

$$\int_a^{\ell(t)} \frac{d}{dt}\omega(x,t)dx + \ell'(t) = \alpha\frac{d}{dt}\int_A^t \frac{\omega(a,\tau)}{(t-\tau)^{1-\lambda}}d\tau - \frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_A^t \frac{\omega_x(a,\tau)}{(t-\tau)^{1-\lambda}}d\tau + \mu\omega(a,\tau) \tag{2.11}$$

Based on the hypothesis (H.1), if $a \to \ell(t)$, the first term vanishes, and as a result, we get Eq.(2.9), the operator $I_A^{1-\lambda}$ is applied to both sides of Eq.(2.11), and then

$$\begin{aligned}
&\frac{1}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\int_a^{\ell(t)}\frac{d}{d\tau}\omega(x,\tau)dxd\tau + \frac{1}{\Gamma(1-\lambda)}\int_A^t \ell'(\tau)(t-\tau)^{-\lambda}d\tau\\
&= \frac{\alpha}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\frac{d}{d\tau}\int_A^\tau (\tau-\rho)^{\lambda-1}\omega(a,\rho)d\rho d\tau\\
&- \frac{1}{\Gamma(\lambda)}\frac{1}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\frac{d}{d\tau}\int_A^\tau (\tau-\rho)^{\lambda-1}\omega_x(a,\rho)d\rho d\tau\\
&+ \frac{\mu}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\omega(x,\tau)d\tau
\end{aligned} \tag{2.12}$$

According to (H.1) we have $\omega_x(a,\cdot)\in L^\infty(D_a)$ so, the right hand side of Eq.(2.12) can be expressed as :

$$\begin{aligned}
&I_A^{1-\lambda}\alpha\frac{d}{dt}\left(I_A^\lambda\omega(a,.)\right) - I_A^{1-\lambda}\frac{d}{dt}\left(I_A^\lambda\omega_x(a,.)\right) + \mu I_A^{1-\lambda}\omega(a,.)\\
&= \alpha\omega(a,t) - \omega_x(a,t) + \frac{\mu}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\omega(a,\tau)d\tau
\end{aligned} \tag{2.13}$$

Using the Fubini's Theorem to the first part of Eq.(2.12), to obtain

$$\int_a^{\ell(t)} \partial_A^\lambda\omega(x,t)dx + \frac{1}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\ell'(t)d\tau = \alpha\omega(\mathrm{a},\mathrm{t}) - \omega_x(a,t) + \mu I_A^{1-\lambda}\omega(a,). \tag{2.14}$$

Using the substitution $\tau=\ell^{-1}(x)$, we obtain

$$\frac{1}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\ell'(\tau)d\tau = \frac{1}{\Gamma(1-\lambda)}\int_a^{\ell(t)} (t-\tau)^{-\lambda}dx \tag{2.15}$$

and,

$$\begin{aligned}
&\int_a^{\ell(t)} \partial_A^\lambda\omega(x,t)dx + \frac{1}{\Gamma(1-\lambda)}\int_a^{\ell(t)} (t-\tau)^{-\lambda}dx\\
&= \alpha\omega(\mathrm{a},\mathrm{t}) - \omega_x(a,t) + \mu I_A^{1-\lambda}\omega(a,.)
\end{aligned} \tag{2.16}$$

We expect that $\omega_{\mathrm{x}}(\cdot,\mathrm{t})$ might show distinct behavior near the phase transition point. Therefore, it is necessary to proceed with great care. Given that $\varepsilon>0$

and a $< \ell(\mathrm{t}) - \varepsilon$, assumption (H.1) gives

$$-\omega_x(a,t) = \int_a^{\ell(t)-\varepsilon} \omega_{xx}(x,t)dx - \omega_x(\ell(t)-\varepsilon,t) \tag{2.17}$$

$$-\alpha\omega(a,t) = \int_a^{\ell(t)-\varepsilon} \alpha\omega_x(a,t)dx - \alpha\omega(\ell(t)-\varepsilon,t) \tag{2.18}$$

By applying these two identifications in Eq.(2.16), we obtain

$$\begin{aligned}&\int_a^{\ell(t)-\varepsilon}\left[\partial_\tau^\lambda\omega - \omega_{xx} + \alpha\omega_x + \mu\omega + \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}\right]dx \\ &= -\int_{\ell(t)-\varepsilon}^{\ell(t)}\left[\partial_\tau^\lambda\omega(x,t) + \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}\right]dx + \alpha\omega(\ell(t)-\varepsilon,t) \\ &-\omega_x(\ell(t)-\varepsilon,t) + \frac{\mu}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\omega(\ell(\tau)-\varepsilon,\tau)d\tau\end{aligned} \tag{2.19}$$

suppose $\tilde{a}$ any value such that $a \in (\ell(0), \tilde{a})$. T repeat the previous steps using $\tilde{a}$ in place of $a$, then

$$\begin{aligned}&\int_{\tilde{\alpha}}^{\ell(t)-\varepsilon}\left[\partial_\tau^\lambda\omega - \omega_{xx} + \alpha\omega_x + \mu\omega + \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}\right]dx \\ &= -\int_{\ell(t)-\varepsilon}^{\ell(t)}\left[\partial_\tau^\lambda\omega(x,t) + \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}\right]dx + \alpha\omega(\ell(t)-\varepsilon,t) \\ &-\omega_x(\ell(t)-\varepsilon,t) + \frac{\mu}{\Gamma(1-\lambda)}\int_A^t (t-\tau)^{-\lambda}\omega(\ell(\tau)-\varepsilon,\tau)d\tau\end{aligned} \tag{2.20}$$

Subtracting the sides of Eq.(2.19) and Eq.(2.20) we obtain at

$$\int_{\tilde{a}}^{a}\left[\partial_\tau^\lambda\omega - \omega_{xx} + \alpha\omega_x + \mu\omega + \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}\right]dx = 0 \tag{2.21}$$

for arbitrary $a, \tilde{a} \in (\ell(t), \ell(t)-\varepsilon)$ hence, ), it follows that

$$\partial_\tau^\lambda\omega - \omega_{xx} + \alpha\omega_x + \mu\omega + \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)} = 0, \forall x \in (\ell(0), \ell(t)) \tag{2.22}$$

The Eq.(2.1) is satisfied. It still needs to be shown Eq.(2.4). From Eq.(2.19) and Eq.(2.21) we deduce that

$$0 = \mu I_A^{1-\lambda}\omega(\ell(t)-\varepsilon,\mathrm{t}) - \int_{\ell(t)-\varepsilon}^{\ell(t)} \partial_\tau^\lambda\omega(x,t)dx + \alpha\omega(\ell(t)-\varepsilon,t) - \omega_x(\ell(t)-\varepsilon,t)$$

To apply (H.3), we get that

$$\lim_{\varepsilon\to 0^+}\int_{\ell(t)-\varepsilon}^{\ell(t)} \partial_\tau^\lambda\omega(x,t)dx = 0 \tag{2.23}$$

The Eq.(2.23),implies Eq.(2.4), which completes the proof. □

## 3. Self-Similarity Technique for Fractional SP

The self-similarity technique is a mathematical method used to simplify the initial boundary values problems like the Stefan problem. In these problems, the position of the phase-change interface (melting front) evolves over time, and the solution often exhibits scaling properties, see [5, 1].

**Lemma 3.1.** *Suppose that $\omega(x,t)$ be a solution of the Stefan problem (2.5)-(2.9). This solution becomes asymptotically equal to a solution to the problem.*

$$\partial_\tau^\lambda \omega(x,t) - \omega_{xx}(x,t) + (t-\tau)^{-\lambda}/\Gamma(1-\lambda) = 0, \quad \text{in D} \tag{3.1}$$

$$\omega(0,\mathrm{t}) = \varpi, \quad \omega(\ell(t),\mathrm{t}) = 0, \quad \omega_x(\ell(t),\mathrm{t}) = 0, \tag{3.2}$$

$$\ell'(t) = -\frac{1}{\Gamma(\lambda)} \lim_{a\to\ell(t)} \frac{d}{dt}\left(\int_A^t (t-\tau)^{\lambda-1}\omega_x(a,\tau)d\tau\right) \tag{3.3}$$

*Proof.* Now, we apply the recalling technique to simplify the Stefan problem Eqs.(2.5)-(2.9). and calculate it by using rescaling function $\omega_k(x,t)$ and then $\lim_{k\to\infty}\omega_k(x,t) = \omega(x,t)$ which solves Eq.(3.1). Let us introduce parameters $\beta_1,\beta_2,\beta_3$ the rescaling function as follows

$$\omega_k(x,t) = k^{\beta_3}\omega\left(k^{\beta_1}x, k^{\beta_2}t\right) \tag{3.4}$$

Our goal is to find $\beta_1,\beta_2,\beta_3$ and the interface $\ell(t)$ such that $\omega_k = \omega$. By the calculation, if the pair $\{\omega,\eta\}$ is a solution to Eqs.(2.5), (2.8),(2.9), then

$$\partial_\tau^\alpha \omega_k(x,t) - (\omega_k)_{xx}(x,t) + \left(k^{-\beta_2}\mathrm{t}-\tau\right)^{-\lambda}/\Gamma(1-\lambda)$$
$$= -\alpha k^{2\beta_1}\omega_x\left(k^{\beta_1}\mathrm{x}, k^{\beta_2}\mathrm{t}\right) - \mu k^{\beta_3}\omega\left(k^{\beta_1}\mathrm{x}, k^{\beta_2}\mathrm{t}\right),$$

and the same technique as in [1, 3, 18], then

$$\partial_\tau^\lambda \omega(x,t) - \omega_{xx}(x,t) - (t-\tau)^{-\lambda}/\Gamma(1-\lambda) = -\alpha k^{-(\beta_3-\beta_1)}(\omega_k)_x(x,t) + \mu k^{-\beta_3}\omega_k(x,t)$$

Thus, if $\beta_3 = 0$ and $\beta_2 = 2\beta_1/\lambda$. Assuming $\ell(t)$ satisfies $\tau = \ell^{-1}(\mathrm{x}) = k^{-\beta_2}\ell^{-1}\left(k^{\beta_1}\mathrm{x}\right)$. On the condition that $\beta_1 < 0, \lim_{k\to\infty}\omega_k(x,t) = \omega(x,t)$, then

$$\partial_\tau^\lambda \omega(x,t) - \omega_{xx}(x,t) - (t-\tau)^{-\lambda}/\Gamma(1-\lambda) = 0$$

This equation fulfills the diffusive flux under the condition $\ell(0) < \mathrm{a} < \ell(\mathrm{t}) < b$, we get Stefan's condition

$$\frac{d}{dt}\left(\int_a^{\ell(t)} \omega(x,t) + 1dx\right) = -\frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_A^t (t-\tau)^{\lambda-1}\omega_x(a,\tau)d\tau$$

Taking the derivative of the integral on the left-hand side results to

$$\int_a^{\ell(t)} \frac{d}{dt}\omega(x,t)dx + \ell'(t)[\omega(\ell(t),t)+1] = -\frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_A^t (t-\tau)^{\lambda-1}\omega_x(a,\tau)d\tau$$

Using $\omega(\ell(t),t) = 0$, obtain

$$\int_a^{\ell(t)} \frac{d}{dt}\omega(x,t)dx + \ell'(t) = -\frac{1}{\Gamma(\lambda)}\frac{d}{dt}\int_A^t (t-\tau)^{\lambda-1}\omega_x(a,\tau)d\tau$$

Based on assumption (H.1), if $a \to \ell(t)$, the first term vanishes, and consequently, the Eq.(3.1) is satisfied. Thus, the system of Eqs. (3.1)-(3.3) holds and the proof is done. □

**Theorem 3.2.** *Let us assume that $\ell$ is defined by $\ell(\mathrm{t}) = \zeta \mathrm{t}^{\frac{\lambda}{2}}$ for a given positive $\zeta$, and $\omega$ fulfills Eqs.(3.1)-(3.3). Suppose that $\omega$ has following regularity that for some $h > 1$ and every $t > 0$ there hold $\omega_x(\cdot, t) \in L^1(0, \ell(t)), \omega_{xx}(\cdot, t) \in L^1(\ell(t)/h, \ell(t))$. Then, the function $\theta$ defined by*

$$\theta(\zeta) = \omega(1, \zeta), \quad \zeta = tx^{-\frac{2}{\lambda}} \tag{3.5}$$

*fulfills $\theta' \in AC\left(\left[\zeta_0, h^{\frac{2}{\lambda}}\zeta_0\right]\right), \theta \in C^2\left(\zeta_0, h^{\frac{2}{\lambda}}\zeta_0\right)$ and for $\zeta \in \left(\zeta_0, h^{\frac{2}{\lambda}}\zeta_0\right)$, there hold*

$$\frac{1}{\Gamma(1-\lambda)}\int_{\zeta_0}^{\zeta}\frac{\theta'(\rho)}{(\zeta-\rho)^{\lambda}}d\rho - \left(\frac{2}{\lambda}\right)^2\zeta^2\theta''(\zeta) - \frac{2(\lambda+2)}{\lambda^2}\zeta\theta'(\zeta) + \frac{(\zeta-\zeta_0)^{-\lambda}}{\Gamma(1-\lambda)} = 0 \tag{3.6}$$

$$\theta(\zeta_0) = 0 \tag{3.7}$$

$$\left(\frac{\lambda}{2}\right)^2\zeta_0^{-2}\Gamma(\lambda) = \lim_{b\to\zeta_0}\frac{d}{db}\left[\int_{\zeta_0}^{b}(b-\rho)^{\lambda-1}\theta'(\rho)d\rho\right] \tag{3.8}$$

$$\lim_{\zeta\to\zeta_0}(\zeta-\zeta_0)^{\lambda}\theta''(\zeta) = \left(\frac{\lambda}{2}\right)^2\left(\frac{\zeta_0^{-2}}{\Gamma(1-\lambda)}\right) \tag{3.9}$$

$$\theta'(\zeta_0) = 0 \tag{3.10}$$

*Proof.* Let us start by introducing a self-similarity variable. Given the existence and uniqueness of the solution to the Stefan problem Eqs.(3.1), (3.2) and Eq.(3.3), it follows that $\omega_k$ is also solution, where $\omega_k(x,t) = k^{\beta_3}\omega\left(k^{\beta_1}x, k^{\beta_2}t\right)$, and From the rescaling function and the assumed shape function $\theta$ in (3.5), by selecting the parameters without loss of generality as $\beta_1 = -\lambda/2, \beta_2 = 1, \beta_3 = 0$. then we obtain

$$\zeta = \mathrm{t}x^{-\frac{2}{\lambda}} \tag{3.11}$$

and the shape function $\theta$ is defined in Eq.(3.5).
In this theorem, we establish how the expected regularity properties of $\omega$ transforms to the properties of $\theta$. Furthermore, we will reformulate the conditions Eqs.(3.1), (3.2) and Eq.(3.3) in terms of $\theta$ and prove that Eq.(3.3) implies vanishing of derivative of $\theta$ in point $\zeta_0$. Let us begin with a straightforward calculation. By Eq.(3.5)

$$\omega_t(x,t) = \theta'\left(tx^{-\frac{2}{\lambda}}\right)x^{-\frac{2}{\lambda}}, \quad \omega_x(x,t) = -\frac{2}{\lambda}\theta'\left(tx^{-\frac{2}{\lambda}}\right)tx^{\frac{-(\lambda+2)}{\lambda}} \tag{3.12}$$

$$\omega_{xx}(x,t) = \left(\frac{2}{\lambda}\right)^2\theta''\left(tx^{-\frac{2}{\lambda}}\right)\left(tx^{-\frac{2}{\lambda}}\right)^2x^{-2} + \frac{2}{\lambda}\left(\frac{2}{\lambda}+1\right)\theta'\left(tx^{-\frac{2}{\lambda}}\right)\left(tx^{-\frac{2}{\lambda}}\right)x^{-2} \tag{3.13}$$

Using the change of variable $\rho = \tau x^{-\frac{2}{\lambda}}$ we arrive at

$$\partial_\tau^{\lambda}\omega(x,t) = \frac{1}{\Gamma(1-\lambda)}\int_{\zeta_0 x^{\frac{2}{\lambda}}}^{t}(t-\tau)^{-\lambda}\omega_\tau(x,\tau)d\tau$$

Using a variable substitution $\tau$ to $\rho$

$$\partial_{\tau}^{\lambda}\omega(x,t)=\frac{x^{-2}}{\Gamma(1-\lambda)}\int_{\zeta_0}^{t\mathrm{x}^{-\frac{2}{\lambda}}}\left(t\mathrm{x}^{-\frac{2}{\lambda}}-\rho\right)^{-\lambda}\theta'(\rho)d\rho$$

Moreover, we obtain

$$\left(\mathrm{t}-\zeta_0\mathrm{x}^{\frac{2}{\lambda}}\right)^{-\lambda}=\mathrm{x}^{-2}\left(t\mathrm{x}^{-\frac{2}{\lambda}}-\zeta_0\right)^{-\lambda}$$

Applying these findings in equation Eq.(3.1) where $\ell$ is defined as $\ell(\mathrm{t})=\zeta\mathrm{t}^{\frac{\lambda}{2}}$
$\frac{x^{-2}}{\Gamma(1-\lambda)}\int_{\zeta_0}^{tx^{-\frac{2}{\lambda}}}\left(t\mathrm{x}^{-\frac{2}{\lambda}}-\rho\right)^{-\lambda}\theta'(\rho)d\rho=\left(\frac{2}{\lambda}\right)^2\theta''\left(tx^{-\frac{2}{\lambda}}\right)\left(tx^{-\frac{2}{\lambda}}\right)^2x^{-2}$

$$+\frac{2(\lambda+2)}{\lambda^2}\theta'\left(tx^{-\frac{2}{\lambda}}\right)\left(tx^{-\frac{2}{\lambda}}\right)x^{-2}+\mu\theta\left(tx^{-\frac{2}{\lambda}}\right)-\frac{\left(t-\zeta_0x^{\frac{2}{\lambda}}\right)^{-\lambda}}{\Gamma(1-\lambda)}$$

where $\zeta=tx^{-\frac{2}{\lambda}}$, then

$$\frac{1}{\Gamma(1-\lambda)}\int_{\zeta_0}^{\zeta}(\zeta-\rho)^{-\lambda}\theta'(\rho)d\rho=\left(\frac{2}{\lambda}\right)^2\zeta^2\theta''(\zeta)+\frac{2(\lambda+2)}{\lambda^2}\zeta\theta'(\zeta)-\frac{(\zeta-\zeta_0)^{-\lambda}}{\Gamma(1-\lambda)}$$

if $\zeta\to\zeta_0$, then
$\frac{1}{\Gamma(1-\lambda)}\int_{\zeta_0}^{\zeta}(\zeta-\rho)^{-\lambda}\theta'(\rho)d\rho=\left(\frac{2}{\lambda}\right)^2\zeta^2\theta''(\zeta)+\frac{2(\lambda+2)}{\lambda^2}\zeta\theta'(\zeta)$
since that the function $\omega$ vanishes on the free boundary sufficient to prove that Eq.(3.7) holds

$$0=\omega(\ell(t),t)=\omega\left(\zeta_1t^{\frac{\lambda}{2}},t\right)=\theta\left(\zeta_0\right)$$

Using Eq.(3.11). Now, we prove the regularity properties. By Eq.(3.12) ,we obtain

$$\infty>\int_0^{\ell(t)}\left|\omega_x(x,t)\right|dx=\int_{\zeta_0}^{\infty}\left|\theta'(\zeta)\right|d\zeta$$
$$\int_{\zeta_0}^{\infty}\left|\theta'(\zeta)\right|d\zeta<\infty \tag{3.14}$$

Similarly, from Eq. (3.13), then

$$\infty>\int_{\ell(t)/h}^{\ell(t)}\left|\omega_{xx}(x,t)\right|dx=\int_{\zeta_0}^{h^{\frac{2}{\lambda}}\zeta_0}\left|\frac{2}{\lambda}\theta''(\zeta)\zeta^{1+\frac{\lambda}{2}}t^{-\frac{\lambda}{2}}+\left(\frac{2}{\lambda}+1\right)\theta'(\zeta)\zeta^{\frac{\lambda}{2}}t^{-\frac{\lambda}{2}}\right|d\zeta$$
$$\geq\frac{2}{\lambda}\zeta_0^{1+\frac{\lambda}{2}}t^{-\frac{\lambda}{2}}\int_{\zeta_0}^{\frac{2}{\lambda}\zeta_0}\left|\theta''(\zeta)\right|d\zeta-\left(\frac{2}{\lambda}+1\right)h\zeta_0^{\frac{\lambda}{2}}t^{-\frac{\lambda}{2}}\int_{\zeta_0}^{\infty}\left|\theta'(\zeta)\right|d\zeta,t>0$$

thus, we obtain

$$\int_{\zeta_0}^{h^{\frac{2}{\lambda}}\zeta_0}\left|\theta''(\zeta)\right|d\zeta<\infty \tag{3.15}$$

Then the condition $\theta'\in AC\left(\left[\zeta_0,h^{2/\lambda}\zeta_0\right]\right)$ is fulfilled based on the estimates from Eq.(3.15) and Eq.(3.14), and $\theta\in C^2\left(\zeta_0,h^{2/\lambda}\zeta_0\right)$ for the absolute continuity of $\theta'$ in Eq.(3.6). Thus, the proposed regularity result is established.. To rewrite Eq.(3.3)

which refers to Eq.(3.8). To fix $a \in (\ell(t)/h, \ell(t))$ and applyied $\rho = \tau a^{-2/\lambda}$. Furthermore, integration by parts is applied to obtain $\Phi = \ell'(t)$.

$$\begin{aligned}\Phi &= -\frac{1}{\Gamma(\lambda)}\frac{d}{dt}\left(\int_A^t (t-\tau)^{\lambda-1}\omega_x(a,\tau)d\tau\right)\\ &= \frac{a}{\Gamma(\lambda)}\frac{d}{dt}\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda-1}\rho\theta'(\rho)d\rho\end{aligned}$$

After applying integration by parts, we derive

$$\Phi = \frac{2}{\lambda}\frac{a}{\Gamma(\lambda)}\frac{\mathrm{d}}{\mathrm{dt}}\left(\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\frac{\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda}}{\lambda}\left(\rho\theta''(\rho)+\theta'(\rho)\right)d\rho + \frac{\left(ta^{-\frac{2}{\lambda}}-\zeta_0\right)^{\lambda}}{\lambda}\zeta_0\theta'\left(\zeta_0\right)\right)$$

Given the smoothness of the second derivatives of $\theta$ within $\left(\zeta_0, h^{\frac{2}{\lambda}}\zeta_0\right)$, and since $\theta' \in AC\left(\left[\zeta_0, h^{\frac{2}{\lambda}}\zeta_0\right]\right)$, and

$$\lim_{a\to\ell(t)}\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda-1}\theta'(\rho)d\rho = 0$$

then

$$\Phi = \frac{2}{\lambda}\frac{a^{1-\frac{2}{\lambda}}}{\Gamma(\lambda)}\left[\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-p\right)^{\lambda-1}\rho\theta''(\rho)d\rho + \left(ta^{-\frac{2}{\lambda}}-\zeta_0\right)^{\lambda-1}\zeta_0\theta'\left(\zeta_0\right)\right]$$

Applying these findings together with $\ell(\mathrm{t}) = \zeta \mathrm{t}^{\frac{\lambda}{2}}$ into Eq. (3.3), and do some calculation and changing variables $a^{1-2/\lambda} = {\zeta_1}^{1-2/\lambda}t^{\lambda/2-1}$, we observe that

$$\begin{aligned}\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-p\right)^{\lambda-1}\rho\theta''(\rho)d\rho &= -\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda}\theta''(\rho)d\rho + ta^{-\frac{2}{\lambda}}\\ &\times\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda-1}\theta''(\rho)d\rho\end{aligned}$$

And since,

$$\lim_{a\to\ell(t)}\left|\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda}\theta''(\rho)d\rho\right| \le \lim_{a\to\ell(t)}\left(ta^{-\frac{2}{\lambda}}-\zeta_0\right)^{\lambda}\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left|\theta''(\rho)\right|d\rho = 0$$

By applying this convergence, we get

$$\left(\frac{\lambda}{2}\right)^2\zeta_1^{\frac{2}{\lambda}}\Gamma(\lambda) = \lim_{a\to\ell(t)}\left(\int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}}\left(ta^{-\frac{2}{\lambda}}-\rho\right)^{\lambda-1}\theta''(\rho)d\rho + \left(ta^{-\frac{2}{\lambda}}-\zeta_0\right)^{\lambda-1}\zeta_0\theta'\left(\zeta_0\right)\right)$$

where $\zeta_1^{\frac{2}{\lambda}} = \zeta_0^{-1}$, then

$$\left(\frac{\lambda}{2}\right)^2 \zeta_0^{-2}\Gamma(\lambda) = \lim_{a\to\ell(t)} \left( \int_{\zeta_0}^{ta^{-\frac{2}{\lambda}}} \left(ta^{-\frac{2}{\lambda}} - \rho\right)^{\lambda-1} \theta''(\rho)d\rho + \left(ta^{-\frac{2}{\lambda}} - \zeta_0\right)^{\lambda-1} \theta'(\zeta_0) \right)$$

It implies that

$$\left(\frac{\lambda}{2}\right)^2 \zeta_0^{-2}\Gamma(\lambda) = \lim_{b\to\zeta_0} \frac{d}{db} \left( \int_{\zeta_0}^{b} (b-\rho)^{\lambda-1}\theta'(\rho)d\rho \right) \tag{3.16}$$

Hence, we obtain Eq. (3.8). In order to demonstrate Eq. (3.9), we observe that from Eq. (3.6), and because $\theta'$ is AC on some neighborhood of $\zeta_0$ as $\left(\zeta_0, h^{\frac{2}{\lambda}}\zeta_0\right)$, thus, by taking the limit at $\zeta \to \zeta_0$, we obtain

$$\lim_{\zeta\to\zeta_0} (\zeta-\zeta_0)^{\lambda}\,\theta''(\zeta) = \left(\frac{\lambda}{2}\right)^2 \left(\frac{\zeta_0^{-2}}{\Gamma(1-\lambda)}\right) \tag{3.17}$$

From Eq. (3.17), we prove Eq.(3.9). Since

$$\frac{d}{db}\int_{\zeta_0}^{b} (b-\rho)^{\lambda-1}\theta'(\rho)d\rho = \Gamma(\lambda)\partial_{\zeta_0}^{1-\lambda}\theta'(b)$$

fix $\varepsilon > 0$ and $\zeta > \zeta_0$. It follows from Eq. (3.8) that there is some neighborhood of $\zeta_0$ so that

$$\left(\left(\frac{\lambda}{2\zeta_0}\right)^2 - \varepsilon\right) \le \partial_{\zeta_0}^{1-\lambda}\theta'(\zeta) \le \left(\left(\frac{\lambda}{2\zeta_0}\right)^2 + \varepsilon\right).$$

Since $\theta'$ is absolutely continuous, $I_{\zeta_0}^{1-\lambda}\partial_{\zeta_0}^{1-\lambda}\theta' = \theta'$. Using $I_{\zeta_0}^{1-\lambda}$ to the above inequalities, then

$$I_{\zeta_0}^{1-\lambda}\left(\left(\frac{\lambda}{2\zeta_0}\right)^2 - \varepsilon\right) \le I_{\zeta_0}^{1-\lambda}\partial_{\zeta_0}^{1-\lambda}\theta'(\zeta) \le I_{\zeta_0}^{1-\lambda}\left[\left(\frac{\lambda}{2\zeta_0}\right)^2 + \varepsilon\right]$$

where

$$\left(\left(\frac{\lambda}{2\zeta_0}\right)^2 - \varepsilon\right)\frac{(\zeta-\zeta_0)^{1-\lambda}}{\Gamma(2-\lambda)} \le \theta'(\zeta) \le \left[\left(\frac{\lambda}{2\zeta_0}\right)^2 + \varepsilon\right]\frac{(\zeta-\zeta_0)^{1-\lambda}}{\Gamma(2-\lambda)}.$$

Thus, for each $\zeta > \zeta_0$,

$$\left[\left(\frac{\lambda}{2\zeta_0}\right)^2 - \varepsilon\right] \le (\zeta-\zeta_0)^{\lambda-1}\,\Gamma(2-\lambda)\theta'(\zeta) \le \left[\left(\frac{\lambda}{2\zeta_0}\right)^2 + \varepsilon\right].$$

The last two inequalities are equivalent to

$$\lim_{\zeta\to\zeta_0} \frac{\theta'(\zeta)}{(\zeta-\zeta_0)^{1-\lambda}} = \left(\frac{\lambda}{2}\right)^2 \frac{\zeta_0^{-2}}{\Gamma(2-\lambda)}.$$

Thus, in particular $\theta'(\zeta_0) = 0$, then the proof of Theorem 3.1, is done. □

## 4. Applied Results and Discussion

Let consider the problem from theorem 3.2 as follows

$$L\theta = \frac{1}{\Gamma(1-\lambda)}\int_0^{\zeta}\frac{\theta'(\rho)}{(\zeta-\rho)^{\lambda}}d\rho - \left(\frac{2}{\lambda}\right)^2\zeta^2\theta''(\zeta) - \frac{2(\lambda+2)}{\lambda^2}\zeta\theta'(\zeta) + \frac{\zeta^{-\lambda}}{\Gamma(1-\lambda)} = 0 \tag{4.1}$$

$$\theta(0) = 0 \tag{4.2}$$

$$\theta'(0) = 0 \tag{4.3}$$

which has approximated solution $\theta(\zeta) = \zeta^2 e^{-\zeta}, \zeta > 0$ for various values of the fractional order parameter $0 < \lambda < 1$. The equation is evaluated numerically for the following values of $\lambda$: $\lambda = 0.4, 0.6, 0.8, 1.0$. $L\theta$ was computed and plotted over the interval $x \in [0.1, 5]$. The following observations were made for $\lambda = 0.4$ acceptably small error, indicating that the function $\theta$ approximately satisfies (4.1),where $L\theta = 0$. Similarly, $\lambda = 0.6$ suggesting a closer match to (4.1) and so $\lambda = 0.8$ to make very small residual error, see the Figure 2.

- For smaller $\lambda$ values ($\lambda = 0.4$), the residual is larger due to stronger non-local (memory) effects in the fractional integral. This increases deviation from classical behavior.
- As $\lambda$ approaches 1 from below, the fractional integral behaves more like a classical derivative. At $\lambda = 0.8$, the residuals are minimal, indicating that $\theta(\zeta)$ closely satisfies (4.1).

Therefore, the function $\theta(\zeta) = \zeta^2 e^{-\zeta}$ approximately satisfies the problem (4.1)-(2.23) for values $\lambda \in (0, 1)$. The accuracy of this approximation improves as $\lambda$ increases toward 1. Now, from above the solution of (3.1) with the initial

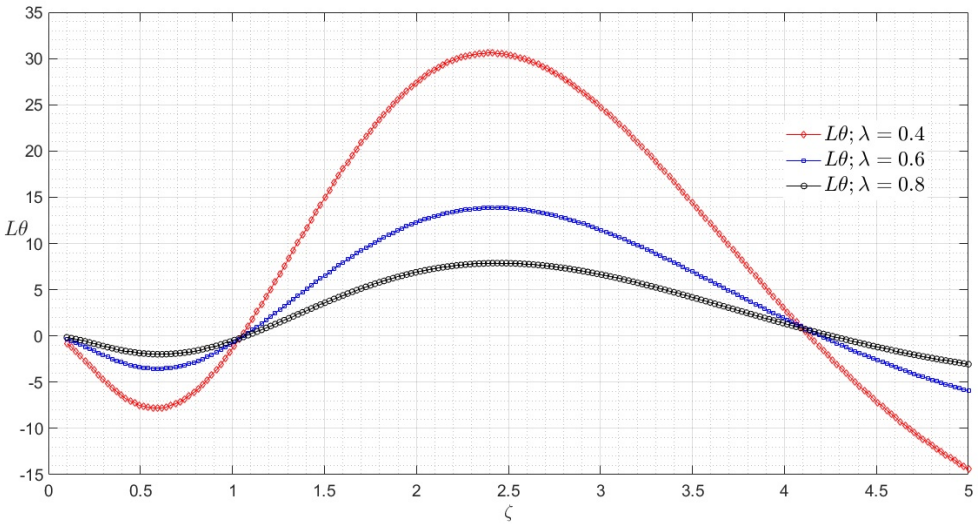



Figure 2. The shape function $\theta(\zeta)$ satisfies (4.1) depending on the values of $\lambda$ $(0.4, 0.6, 0.8)$.

condition $\omega(0,t)=0$ is asymptotically equal to solution of (4.1)-(4.3), then

$$\omega(x,t)=t^2x^{-\frac{4}{\lambda}}e^{-tx^{-\frac{2}{\lambda}}},\ t>0,\ x>0$$

From Figure 3, when the function is defined as $\omega(x,t)$ its behavior changes significantly compared to the case with a positive exponent. The negative power of $x$ introduces a singularity as $x\to 0^+$, causing the function to grow rapidly and potentially diverge near the origin. This effect becomes more pronounced as the parameter $\lambda$ decreases, since the exponent $-2/\lambda$ increases in magnitude.

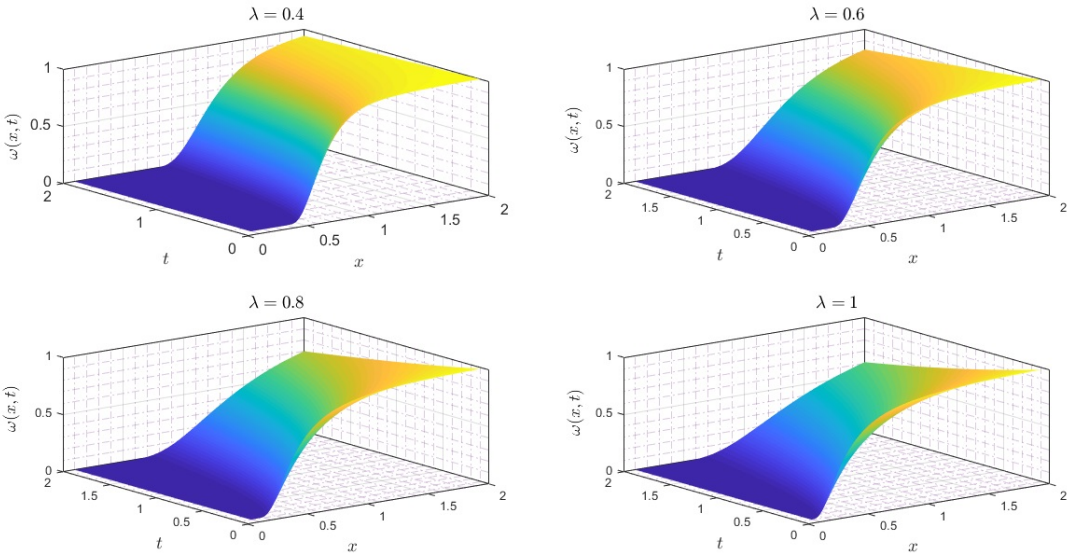



FIGURE 3. The solution $\omega(x,t)$ of the Fractional Stefan problem (3.1)-(3.3) changed w.r.t. values of $\lambda(0.4,0.6,0.8)$.

Consequently, for smaller values of $\lambda$, such as 0.7 or 0.8, the function exhibits a sharp and steep peak near small $x$, while for larger $\lambda$ values, like 1, the growth remains significant but more moderate. As $x$ increases, the exponent $x^{-2/\lambda}$ diminishes, and the function smoothly decays toward unity, creating a surface that flattens along the $x$-axis. This contrasting behavior—rapid blow-up at small $x$ and flattening at larger $x$—results in a pronounced ridge or spike in the 3D surface plots, particularly evident when $\lambda$ is small. This sensitivity to small values of $x$ highlights the critical role of $\lambda$ in controlling the spatial scale and steepness of the function, and it suggests a strong localization effect around the origin in models where such functions arise. The interface function $\ell(t)=x_0t^{\frac{\lambda}{2}}$, defined on the interval $0\le t<2$, describes a power-law growth influenced by both the initial position parameter $x_0$ and the fractional exponent $\frac{\lambda}{2}$. When considering values of $\lambda=0.4,0.6,0.8$, and 1, the corresponding exponents $\frac{\lambda}{2}=0.2,0.3,0.4$, and 0.5 reflect increasingly faster but still sublinear growth behaviors. For smaller $\lambda$, such as $\lambda=0.4$, the interface grows very slowly, especially near $t=0$. As $\lambda$ increases, the interface becomes more responsive to changes in time, showing a more noticeable rise. The parameter $x_0$ acts as a linear scaling factor, modifying

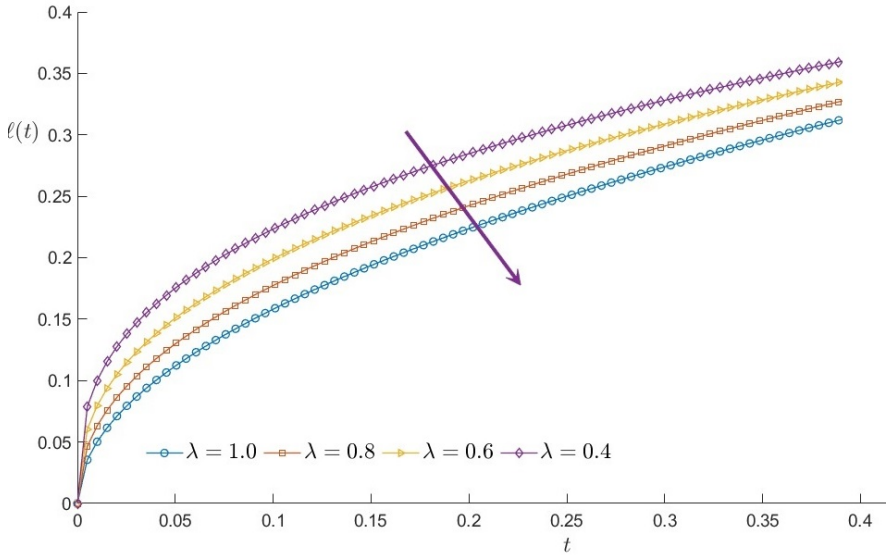



FIGURE 4. The interface function $\ell(t)$ varies in shape depending on the value of the parameter $\lambda$.

the overall magnitude of the interface function without altering its shape; larger values of $x_0$ produce proportionally higher interface positions.

## 5. Conclusion

This work presents a comprehensive analysis of a time-fractional Stefan problem with an advective-diffusive flux, formulated through the fractional Diffusion–Convection–Reaction equation and applied to a realistic ice-melting scenario. By employing a rescaling technique, the governing fractional PDE was reduced to a self-similar form, enabling the derivation of approximate solutions under both Riemann–Liouville and Caputo formulations. The results confirm that the proposed interface function satisfies the necessary boundary and Stefan conditions, accurately representing the evolution of the moving boundary in fractional phase-change processes. Overall, the study advances the theoretical understanding of fractional-order thermal models and provides a robust framework for analyzing anomalous diffusion and memory effects in engineering and applied sciences.

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