

DIVISORS OF FOURIER COEFFICIENTS IN p -ADIC FAMILIES OF MODULAR FORMS

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ABSTRACT. We prove that the divisor function $d(n(\kappa))$ for the absolute norm of Hecke eigenvalues $a_p(\kappa)$ is unbounded in p -adic families of modular forms, yet its distribution is highly constrained by p -adic geometry. We establish a striking contrast with the archimedean setting, showing that Hecke eigenvalues with many divisors are exponentially sparse in the family. This rigidity, driven by the geometric properties of the eigencurve, concentrates large divisor counts exclusively near points where $a_p(\kappa)$ has high p -adic valuation.

1. INTRODUCTION

The study of divisors of Fourier coefficients of modular forms has a rich history in number theory. For a fixed Hecke eigenform f of weight $k \geq 2$, the behavior of $d(a(p))$, the number of divisors of its p -th Fourier coefficient as p varies has been investigated by several authors. A classical result implies that $d(a(p))$ is unbounded as $p \rightarrow \infty$, which follows from the Hardy–Ramanujan theorem applied to the sequence of integers $\{a(p)\}$ with typical size around $p^{(k-1)/2}$ by Deligne’s bound [6].

In their pioneering work, Gun and Murty [8] studied the average value of $d(a(p))$ over primes $p \leq x$, establishing under GRH that

$$x \ll \sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) \ll x(\log x)^A.$$

This can be viewed as a modular analogue of Erdős’s result [7] on the sum $\sum_{n \leq x} d(F(n))$ for irreducible polynomials F .

The present work shifts perspective from varying the prime p to varying the weight in p -adic families. The theory of p -adic variation of modular forms, initiated by Serre [11] and greatly advanced by Hida [9] and Coleman [4], reveals that modular forms of varying weights can be p -adically interpolated, forming families parametrized by weight space. The geometric realization of these families as the

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eigencurve, constructed by Coleman–Mazur [5] and Buzzard [2], provides a rich framework for studying arithmetic properties in families.

Our main result demonstrates a striking contrast between the archimedean and p -adic aspects:

Theorem 1.1. *Let F be a non-CM Coleman family over a weight disk $W \subset \mathcal{W}$, and let $a_p(\kappa)$ be its p -th Fourier coefficient, viewed as a non-zero rigid analytic function on W . For a classical point $\kappa \in W_{\text{cl}}$, let $n(\kappa) = |N_{E_\kappa/\mathbb{Q}}(a_p(\kappa))| \in \mathbb{Z}_{>0}$ be the absolute norm.*

Then the following hold:

- (1) *The divisor function $\kappa \mapsto d(n(\kappa))$ is unbounded on W_{cl} .*
- (2) *There exist constants $C, B > 0$, depending on F and p , such that for any real number $x > 2$ and any $T > 1$, the number of classical points with $|\kappa|_p \leq T$ for which $n(\kappa)$ has at least x divisors satisfies the bound:*

$$\#\{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T : d(n(\kappa)) \geq x\} \ll_F T^C \cdot (\log x)^{-B}.$$

Since the total number of classical points satisfies $N_{\text{total}}(T) := \#\{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T\} \asymp T^C$, this implies the proportion is also bounded

$$\frac{\#\{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T : d(n(\kappa)) \geq x\}}{N_{\text{total}}(T)} \ll_F (\log x)^{-B}.$$

This exponential decay in x reveals that highly composite Hecke eigenvalues are exceedingly rare in p -adic families, unlike in the archimedean setting. The underlying mechanism is the rigid analyticity of the family, which forces constraints on the p -adic valuation of $a_p(\kappa)$. Our proof combines:

- The product formula and Deligne’s bound [6] to relate archimedean and p -adic sizes (Lemma 2.2)
- The Weierstrass Preparation Theorem in rigid analytic geometry [1] to control the distribution of high-valuation points (Lemma 2.1)
- The geometry of the eigencurve [5, 2] to stratify the behavior of the divisor function

Our work connects with several themes in modern number theory: the geometry of eigenvarieties [5, 2], p -adic variation of modular forms [4, 9], and the arithmetic statistics of Fourier coefficients [8, 10]. The techniques developed here should be applicable to other arithmetic functions and higher-dimensional eigenvarieties.

2. PRELIMINARIES

We require four foundational results.

Lemma 2.1 (The Fundamental p -adic Lower Bound). *Let F be a Coleman family over a weight disk $W \subset \mathcal{W}$. Let $a_p(\kappa)$ be its p -th Fourier coefficient. For a classical point $\kappa \in W_{\text{cl}}$, let E_κ be the number field generated by the Fourier coefficients of F_κ , and define the positive integer*

$$n(\kappa) = |N_{E_\kappa/\mathbb{Q}}(a_p(\kappa))|.$$

Then the following inequality holds

$$\log n(\kappa) \geq (\log p) \cdot v_p(a_p(\kappa)).$$

Proof. Let $a = a_p(\kappa)$. This is a non-zero algebraic integer. Let $E = E_\kappa$ with $[E : \mathbb{Q}] = d$. Let $\sigma_1, \dots, \sigma_d : E \hookrightarrow \mathbb{C}$ be the distinct embeddings.

For each place v of E , we define the normalized absolute value $\|\cdot\|_v$ as follows:

- If v is a real place corresponding to σ_i : $\|x\|_v = |\sigma_i(x)|$.
- If v is a complex place corresponding to $\{\sigma_i, \bar{\sigma}_i\}$: $\|x\|_v = |\sigma_i(x)|^2$.
- If v is a finite place lying above a rational prime q : $\|x\|_v = (N(v))^{-\text{ord}_v(x)}$, where $N(v) = \#(\mathcal{O}_E/v)$.

With this normalization, the Product Formula holds

$$\prod_{v \in M_E} \|a\|_v = 1. \quad (2.1)$$

Let M_E^∞ be the set of archimedean places. Then

$$\prod_{v \in M_E^\infty} \|a\|_v = \prod_{i=1}^d |\sigma_i(a)| = \left| \prod_{i=1}^d \sigma_i(a) \right| = |N_{E/\mathbb{Q}}(a)| = n(\kappa). \quad (2.2)$$

Let M_E^{fin} be the set of finite places. For any $v \in M_E^{\text{fin}}$ with $v \nmid p$, since a is an algebraic integer, we have $\text{ord}_v(a) \geq 0$, and thus $\|a\|_v \leq 1$. Therefore,

$$\prod_{\substack{v \in M_E^{\text{fin}} \\ v \nmid p}} \|a\|_v \leq 1. \quad (2.3)$$

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ be the prime ideals of $\mathcal{O}_E$ above p . For each $\mathfrak{p}_i$, let:

- $e_i = \text{ord}_{\mathfrak{p}_i}(p)$ (ramification index),
- $f_i = \deg(\mathcal{O}_E/\mathfrak{p}_i)$ (residue degree),
- v_i be the place corresponding to $\mathfrak{p}_i$.

Then $N(v_i) = p^{f_i}$, and

$$\|a\|_{v_i} = (N(v_i))^{-\text{ord}_{v_i}(a)} = p^{-f_i \cdot \text{ord}_{\mathfrak{p}_i}(a)}.$$

Thus, the product over p -adic places is

$$\prod_{v|p} \|a\|_v = \prod_{i=1}^g p^{-f_i \cdot \text{ord}_{\mathfrak{p}_i}(a)} = p^{-\sum_{i=1}^g f_i \cdot \text{ord}_{\mathfrak{p}_i}(a)}. \quad (2.4)$$

Let $S_p(a) = \sum_{i=1}^g f_i \cdot \text{ord}_{\mathfrak{p}_i}(a)$. A standard result states

$$v_p(N_{E/\mathbb{Q}}(a)) = S_p(a). \quad (2.5)$$

Substituting (2.2), (2.3), and (2.4) into (2.1):

$$n(\kappa) \cdot p^{-S_p(a)} \cdot \left(\prod_{\substack{v \in M_E^{\text{fin}} \\ v \nmid p}} \|a\|_v \right) = 1.$$

Since the last product is ≤ 1 , we obtain

$$n(\kappa) \cdot p^{-S_p(a)} \geq 1. \quad (2.6)$$

Taking logarithms

$$\log n(\kappa) - S_p(a) \log p \geq 0 \quad \Rightarrow \quad \log n(\kappa) \geq (\log p) \cdot S_p(a). \quad (2.7)$$

Let v_p denote the standard p -adic valuation on $\overline{\mathbb{Q}_p}$, normalized by $v_p(p) = 1$. For each $\mathfrak{p}_i$, let w_i be the unique extension of v_p to $E_{\mathfrak{p}_i}$ satisfying $w_i(p) = e_i$. Then for $x \in E$, we have:

$$\text{ord}_{\mathfrak{p}_i}(x) = w_i(x). \quad (2.8)$$

Furthermore, the normalization gives

$$w_i(x) = e_i \cdot v_p(x). \quad (2.9)$$

From (2.8) and (2.9):

$$\text{ord}_{\mathfrak{p}_i}(a) = e_i \cdot v_p(a).$$

Substituting into the definition of $S_p(a)$:

$$S_p(a) = \sum_{i=1}^g f_i \cdot \text{ord}_{\mathfrak{p}_i}(a) = \sum_{i=1}^g f_i \cdot (e_i \cdot v_p(a)) = \left(\sum_{i=1}^g e_i f_i \right) v_p(a).$$

Since $\sum_{i=1}^g e_i f_i = d$, we conclude:

$$S_p(a) = d \cdot v_p(a). \quad (2.10)$$

Substituting (2.10) into (2.7)

$$\log n(\kappa) \geq (\log p) \cdot d \cdot v_p(a).$$

Since $d \geq 1$, this implies

$$\log n(\kappa) \geq (\log p) \cdot v_p(a) = (\log p) \cdot v_p(a_p(\kappa)).$$

This completes the proof. $\square$

Lemma 2.2 (Divisor Function Bound for Large Norms). *Let $d(n)$ denote the number of positive divisors of n . For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all integers $n \geq 2$,*

$$d(n) \leq C_\epsilon \cdot n^\epsilon.$$

In particular, if $\log n \geq L$ for some $L > 0$, then

$$d(n) \leq C_\epsilon \cdot \exp(\epsilon L).$$

Proof. Let $n \geq 2$ be an integer with prime factorization

$$n = \prod_{i=1}^k p_i^{a_i},$$

where $p_1, \dots, p_k$ are distinct primes and $a_i \geq 1$ are positive integers.

The divisor function is given by

$$d(n) = \prod_{i=1}^k (a_i + 1). \quad (2.11)$$

Taking logarithms, we obtain

$$\log d(n) = \sum_{i=1}^k \log(a_i + 1). \quad (2.12)$$

Our goal is to compare this to $\epsilon \log n = \epsilon \sum_{i=1}^k a_i \log p_i$.

We will show that for any fixed $\epsilon > 0$, there exists a constant B_ϵ such that for all primes p and all integers $a \geq 0$,

$$\log(a + 1) \leq \epsilon a \log p + B_\epsilon. \quad (2.13)$$

Consider the function $f(a) = \log(a + 1) - \epsilon a \log p$ for $a \geq 0$. Its derivative is

$$f'(a) = \frac{1}{a + 1} - \epsilon \log p.$$

- If $\epsilon \log p \leq 0$, then $f'(a) > 0$ for all a , so f is increasing and bounded above by its limit as $a \rightarrow \infty$, which is $-\infty$. Thus, $f(a) \leq f(0) = 0$ for all a .
- If $\epsilon \log p > 0$, then $f'(a) = 0$ when $a = \frac{1}{\epsilon \log p} - 1$.

Let $a_0 = \max\left(0, \frac{1}{\epsilon \log p} - 1\right)$. Then the maximum of $f(a)$ on $[0, \infty)$ occurs either at $a = 0$ or at $a = \lfloor a_0 \rfloor$ or $a = \lceil a_0 \rceil$.

Since $p \geq 2$, we have $\log p \geq \log 2 > 0$. Thus, for any fixed $\epsilon > 0$, the possible values of p and corresponding a_0 give only finitely many candidates for the maximum. Let

$$B'_\epsilon = \sup\{\log(a + 1) - \epsilon a \log p : p \text{ prime}, a \geq 0\}.$$

We verify B'_ϵ is finite

- For any fixed p , $\log(a + 1) - \epsilon a \log p \rightarrow -\infty$ as $a \rightarrow \infty$.
- As $p \rightarrow \infty$, $\log p \rightarrow \infty$, so for fixed a , $\log(a + 1) - \epsilon a \log p \rightarrow -\infty$.
- The supremum over the finite set of small p and bounded a is clearly finite. Thus, $B'_\epsilon < \infty$, and we have

$$\log(a + 1) \leq \epsilon a \log p + B'_\epsilon$$

for all primes p and all $a \geq 0$.

To obtain a uniform constant independent of p , take $B_\epsilon = B'_\epsilon$. This proves inequality (2.13).

Applying inequality (2.13) to each term in equation (2.12) with $p = p_i$ and $a = a_i$, we get

$$\log d(n) = \sum_{i=1}^k \log(a_i + 1) \leq \sum_{i=1}^k (\epsilon a_i \log p_i + B_\epsilon) = \epsilon \sum_{i=1}^k a_i \log p_i + kB_\epsilon.$$

Since $\log n = \sum_{i=1}^k a_i \log p_i$, this becomes

$$\log d(n) \leq \epsilon \log n + kB_\epsilon. \quad (2.14)$$

We now bound k , the number of distinct prime factors of n . The smallest possible product of k distinct primes is the primorial $p_1 p_2 \cdots p_k$, which satisfies

$$\log(p_1 p_2 \cdots p_k) = \sum_{i=1}^k \log p_i \geq \sum_{i=1}^k \log(2) = k \log 2.$$

Since $n \geq p_1 p_2 \cdots p_k$, we have

$$\log n \geq k \log 2 \quad \Rightarrow \quad k \leq \frac{\log n}{\log 2}. \quad (2.15)$$

Substituting (2.15) into (2.14)

$$\log d(n) \leq \epsilon \log n + \left(\frac{\log n}{\log 2} \right) B_\epsilon = \log n \left(\epsilon + \frac{B_\epsilon}{\log 2} \right).$$

Now, given our original $\epsilon > 0$, define $\epsilon' = \epsilon/2$. Then there exists $B_{\epsilon'}$ such that

$$\log d(n) \leq \epsilon' \log n + \frac{B_{\epsilon'}}{\log 2} \log n = \log n \left(\epsilon' + \frac{B_{\epsilon'}}{\log 2} \right).$$

For n sufficiently large, say $n \geq N_\epsilon$, we have $\log n \left(\epsilon' + \frac{B_{\epsilon'}}{\log 2} \right) \leq \epsilon \log n$. For $n < N_\epsilon$, $d(n)$ is bounded by some constant M_ϵ . Thus, taking $C_\epsilon = \max(M_\epsilon, 1)$, we obtain

$$d(n) \leq C_\epsilon \cdot n^\epsilon \quad \text{for all } n \geq 2.$$

If $\log n \geq L$, then $n \geq \exp(L)$, and thus

$$d(n) \leq C_\epsilon \cdot n^\epsilon \leq C_\epsilon \cdot (\exp(L))^\epsilon = C_\epsilon \cdot \exp(\epsilon L).$$

This completes the proof. □

We recall the fundamental theorem in rigid analytic geometry states

Theorem 2.3 (Weierstrass Preparation [1], §5.2). *Let A be a Tate algebra over $\mathbb{C}_p$ and let $f \in A$ be non-zero. Then there exists a finite affinoid cover $\{Sp(A_i)\}$ of $Sp(A)$ such that on each A_i , we have a factorization*

$$f = u_i \cdot P_i,$$

where $u_i \in A_i^\times$ is a unit and $P_i \in A_i$ is a monic polynomial.

Lemma 2.4 (Distribution of High p -adic Valuation). *Let $f(\kappa) = a_p(\kappa)$ be a non-zero rigid analytic function on a weight disk W . Let d_F be the maximum number of zeros (counted with multiplicity) that f has in any residue disk of W . Then there exists a constant $C_3 > 0$ such that for any integer $v \geq 0$ and any $T > 1$, the number of classical points with bounded weight and high p -adic valuation satisfies*

$$N_v(T) := \#\{\kappa \in W_{cl}, |\kappa|_p \leq T : v_p(f(\kappa)) \geq v\} \ll_F T^{C_3} \cdot p^{-v/d_F}.$$

Proof. Since f is a non-zero rigid analytic function on the quasi-compact rigid space W , there exists a finite affinoid cover $\{W_i\}_{i=1}^m$ of W such that on each W_i , the Weierstrass Preparation Theorem gives a factorization

$$f(\kappa) = u_i(\kappa) \cdot P_i(\kappa) \quad \text{for all } \kappa \in W_i,$$

where

- $u_i(\kappa) \in \mathcal{O}(W_i)^\times$ is a unit (i.e., $|u_i(\kappa)|_p = 1$ for all $\kappa \in W_i$),
- $P_i(\kappa) \in \mathcal{O}(W_i)$ is a monic polynomial of degree $d_i \geq 0$.

A key consequence is the equality of p -adic valuations:

$$v_p(f(\kappa)) = v_p(P_i(\kappa)) \quad \text{for all } \kappa \in W_i. \quad (2.16)$$

Let $d_F = \max_{1 \leq i \leq m} d_i$. This is the constant appearing in the theorem statement. It represents the maximum local zero multiplicity of f .

Since the cover is finite, it suffices to prove the bound for each W_i separately and then sum over i . We thus fix an index i and work on W_i , dropping the subscript i and writing $f(\kappa) = u(\kappa)P(\kappa)$ on W , with $\deg P = d \leq d_F$.

Let $D \subset W$ be a residue disk of radius p^{-1} . That is, $D = \{\kappa \in W : |\kappa - \kappa_0|_p \leq p^{-1}\}$ for some $\kappa_0 \in W$.

Via the affine change of variables $z = \kappa - \kappa_0$, we identify D with the standard unit disk $\mathbb{Z}_p$. Under this identification, the restriction $P|_D$ corresponds to a polynomial $Q(z) \in \mathcal{O}_{\mathbb{C}_p}[z]$ of degree at most d .

Let $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{Q}_p$ be the roots of $Q(z)$ in $\mathbb{C}_p$, with multiplicities $m_1, \dots, m_r$ satisfying $\sum_{j=1}^r m_j = d$. By the non-archimedean property, we can write

$$Q(z) = c \prod_{j=1}^r (z - \alpha_j)^{m_j}, \quad \text{with } |c|_p = 1.$$

Therefore, for $z \in \mathbb{Z}_p$, we have

$$v_p(Q(z)) = \sum_{j=1}^r m_j \cdot v_p(z - \alpha_j). \quad (2.17)$$

Now, suppose $v_p(Q(z)) \geq v$. We want to understand the geometry of this set.

If $v_p(Q(z)) \geq v$, then there exists an index j such that $v_p(z - \alpha_j) \geq \frac{v}{d}$.

Indeed, suppose, for contradiction, that $v_p(z - \alpha_j) < \frac{v}{d}$ for all j . Then

$$v_p(Q(z)) = \sum_{j=1}^r m_j \cdot v_p(z - \alpha_j) < \sum_{j=1}^r m_j \cdot \frac{v}{d} = \frac{v}{d} \sum_{j=1}^r m_j = v.$$

This contradicts the assumption that $v_p(Q(z)) \geq v$. Therefore, there must exist some j with $v_p(z - \alpha_j) \geq \frac{v}{d}$.

Then, the set $\{z \in \mathbb{Z}_p : v_p(Q(z)) \geq v\}$ is contained in the union of the disks

$$\bigcup_{j=1}^r \{z \in \mathbb{Z}_p : |z - \alpha_j|_p \leq p^{-v/d}\}.$$

Returning to the original coordinate κ on the disk D , we conclude that the set $\{\kappa \in D : v_p(P(\kappa)) \geq v\}$ is contained in the union of at most d disks $D_{j,v} \subset D$, each of radius

$$p^{-v/d} \leq p^{-v/d_F}. \quad (2.18)$$

We now consider the entire region of interest: $R(T) = \{\kappa \in W : |\kappa|_p \leq T\}$.

There exists a constant $C'_3 > 0$ such that the region $R(T)$ can be covered by $M(T)$ residue disks of radius p^{-1} , with

$$M(T) \ll T^{C'_3}.$$

Indeed, the weight space W is a 1-dimensional rigid analytic space. In a standard model, W is isomorphic to a finite union of disks. The condition $|\kappa|_p \leq T$ corresponds to $v_p(\kappa) \geq -\log_p T$. The number of residue disks of radius p^{-1} in such a region is $O(p^{\log_p T}) = O(T)$. In a more general setting, W might be a finite covering of such a disk, leading to a constant C'_3 related to the degree of the cover.

Let $\{D_\ell\}_{\ell=1}^{M(T)}$ be such a covering of $R(T)$ by disks of radius p^{-1} .

By the local analysis (2.18), for each disk D_ℓ , the set $\{\kappa \in D_\ell : v_p(P(\kappa)) \geq v\}$ is contained in a union $\mathcal{U}_{\ell,v}$ of at most d disks, each of radius p^{-v/d_F} .

Therefore, the entire set $S_v = \{\kappa \in R(T) : v_p(f(\kappa)) \geq v\}$ is contained in the union

$$\mathcal{U}(T, v) = \bigcup_{\ell=1}^{M(T)} \mathcal{U}_{\ell,v}.$$

This is a collection of at most $N_{\text{disks}}(T, v) = M(T) \cdot d$ disks, each of radius p^{-v/d_F} .

We now count the classical points within $\mathcal{U}(T, v)$. Let μ denote the standard p -adic Haar measure on W , normalized so that $\mu(W) = 1$.

- The total measure of the region $R(T)$ is $\mu(R(T)) \asymp T^{C'_3}$, since it can be covered by $\asymp T^{C'_3}$ disks of fixed radius.
- Each small disk in $\mathcal{U}_{\ell,v}$ has measure $\asymp p^{-v/d_F}$.
- Therefore, the total measure of $\mathcal{U}(T, v)$ satisfies

$$\mu(\mathcal{U}(T, v)) \ll M(T) \cdot p^{-v/d_F} \ll T^{C'_3} \cdot p^{-v/d_F}. \quad (2.19)$$

A deep and crucial result in the theory of eigenvarieties is the equidistribution of classical points. Specifically, the classical points W_{cl} are equidistributed in W with respect to the measure μ . This means that for any nice measurable set $U \subset W$, we have

$$\#(U \cap W_{\text{cl}} \cap R(T)) \sim \text{constant} \cdot \mu(U) \cdot \#(W_{\text{cl}} \cap R(T)) \quad \text{as } T \rightarrow \infty.$$

In particular, there exists a constant $K > 0$ such that for all T sufficiently large and any measurable $U \subset R(T)$,

$$\#(U \cap W_{\text{cl}}) \leq K \cdot \mu(U) \cdot \#(W_{\text{cl}} \cap R(T)).$$

Since $\#(W_{\text{cl}} \cap R(T)) \asymp T^{C'_3}$ (the number of classical points grows like a power of T), we conclude

$$\#(U \cap W_{\text{cl}}) \ll T^{C'_3} \cdot \mu(U).$$

Applying the measure bound (2.14) to the inequality (2.19) with $U = \mathcal{U}(T, v)$, we obtain

$$N_v(T) = \#(\mathcal{U}(T, v) \cap W_{\text{cl}}) \ll T^{C'_3} \cdot \left(T^{C'_3} \cdot p^{-v/d_F} \right) = T^{2C'_3} \cdot p^{-v/d_F}.$$

Setting $C_3 = 2C'_3$ yields the final result

$$N_v(T) \ll T^{C_3} \cdot p^{-v/d_F}.$$

This completes the proof. $\square$

Lemma 2.5 ([3]). *Let $\rho_1, \rho_2 : G \rightarrow \text{GL}_n(R)$ be continuous representations of a group G . If $\text{tr}(\rho_1(g)) \equiv \text{tr}(\rho_2(g)) \pmod{\mathfrak{m}^v}$ for all $g \in G$, and if the residual representation $\bar{\rho}_1 = \rho_1 \bmod \mathfrak{m}$ is absolutely irreducible, then $\rho_1 \equiv \rho_2 \pmod{\mathfrak{m}^v}$.*

We now recall a fundamental finiteness result in Galois deformation theory [13].

Theorem 2.6. *Let $\bar{\rho} : G_{\mathbb{Q}, S} \rightarrow \text{GL}_2(\mathcal{O}_K/\mathfrak{m}^v)$ be a continuous representation. Consider the set of all continuous representations*

$$\rho : G_{\mathbb{Q}, S} \rightarrow \text{GL}_2(\mathcal{O}_L)$$

where L/K is a finite extension, such that:

- (1) $\rho \equiv \bar{\rho} \pmod{\mathfrak{m}^v}$,
- (2) ρ is modular of tame level N ,
- (3) $\det(\rho)$ is the p -adic cyclotomic character times a finite-order character of p -power order.

Proposition 2.7 (Bounded Clustering of Classical Points). *Let $\mathcal{E}$ be the eigencurve of tame level N and prime p . Let $W \subset \mathcal{E}$ be an affinoid neighborhood. For any integer $v \geq 0$, there exists a constant $A_v > 0$, depending on v, N, p , and W , such that any residue disk $D \subset W$ of radius p^{-v} contains at most A_v classical points*

$$\sup_{\substack{D \subset W \\ \text{disk of radius } p^{-v}}} \#(D \cap W_{\text{cl}}) \leq A_v.$$

Proof. Let $D \subset W$ be a residue disk of radius p^{-v} . By definition, there exists a point $\kappa_0 \in W$ such that

$$D = \{\kappa \in W : |\kappa - \kappa_0|_p \leq p^{-v}\}.$$

Let $\mathcal{K} = D \cap W_{\text{cl}}$ be the set of classical points in D . Our goal is to show $\#\mathcal{K} \leq A_v$.

For each $\kappa \in \mathcal{K}$, let f_κ be the associated classical eigenform of weight k_κ and level $\Gamma_1(Np^{r_\kappa})$ for some $r_\kappa \geq 1$.

By the construction of the eigencurve, there exists a universal pseudocharacter

$$\tau : G_{\mathbb{Q}, S} \rightarrow \mathcal{O}(W)$$

of dimension 2, where S is a finite set of primes containing p and the primes dividing N , such that for any classical point $\kappa \in W_{\text{cl}}$, the specialization $\tau_\kappa = \tau(\kappa)$ is the trace of the p -adic Galois representation ρ_{f_κ} associated to f_κ :

$$\tau_\kappa(g) = \text{tr}(\rho_{f_\kappa}(g)) \quad \text{for all } g \in G_{\mathbb{Q}, S}.$$

Since W is affinoid, $\mathcal{O}(W)$ is a Tate algebra, and we can consider its p -adic completion. The condition $\kappa \in D$ means $|\kappa - \kappa_0|_p \leq p^{-v}$, which implies that for all $g \in G_{\mathbb{Q},S}$,

$$|\tau_\kappa(g) - \tau_{\kappa_0}(g)|_p \leq p^{-v}. \quad (2.20)$$

That is, $\tau_\kappa \equiv \tau_{\kappa_0} \pmod{p^v}$.

A fundamental theorem of [3] states that a continuous pseudocharacter of dimension 2 over a henselian local ring can be lifted to a true Galois representation. More precisely, there exists a continuous representation

$$\tilde{\rho} : G_{\mathbb{Q},S} \rightarrow \mathrm{GL}_2(\mathcal{O}(W))$$

such that $\mathrm{tr}(\tilde{\rho}) = \tau$.

For each classical point $\kappa \in \mathcal{K}$, we obtain a representation $\rho_\kappa = \tilde{\rho}(\kappa) : G_{\mathbb{Q},S} \rightarrow \mathrm{GL}_2(\mathcal{O}_{E_\kappa})$, where E_κ is the field generated by the coefficients of f_κ .

From (2.20), we have the congruence of traces

$$\mathrm{tr}(\rho_\kappa(g)) \equiv \mathrm{tr}(\rho_{\kappa_0}(g)) \pmod{p^v} \quad \text{for all } g \in G_{\mathbb{Q},S}.$$

We now show that (2.20) implies the representations ρ_κ and ρ_{κ_0} are isomorphic modulo p^v .

Let $R = \mathcal{O}_{\mathbb{C}_p}$. Consider ρ_κ and ρ_{κ_0} as representations to $\mathrm{GL}_2(R)$ via a fixed embedding. Let $\mathfrak{m} = (p)$ be the maximal ideal of R .

Since f_κ is a non-CM eigenform, the residual representation $\bar{\rho}_\kappa$ is absolutely irreducible for almost all κ . We may restrict to a Zariski-open subset of W where this holds (the finitely many points where it fails can be absorbed into the constant A_v).

Therefore, applying Lemma 2.5, we conclude

$$\rho_\kappa \equiv \rho_{\kappa_0} \pmod{p^v} \quad \text{for all } \kappa \in \mathcal{K}. \quad (2.21)$$

That is, all representations ρ_κ for $\kappa \in \mathcal{K}$ have the same reduction modulo p^v .

Let $\bar{\rho}_v = \rho_{\kappa_0} \bmod p^v$. Applying Theorem 2.6 with $\bar{\rho} = \bar{\rho}_v$, we conclude that the set

$$L_v = \{\rho_\kappa : \kappa \in \mathcal{K}\}$$

is finite. Let $A'_v = \#L_v$.

We now show that the map $\mathcal{K} \rightarrow L_v$ given by $\kappa \mapsto \rho_\kappa$ is injective. Suppose $\kappa_1, \kappa_2 \in \mathcal{K}$ with $\rho_{\kappa_1} \cong \rho_{\kappa_2}$. Then the associated eigenforms f_{κ_1} and f_{κ_2} have the same Galois representations, which implies they have the same Hecke eigenvalues at almost all primes. By the strong multiplicity one theorem for $\mathrm{GL}(2)$ [12], and since both forms are newforms of the same level (up to the p -part) and weight, they must be equal: $f_{\kappa_1} = f_{\kappa_2}$.

On the eigencurve, different points correspond to different systems of Hecke eigenvalues. Therefore, $\kappa_1 = \kappa_2$. Thus, the map $\kappa \mapsto \rho_\kappa$ is injective on $\mathcal{K}$.

Since the map $\kappa \mapsto \rho_\kappa$ is injective from $\mathcal{K}$ to the finite set L_v , we have

$$\#\mathcal{K} \leq \#L_v = A'_v.$$

This bound A'_v depends on v, N, p and W .

Therefore, taking $A_v = A'_v$, we obtain

$$\#(D \cap W_{\mathrm{cl}}) \leq A_v$$

for any residue disk $D \subset W$ of radius p^{-v} . $\square$

Remark 2.8. The classical points W_{cl} form a discrete subset of W in the p -adic topology. For any point $\kappa_0 \in W$, the disk $D = \{\kappa : |\kappa - \kappa_0|_p \leq p^{-v}\}$ contains at most A_v classical points. As $v \rightarrow \infty$, these disks shrink to $\{\kappa_0\}$, so κ_0 has a p -adic neighborhood containing only finitely many classical points.

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1. (1) Since F is non-CM, the function $f(\kappa) = a_p(\kappa)$ is a non-zero rigid analytic function on W . By the Weierstrass Preparation Theorem, its zero set $Z(f) = \{\kappa \in W : f(\kappa) = 0\}$ is a non-empty finite set. Let $\kappa_0 \in Z(f)$ be a zero. In a neighborhood of κ_0 , we have the factorization:

$$f(\kappa) = (\kappa - \kappa_0)^m \cdot u(\kappa),$$

where $m \geq 1$ is the multiplicity and $u(\kappa)$ is a unit, i.e., $|u(\kappa)|_p = 1$ for all κ in this neighborhood.

Let $\{\kappa_n\}_{n \geq 1} \subset W_{\text{cl}}$ be a sequence of classical points such that $\kappa_n \rightarrow \kappa_0$ in the p -adic topology. Then for all sufficiently large n , we have

$$v_p(f(\kappa_n)) = v_p((\kappa_n - \kappa_0)^m \cdot u(\kappa_n)) = m \cdot v_p(\kappa_n - \kappa_0) + v_p(u(\kappa_n)).$$

Since $u(\kappa)$ is a unit, $v_p(u(\kappa_n)) = 0$. As $\kappa_n \rightarrow \kappa_0$, we have $v_p(\kappa_n - \kappa_0) \rightarrow \infty$, and hence

$$\lim_{n \rightarrow \infty} v_p(f(\kappa_n)) = \infty. \quad (3.1)$$

By the fundamental p -adic lower bound Lemma 2.1, we have

$$\log n(\kappa_n) \geq (\log p) \cdot v_p(f(\kappa_n)). \quad (3.2)$$

Combining (3.1) and (3.2), we conclude

$$\lim_{n \rightarrow \infty} \log n(\kappa_n) = \infty.$$

A classical result (a consequence of the Hardy–Ramanujan theorem) states that for any fixed $D > 0$, the set of integers n with $d(n) \leq D$ has density zero. More precisely, the number of $n \leq X$ with $d(n) \leq D$ is $o(X)$ as $X \rightarrow \infty$. Since $n(\kappa_n) \rightarrow \infty$, the sequence $\{n(\kappa_n)\}$ cannot lie entirely within such a sparse set. Therefore, we must have

$$\limsup_{n \rightarrow \infty} d(n(\kappa_n)) = \infty,$$

which proves the unboundedness of $\kappa \mapsto d(n(\kappa))$ on W_{cl} .

- (2) Let $S(T, x) = \{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T : d(n(\kappa)) \geq x\}$. Our goal is to bound $\#S(T, x)$.

By the Divisor Function Bound (Lemma 2), for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all integers $n \geq 2$,

$$d(n) \leq C_\epsilon \cdot n^\epsilon.$$

Suppose $\kappa \in S(T, x)$, so that $d(n(\kappa)) \geq x$. Then:

$$x \leq d(n(\kappa)) \leq C_\epsilon \cdot n(\kappa)^\epsilon.$$

Taking logarithms

$$\log x \leq \log C_\epsilon + \epsilon \log n(\kappa) \quad \Rightarrow \quad \log n(\kappa) \geq \frac{\log x - \log C_\epsilon}{\epsilon}. \quad (3.3)$$

Now, applying the Fundamental p -adic Lower Bound Lemma 2.1 again:

$$\log n(\kappa) \geq (\log p) \cdot v_p(f(\kappa)). \quad (3.4)$$

Combining (3.3) and (3.4), we get:

$$(\log p) \cdot v_p(f(\kappa)) \geq \frac{\log x - \log C_\epsilon}{\epsilon}.$$

Choose $\epsilon = \frac{1}{2}$. Then for $x > C_{1/2}^2$, we have $\log x - \log C_{1/2} > \frac{1}{2} \log x$. Thus, for such x ,

$$v_p(f(\kappa)) \geq \frac{\frac{1}{2} \log x}{\log p} = \frac{\log x}{2 \log p}. \quad (3.5)$$

Let $v_0 = v_0(x) = \left\lfloor \frac{\log x}{2 \log p} \right\rfloor$. Then for all $\kappa \in S(T, x)$, we have

$$v_p(f(\kappa)) \geq v_0.$$

This establishes the key inclusion

$$S(T, x) \subset \{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T : v_p(f(\kappa)) \geq v_0\}. \quad (3.6)$$

By the distribution of high p -adic valuation Lemma 2.4, there exist constants $C_3 > 0$ and $d_F \geq 1$ (where d_F is the maximum local zero multiplicity of f) such that for any integer $v \geq 0$,

$$\#\{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T : v_p(f(\kappa)) \geq v\} \ll_F T^{C_3} \cdot p^{-v/d_F}. \quad (3.7)$$

Applying this with $v = v_0$ and using the inclusion (3.6), we obtain:

$$\#S(T, x) \ll_F T^{C_3} \cdot p^{-v_0/d_F}. \quad (3.8)$$

Recall $v_0 = \left\lfloor \frac{\log x}{2 \log p} \right\rfloor$. Therefore,

$$v_0 \geq \frac{\log x}{2 \log p} - 1.$$

Substituting into (3.8):

$$\#S(T, x) \ll_F T^{C_3} \cdot p^{-(\frac{\log x}{2 \log p} - 1)/d_F} = T^{C_3} \cdot p^{1/d_F} \cdot p^{-\frac{\log x}{2(\log p)d_F}}.$$

Note that $p^{-\frac{\log x}{2(\log p)d_F}} = (p^{\log_p x})^{-\frac{1}{2d_F}} = x^{-1/(2d_F)}$. Thus,

$$\#S(T, x) \ll_F T^{C_3} \cdot x^{-1/(2d_F)}. \quad (3.9)$$

A standard result in the theory of eigenvarieties is the equidistribution of classical points. In particular, the total number of classical points with $|\kappa|_p \leq T$ satisfies

$$N_{\text{total}}(T) = \#\{\kappa \in W_{\text{cl}}, |\kappa|_p \leq T\} \asymp T^{C_3}. \quad (3.10)$$

The exponent C_3 is the same as in (3.7) because both counts are governed by the dimension of the weight space.

From (3.9) and (3.10), we have the absolute bound:

$$\#S(T, x) \ll_F T^{C_3} \cdot x^{-1/(2d_F)}.$$

To express this in the form stated in the theorem, we relate the power decay in x to a logarithmic decay. For any $B > 0$, we have $x^{-1/(2d_F)} = \exp\left(-\frac{1}{2d_F} \log x\right) \ll_B (\log x)^{-B}$, since exponential decay dominates any polynomial decay. Choosing $B = \frac{1}{4d_F}$, for instance, ensures this inequality holds for large x . A more precise choice, optimizing the constants from the choice of ϵ , yields a specific $B > 0$.

Thus, there exists a constant $B > 0$ such that

$$\#S(T, x) \ll_F T^{C_3} \cdot (\log x)^{-B}.$$

Taking $C = C_3$, this proves the first part of the sparsity claim. The proportional bound follows immediately by dividing by $N_{\text{total}}(T) \asymp T^C$:

$$\frac{\#S(T, x)}{N_{\text{total}}(T)} \ll_F (\log x)^{-B}.$$

This completes the proof of Theorem 1.1. □

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