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REFINED MONOTONICITY-BASED APPROACH TO OSCILLATION IN SECOND-ORDER HALF-LINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. This study aims to establish advanced oscillation criteria for second-order half-linear delay differential equations, thus broadening the existing theoretical framework. Our method relies on establishing stronger monotonicity properties for the solutions than those previously available. We employ a primary approach in our analysis: a comparison principle involving first-order delay differential inequalities. An illustrative example is presented to demonstrate the superiority of the proposed method compared to existing results.

Keywords. delay differential equations, oscillation criteria, second-order, half-linear
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1. Introduction

Differential equations (DEs) are a fundamental language for modeling and predicting dynamic phenomena across science and engineering [4, 9, 17, 37]. This work focuses on a specific class called Functional Differential Equations (FDEs), particularly Delay Differential Equations (DDEs) that incorporate time delays. A more versatile framework is provided by Neutral Delay Differential Equations (NDEs), which generalize DDEs by allowing the current rate of change to depend on past rates [16, 27]. The idea of DDEs arises from the fact that many real-world processes do not happen instantaneously. This is especially clear in economics, where where delays are not anomalies but inherent features of the system. DDEs help model these built-in time lags. For example, the delay between a central bank changing interest rates and the effect on inflation and investment, the time needed for companies to increase production, or the gap between ordering and receiving goods. As noted in modern texts such as Smith [36], neglecting these delays results in models that are overly simplistic and may severely misrepresent system stability. The parallel development of DDE theory and economics is not coincidental. Kalecki's pioneering work [20] on business cycle dynamics, which introduced an investment delay to account for endogenous oscillations, actually

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anticipated the formal development of DDE theory. This connection highlights a deep historical link: economists were among the earliest to recognize the essential role of past states and memory in shaping dynamic behavior. Oscillation—the repetitive variation around an equilibrium—is a crucial dynamic behavior in systems with delays, relevant to fields from mechanics to ecology. Consequently, a major focus of DDE research is establishing criteria to guarantee that all solutions oscillate. This includes significant work on higher-order [1, 8, 30, 34] and multi-order equations [11, 26, 29]. Recently, attention has shifted prominently to second-order DDEs due to their importance in applied models, leading to refined oscillation criteria by researchers like Baculíková and Džurina [31] and Li et al. [25]. This falls within oscillation theory, a key part of the qualitative study of DDEs, which determines whether solutions are oscillatory (with infinitely many zeros) or non-oscillatory see; [28, 33]. The half-linear form, a notable extension of the classical second-order linear differential equation, forms the core of this investigation. Additional background can be found in the classical literature and recent studies cited in [14, 15, 32].

The present work investigates the oscillatory behavior of second-order half-linear DDEs

$$\left(a\left(u\right)\left(\vartheta'\left(u\right)\right)^{\alpha}\right)' + \sum_{j=1}^{m} h_j\left(u\right)\vartheta^{\alpha}\left(g_j\left(u\right)\right) = 0, \tag{1.1}$$

where $u \geq u_0$, and m belongs to the set of natural numbers, it is assumed that

(A1) $\alpha \geq 1$ is a ratio of two odd integers, h_j is a continuous function on $([u_0,\infty))$ satisfying $h_j\left(u\right) \geq 0$;

(A2) $a \in \mathbf{C}\left([u_0,\infty)\right)$, $a\left(u\right) > 0$, and $\varrho\left(\infty\right) = \infty$, where for $u_1 \geq u_0$,

$$\varrho\left(u\right) = \int_{u_1}^{u} a^{-1/\alpha}\left(\eta\right)\mathrm{d}\eta; \tag{1.2}$$

(A3) $g_j \in \mathbf{C}\left([u_0,\infty)\right)$, $g_j\left(u\right) \leq u$, $g_j'\left(u\right) \geq 0$, and $\lim_{u\to\infty} g_j\left(u\right) = \infty$, for $j = 1, 2, ..., m$.

A solution $\vartheta \in C^2\left([u_\vartheta,\infty)\right)$ for some $u_\vartheta \geq u_0$ is considered a solution of Eq (1.1), if it satisfies the equation on $[u_\vartheta,\infty)$. We confine our analysis to solutions satisfying $\sup\{|\vartheta\left(u\right)| : u \geq T\} > 0$ for all $T \geq T_\vartheta$. A solution is referred to as oscillatory when its zeros are unbounded above on $[u_0,\infty)$; if not, the solution possesses a nonoscillatory behavior. Eq (1.1) is considered oscillatory if all its solutions are oscillatory.

In particular situations, the asymptotic nature of Eq (1.1) can be suitably examined by comparing it with more tractable forms, particularly first-order delay differential inequality (1.1) and its extension as a neutral differential equation (NDE), see references [6, 7]. This comparison strategy is a foundational and highly effective technique in oscillation theory. It enables the intricate qualitative properties of the second-order half-linear equation to be transformed into a more manageable analysis involving a first-order delay differential inequality. The earliest advances on second-order DEs came from Koplatadze [21] and Wei [39], who demonstrated that equation

$$\vartheta''\left(u\right) + h\left(u\right)\vartheta\left(g\left(u\right)\right) = 0$$

oscillates if

$$\liminf_{u\to\infty}\int_{g(u)}^{u} g(\eta)\,h(\eta)\,\mathrm{d}\eta > \frac{1}{\mathrm{e}}. \tag{1.3}$$

Incorporating the term $a(u)(\vartheta'(u))^{\alpha}$ which is damping term, marks a notable generalization, and its influence has been the subject of extensive research. Došlý and Řehák [12] developed comparison theorems for equations with power-type nonlinearities, while Agarwal et al. [3] provided oscillation criteria for equations involving non-monotone operators. The delay term $g(u)$ adds extra complexity to the analysis. If $g(u)\le u$, the equation has a delayed argument, while $g(u)\ge u$ gives an advanced argument. The mixed case, where $g(u)$ can be either, was studied by Kusano and Naito [22]. For nonlinear delay equations, Ladde et al. [38]. Recently, Li et al. [24] extended these results to equations with neutral terms and variable exponents. Among simpler models, the half-linear second-order DE is particularly suitable for comparison with (1.1). In [13], sharp and straightforward criteria for Eq (1.1) were derived by Džurina and Jadlovská, extending previous results while avoiding the assumption of increasing delays in the noncanonical scenario, where $\varrho(u)=\int_{u_1}^{u} a^{-1/\alpha}(\eta)\,\mathrm{d}\eta<\infty$. In the canonical case (1.2), Santra in [35] considered two cases depending on whether $\alpha>\gamma$ or $\alpha<\gamma$ and developed comprehensive criteria that are both necessary and sufficient to determine the oscillatory behavior of the equation

$$\left(a(u)(\vartheta'(u))^{\gamma}\right)' + h(u)\,\vartheta^{\alpha}(g(u)) = 0, \tag{1.4}$$

in the case of $\alpha\ge 1$, Jadlovská and Džurina [19] proposed oscillation conditions for a half-linear NDDEs

$$\left(a(u)\left|\vartheta'(u)\right|^{\alpha-1}\vartheta'(u)\right)' + h(u)\left|\vartheta(g(u))\right|^{\alpha-1}\vartheta(g(u)) = 0,$$

A sharp result was obtained; however, for $\alpha<1$, the result was slightly less precise.

In this case (1.2), Grace et al [18] proposed criteria to determine the oscillation of solutions in a class of second-order half-linear neutral DDE

$$\left(a(u)(z'(u))^{\alpha}\right)' + h(u)\,\vartheta^{\alpha}(g(u)) = 0, \tag{1.5}$$

where, $z(u)=\vartheta(u)+p(u)\,\vartheta(\tau(u))$. Subsequent methodological advances have significantly improved oscillation criteria. These include a refined Riccati transformation that yielded improved results for both neutral and nonneutral equations, an iterative technique to enhance monotonicity properties [18], and the establishment of sufficient conditions for the noncanonical case [2]. These approaches, which overcome limitations of earlier methods, have been further explored and extended for (1.1) and its various generalizations in several papers [10, 23]. Analyzing (1.1) is challenging due to its nonlinearity and delay term, but determining whether its solutions oscillate is crucial for understanding the system's long-term behavior. This work examines the oscillation and monotonicity of positive solutions to Eq. (1.1) using comparison principles derived from first-order delay differential inequalities. These findings deliver refined extensions applicable delay differential equations. Our method specifically relies on developing stronger monotonicity properties for the equation compared to those previously established. As a result, this work formulates advanced oscillation conditions for second-order half-linear DDEs, thus broadening the scope of existing theoretical results. Furthermore, an illustrative example is presented to demonstrate the superiority of the proposed method compared to previously established results.

2. Main Results

Lemma 2.1. *Assuming $\vartheta(u)$ of Eq (1.1) is eventually positive, it follows $\vartheta'(u) > 0$ that $(a(u)(\vartheta'(u))^{\alpha})' \leq 0$,*

$$\left(\frac{\vartheta(u)}{\varrho(u)}\right)' < 0, \tag{2.1}$$

and

$$\left(a^{1/\alpha}(u)\vartheta'(u)\right)' + \frac{1}{\alpha}\sum_{j=1}^{m}\varrho^{\alpha-1}(g_j(u))h_j(u)\vartheta(g_j(u)) \leq 0. \tag{2.2}$$

Proof. Let $\vartheta(u)$ denote a solution of Eq (1.1) such that we have $\vartheta(u) > 0$ for some $u_1 \geq u_0$, and $\vartheta(g_j(u)) > 0$ for $u \geq u_1$. Consequently, Eq (1.1) implies $(a(u)(\vartheta'(u))^{\alpha})' \leq 0$. Therefore, since $a(u)(\vartheta'(u))^{\alpha}$ is nonincreasing, Thus, $\vartheta'(u)$ maintains a constant sign for large u. Using the canonical condition $\varrho(\infty) = \infty$, we obtain that $\vartheta'(u) > 0$. Moreover, we see that

$$\left(a(u)(\vartheta'(u))^{\alpha}\right)' = \left(\left[a^{1/\alpha}(u)\vartheta'(u)\right]^{\alpha}\right)' = \alpha\left[a^{1/\alpha}(u)\vartheta'(u)\right]^{\alpha-1}\left(a^{1/\alpha}(u)\vartheta'(u)\right)',$$

and the above differential equation gives

$$\alpha\left[a^{1/\alpha}(u)\vartheta'(u)\right]^{\alpha-1}\left(a^{1/\alpha}(u)\vartheta'(u)\right)' = -\sum_{j=1}^{m}h_j(u)\vartheta^{\alpha}(g_j(u)).$$

So

$$\left(a^{1/\alpha}(u)\vartheta'(u)\right)' = -\frac{1}{\alpha}\left[a^{1/\alpha}(u)\vartheta'(u)\right]^{1-\alpha}\sum_{j=1}^{m}h_j(u)\vartheta^{\alpha}(g_j(u)). \tag{2.3}$$

Since $(a(u)(\vartheta'(u))^{\alpha})' \leq 0$, we find that

$$\vartheta(u) \geq \int_{u_1}^{u}\frac{a^{1/\alpha}(\eta)\vartheta'(\eta)}{a^{1/\alpha}(\eta)}\mathrm{d}\eta \geq a^{1/\alpha}(u)\vartheta'(u)\int_{u_1}^{u}\frac{1}{a^{1/\alpha}(\eta)}\mathrm{d}\eta = \varrho(u)a^{1/\alpha}(u)\vartheta'(u), \tag{2.4}$$

and

$$\begin{aligned}\left(\frac{\vartheta(u)}{\varrho(u)}\right)' &= \frac{\varrho(u)\vartheta'(u) - a^{-1/\alpha}(u)\vartheta(u)}{\varrho^2(u)}\\ &= \frac{1}{\varrho^2(u)a^{1/\alpha}(u)}\left(\varrho(u)a^{1/\alpha}(u)\vartheta'(u) - \vartheta(u)\right)\\ &< 0.\end{aligned}$$

Employing (2.4), we arrive at

$$\vartheta(g_j(u)) \geq \varrho(g_j(u))a^{1/\alpha}(g_j(u))\vartheta'(g_j(u)) \geq \varrho(g_j(u))a^{1/\alpha}(u)\vartheta'(u),$$

or

$$\frac{\vartheta\left(g_j\left(u\right)\right)}{\varrho\left(g_j\left(u\right)\right)} \geq a^{1/\alpha}\left(u\right)\vartheta'\left(u\right). \tag{2.5}$$

Applying (1.1) and (2.5) to (2.3), we arrive at

$$\begin{aligned}\left(a^{1/\alpha}\left(u\right)\vartheta'\left(u\right)\right)' &\leq \frac{1}{\alpha}\left[\frac{\vartheta\left(g_j\left(u\right)\right)}{\varrho\left(g_j\left(u\right)\right)}\right]^{1-\alpha}\left(a\left(u\right)\left(\vartheta'\left(u\right)\right)^{\alpha}\right)' \\ &= -\frac{1}{\alpha}\left[\frac{\vartheta\left(g_j\left(u\right)\right)}{\varrho\left(g_j\left(u\right)\right)}\right]^{1-\alpha}\sum_{j=1}^{m} h_j\left(u\right)\vartheta^{\alpha}\left(g_j\left(u\right)\right) \\ &\leq -\frac{1}{\alpha}\sum_{j=1}^{m}\varrho^{\alpha-1}\left(g_j\left(u\right)\right)h_j\left(u\right)\vartheta\left(g_j\left(u\right)\right).\end{aligned}$$

Hence, the proof is concluded. □

This is the first-level (standard) monotonicity property.

Lemma 2.2. *Considering an eventually positive solution $\vartheta\left(u\right)$ of Eq (1.1) and*

$$\int_{u_1}^{\infty}\sum_{j=1}^{m}\varrho^{-\alpha}\left(g_j\left(\eta\right)\right)h_j\left(\eta\right)\mathrm{d}\eta = \infty. \tag{2.6}$$

Then

$$\lim_{u\to\infty}\frac{\vartheta\left(u\right)}{\varrho\left(u\right)} = 0.$$

Proof. Assume that $\vartheta\left(u\right)$ satisfies Eq (1.1) and $\vartheta\left(u\right) > 0$ for some $u_1 \geq u_0$. By Lemma 2.1, it follows that the function $\vartheta\left(u\right)/\varrho\left(u\right)$ is positive and decreases monotonically. Consequently, $\lim_{u\to\infty}\vartheta\left(u\right)/\varrho\left(u\right) = \ell \geq 0$, and for some $u_1 \geq u_0$ it holds that $\vartheta\left(u\right) \geq \ell\varrho\left(u\right)$ whenever $u \geq u_1$. Hence

$$\begin{aligned}\left(a^{1/\alpha}\left(u\right)\varrho^2\left(u\right)\left(\frac{\vartheta\left(u\right)}{\varrho\left(u\right)}\right)'\right)' &= \left(\varrho\left(u\right)a^{1/\alpha}\left(u\right)\vartheta'\left(u\right) - \vartheta\left(u\right)\right)' \\ &= \varrho\left(u\right)\left[a^{1/\alpha}\left(u\right)\vartheta'\left(u\right)\right]' + a^{-1/\alpha}\left(u\right)a^{1/\alpha}\left(u\right)\vartheta'\left(u\right) - \vartheta'\left(u\right) \\ &= \varrho\left(u\right)\left[a^{1/\alpha}\left(u\right)\vartheta'\left(u\right)\right]',\end{aligned}$$

which with (2.2) gives

$$\left(a^{1/\alpha}\left(u\right)\varrho^2\left(u\right)\left(\frac{\vartheta\left(u\right)}{\varrho\left(u\right)}\right)'\right)' \leq -\frac{1}{\alpha}\varrho\left(u\right)\sum_{j=1}^{m}\varrho^{\alpha-1}\left(g_j\left(u\right)\right)h_j\left(u\right)\vartheta\left(g_j\left(u\right)\right). \tag{2.7}$$

integrating Eq (2.7) over u_1 to u, yields

$$a^{1/\alpha}\left(u\right)\varrho^2\left(u\right)\left(\frac{\vartheta\left(u\right)}{\varrho\left(u\right)}\right)' + L \leq -\frac{1}{\alpha}\int_{u_1}^{u}\sum_{j=1}^{m}\varrho\left(\eta\right)\varrho^{\alpha-1}\left(g_j\left(\eta\right)\right)h_j\left(\eta\right)\vartheta\left(g_j\left(\eta\right)\right)\mathrm{d}\eta,$$

where

$$L := -\left[a^{1/\alpha}\left(u\right)\varrho^2\left(u\right)\left(\frac{\vartheta\left(u\right)}{\varrho\left(u\right)}\right)'\right]_{u=u_1} > 0.$$

Setting $w(u) := \vartheta(u)/\varrho(u)$, we get

$$w'(u) + \frac{1}{\alpha}\frac{1}{a^{1/\alpha}(u)\,\varrho^2(u)}\int_{u_1}^{u}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,w(g_j(\eta))\,\mathrm{d}\eta \tag{2.8}$$

$$\leq \frac{-L}{a^{1/\alpha}(u)\,\varrho^2(u)} < 0, \tag{2.9}$$

Integrating (2.8) from u_1 to ∞ gives

$$\begin{aligned}\ell - w(u_1) &< -\frac{\ell}{\alpha}\int_{u_1}^{\infty}\frac{1}{a^{1/\alpha}(u)\,\varrho^2(u)}\int_{u_1}^{u}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,\mathrm{d}\eta\mathrm{d}u\\ &= -\frac{\ell}{\alpha}\int_{u_1}^{\infty}\sum_{j=1}^{m}\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,\mathrm{d}\eta,\end{aligned}$$

The above contradicts Eq (2.6), implying $\ell = 0$. Therefore, the proof is complete. □

2.1. Oscillation Theorems.

Notation 2.3. *The following functions is introduced to make our notation simpler.*

$$M(u) = \frac{1}{\alpha a^{1/\alpha}(u)\,\varrho(u)}\int_{0}^{u}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,d\eta,$$

$$I_1 = \int_{0}^{u}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,w(g_j(\eta))\,\mathrm{d}\eta,$$

$$I_2 = \frac{1}{\alpha}\int_{0}^{u_1}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,w(g_j(\eta))\,\mathrm{d}\eta$$

Theorem 2.4. *Consider (2.6). If*

$$\liminf_{u\to\infty}\int_{g(u)}^{u}\frac{1}{\alpha a^{1/\alpha}(s)\,\varrho^2(s)}\int_{0}^{s}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,\mathrm{d}\eta\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.10}$$

It follows that Eq (1.1) is oscillatory.

Proof. For the purpose of contradiction, let us suppose that Eq (1.1) admits a solution which becomes positive and strictly increasing. It follows that $w(u) = \vartheta(u)/\varrho(u)$ obeys the inequality (2.8), for which

$$w'(u) + \frac{1}{\alpha}\frac{1}{a^{1/\alpha}(u)\,\varrho^2(u)}\left(\int_{u_1}^{u}\sum_{j=1}^{m}\varrho(\eta)\,\varrho^{\alpha}(g_j(\eta))\,h_j(\eta)\,w(g_j(\eta))\,\mathrm{d}\eta + L\right) = 0,$$

which is

$$w'(u) + \frac{1}{\alpha a^{1/\alpha}(u)\,\varrho^2(u)}\left(I_1 - I_2 + L\right)$$

From this, it is clear that $w(u)$ satisfies the following inequality as a positive solution

$$w'(u) + M(u) \sum_{j=1}^{m} w(g_j(\eta))\, \mathrm{d}\eta \leq 0, \tag{2.11}$$

Since this leads to a contradiction with Eq (2.10), the proof is complete. □

We will enhance (2.1) by introducing the next lemma.

Lemma 2.5. *Assuming that (2.6) is satisfied, it follows that for any nonoscillatory solution $\vartheta(u)$ of Eq (1.1), the function $w(u) := \vartheta(u) / \varrho(u)$ holds for*

$$(w(u)\, \mu(u))' \leq 0. \tag{2.12}$$

Proof. Consider a solution $\vartheta(u)$ of Eq (1.1) satisfying $\vartheta(u) > 0$. It follows that $w(u)$ satisfies

$$w'(u) + M(u)\, w(u) \leq 0, \tag{2.13}$$

The integrating factor is defined as

$$\mu(u) = e^{\int M(u) du},$$

by multiplying (2.13) by $\mu(u)$,we get

$$\mu(u)\, w'(u) + \mu(u)\, M(u)\, w(u) \leq 0,$$

It then follows that

$$(\mu(u)\, w(u))' \leq 0.$$

Hence, the proof is concluded. □

Remark 2.6. The previous lemma represents the central refinement of our approach. It establishes a second-level monotonicity for the function $\mu(u)\, w(u)$This is a stronger property than the one given in Lemma 2.1, as it incorporates the cumulative effect of the coefficient $M(u)$ through the integrating factor $\mu(u)$. This iterative step is the mechanism that allows us to derive sharper estimates in the subsequent oscillation theorem, ultimately leading to a less conservative criterion.

Theorem 2.7. *Assume that condition (2.6) is satisfied. If*

$$\liminf_{u\to\infty} \int_{g(u)}^{u} \frac{\mu(g(s))}{\alpha a^{1/\alpha}(s)\, \varrho^2(s)} \int_{u_1}^{s} \sum_{j=1}^{m} \frac{\varrho(\eta)\, \varrho^{\alpha}(g_j(\eta))\, h_j(\eta)}{\mu(g_j(\eta))} \mathrm{d}\eta \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.14}$$

then Eq (1.1) is oscillatory.

Proof. Suppose, to the contrary, that Eq (1.1) admits an eventually positive solution. It then follows from Eq (2.8) that

$$w'(u) + \frac{1}{\alpha} \frac{1}{a^{1/\alpha}(u)\, \varrho^2(u)} \int_{u_1}^{u} \sum_{j=1}^{m} \varrho(\eta)\, \varrho^{\alpha}(g_j(\eta))\, h_j(\eta)\, w(g_j(\eta))\, \mathrm{d}\eta \leq 0,$$

applying our refined monotonicity property from (2.12) in Lemm 2.5 implies that $w(u)\mu(u)$ decreasing, thereby showing that $w(u)$ fulfills the differential inequality as a positive solution

$$w'(u)+\frac{\mu(g(u))}{\alpha a^{1/\alpha}(u)\varrho^{2}(u)}\int_{u_1}^{u}\sum_{j=1}^{m}\frac{\varrho(\eta)\varrho^{\alpha}(g_j(\eta))h_j(\eta)w(g_j(\eta))}{\mu(g_j(\eta))}\mathrm{d}\eta\leq 0,$$

Since this leads to a contradiction with Eq (2.14), therefore, the proof is established. □

Corollary 2.8. *Let Eq (2.6) holds and* $\lim_{u\to\infty}\frac{\mu(g(u))}{a^{1/\alpha}(u)}=0$. *If*

$$\liminf_{u\to\infty}\int_{g(u)}^{u}\frac{\mu(g(s))}{\alpha a^{1/\alpha}(s)\varrho^{2}(s)}\int_{0}^{s}\sum_{j=1}^{m}\frac{\varrho(\eta)\varrho^{\alpha}(g_j(\eta))h_j(\eta)}{\mu(g_j(\eta))}\mathrm{d}\eta\mathrm{d}s>\frac{1}{\mathrm{e}}. \tag{2.15}$$

hence Eq (1.1) is oscillatory.

Example 2.9. Let us consider the delay Euler differential equation

$$\vartheta''(u)+\frac{h_0}{u^2}\sum_{j=1}^{m}\vartheta(\delta_j u)=0, \tag{2.16}$$

where $a(u)=1$, $\alpha=1$, $\delta_j\in(0,1)$, $g_j(u)=\delta_j u$ and $h_0>0$, such that $h_j(u)=\frac{h_0}{u^2}$ for $j=1,2,....m$. Assume that $\delta=\min\{\delta_j,\ j=1,2,....m\}$. We note that

$$\varrho(u)=\int_{u_1}^{u}\mathrm{d}\eta=u,$$

and

$$\begin{aligned} M(u) &= \frac{mh_0\delta}{u}\sum_{j=1}^{m}\int_{0}^{u}d\eta\mathrm{d}\eta \\ &= mh_0\delta \end{aligned}$$

hence

$$\mu(u)=e^{mh_0\delta},$$

therefore, by using corollary 2.8 we see condition (2.15) simplifies to

$$\liminf_{u\to\infty}\int_{\delta u}^{u}\frac{\mathrm{e}^{m\delta h_0}}{s^2}\int_{0}^{s}\sum_{j=1}^{m}\frac{m\delta h_0}{\mathrm{e}^{m\delta h_0}}\mathrm{d}\eta\mathrm{d}s>\frac{1}{\mathrm{e}}$$

$$\liminf_{u\to\infty} m\delta h_0\int_{\delta u}^{u}\frac{1}{s^2}\int_{0}^{s}\mathrm{d}\eta\mathrm{d}s>\frac{1}{\mathrm{e}}$$

$$\liminf_{u\to\infty} m\delta h_0\int_{\delta u}^{u}\frac{1}{s}\mathrm{d}s>\frac{1}{\mathrm{e}}$$

Thus, we see that Equation (2.16) is oscillatory if

$$h_0>\frac{1}{m\delta\mathrm{e}\ln\frac{1}{\delta}}. \tag{2.17}$$

While, by using Theorem 2.4 in [5] we see that Equation (2.16) is oscillatory if

$$h > \frac{1}{4\delta} \tag{2.18}$$

Remark 2.10. If we take special case of Eq (2.16) with $\delta = \frac{1}{\mathrm{e}}$, $m = 4$ then we see that condition (2.17) becomes

$$h_0 > \frac{1}{4 \ln e} \approx 0.25, \tag{2.19}$$

and see that condition (2.18) becomes

$$h > \frac{\mathrm{e}}{4} \approx 0.679. \tag{2.20}$$

It can be easily observed that our Criterion (2.19) supports the most efficient condition (2.20).

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