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SOLVABILITY OF CAPUTO-HADAMARD FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING THE P-LAPLACIAN OPERATOR

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ABSTRACT. The aim of this work is to study a class of Caputo–Hadamard fractional integro-differential equations involving the p-Laplacian operator with integral boundary conditions. Existence of solutions is obtained by using fixed-point techniques, and two examples are given to illustrate our outcomes.

Keywords. Caputo-Hadamard derivative, p-Laplacian operator, fractional integro-differential equations, fixed point theorem
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1. INTRODUCTION

In this work, we study the following fractional boundary value problem:

$$\begin{cases} \mathcal{D}^{\beta}\left(\phi_p\left(\mathcal{D}^{\alpha} z(t)\right)\right) = f(t, z(t)) + \int_1^t k(t, s, z(s))ds, \quad t \in I := [1, e] \\ z(1) = 0, \quad z(e) = \int_1^e u(\theta) z(\theta) d\theta, \quad \mathcal{D}^{\alpha} z(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2,\ 0 < \beta \leq 1$, $\mathcal{D}^{\alpha}$ and $\mathcal{D}^{\beta}$ are the Caputo-Hadamard derivative, $\phi_p(s) = |s|^{p-2}s$ is the p-Laplacian operator such that $\frac{1}{p} + \frac{1}{q} = 1,\ p > 1$, and $\phi_p^{-1}(s) = \phi_q(s)$. $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $k : I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions, and $u \in L^1([1, e], \mathbb{R})$.

The study of fractional differential equations has seen remarkable growth, both theoretically and practically. For more details, we direct the reader to [7, 9, 11] and the references therein.

Recently, several researchers have investigated the solvability of various classes of fractional differential equations with integral boundary conditions (see, e.g.,[1, 2]). In particular, fractional problems involving the p-Laplacian operator have been extensively analyzed, leading to significant advances in existence and multiplicity results (see, e.g., [12, 13]). Moreover, a growing body of research has focused on equations incorporating the Caputo–Hadamard fractional derivative, for which existence results, uniqueness criteria, and qualitative properties of solutions have been established (see, e.g., [4, 6]).

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The integro-differential equation is a fundamental mathematical tool in applied sciences and is widely used to model phenomena in physics, engineering, and biology, such as thermal conduction in memory materials, viscoelastic behavior, population dynamics, neural networks, control theory, electrodynamics of complex media, compartmental systems, and nuclear reactors. Some results are presented in this way (see, e.g., [3, 10, 14]).

The present work considers a fractional boundary value problem whose structure has not been examined in this specific form. Specifically, it combines the Caputo–Hadamard fractional derivative with the nonlinear p-Laplacian operator, together with the integro-differential term and the nonlocal integral boundary condition

$$z(e) = \int_1^e u(\theta) z(\theta)\, d\theta,$$

leading to an operator framework different from those treated in earlier works. This hybrid structure requires careful analysis of the interactions between the operators and the integral boundary condition. The contribution of this paper is therefore to extend classical fixed-point techniques to this new class of fractional integro-differential problems and to establish solvability conditions appropriate for this mixed fractional framework.

The paper proceeds as follows. Section 2 introduces the necessary preliminaries. Section 3 presents the main theorems. Section 4 provides two examples to illustrate the obtained theory.

2. Preliminaries

This section gathers the essential definitions, preliminary lemmas, and fixed-point theorems that form the basis of our main results.

Definition 2.1. ([9]) Let $\gamma > 0$ and let $\xi \in L^1([1,\infty))$. The Hadamard fractional integral of order γ for the function ξ is given by

$$\mathcal{I}^\gamma \xi(t) = \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{s}\right)^{\gamma-1} \frac{\xi(s)}{s} ds,$$

whenever the integral is well defined.

Definition 2.2. ([8]). For a given function $\xi \in AC_\delta^n[1,\infty)$, the Caputo-Hadamard fractional derivative of order $\gamma > 0$ is defined by

$$\mathcal{D}^\gamma \xi(t) = \frac{1}{\Gamma(n-\gamma)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\gamma-1} \delta^n \xi(s) \frac{ds}{s}, \quad n-1 < \gamma < n,$$

where $\delta = t\frac{d}{dt}$, $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ.

Lemma 2.3. *([8]) Let $\gamma > 0$ and set $n = [\gamma] + 1$. If $\xi \in AC_\delta^n[1,\infty)$, then the following identity holds:*

$$\mathcal{I}^\gamma \big(\mathcal{D}^\gamma \xi\big)(t) = \xi(t) - \sum_{k=0}^{n-1} \frac{\delta^k \xi(1)}{k!} (\log t)^k, \quad n-1 < \gamma < n.$$

Lemma 2.4. *([12]) Let ϕ_p denote the p-Laplacian operator. Then, for all $x, y \in \mathbb{R}$, we have:*

(1) *If $1 < p \leq 2$, $xy > 0$, and $|x|, |y| \geq m > 0$, then*
$$|\phi_p(x) - \phi_p(y)| \leq (p-1)m^{p-2}|x-y|.$$
(2) *If $p > 2$, $xy > 0$, and $|x|, |y| \leq M$, then*
$$|\phi_p(x) - \phi_p(y)| \leq (p-1)M^{p-2}|x-y|. \tag{2.1}$$

Lemma 2.5. *([5]) Let $\mathcal{F} : Y \to Y$ be a contraction mapping, where (Y, d) is a non-empty complete metric space. Then $\mathcal{F}$ has a unique fixed point in Y.*

Lemma 2.6. *([5]) Let Ω be a nonempty, closed, bounded, and convex subset of a Banach space Y, and let $\mathcal{F} : \Omega \to \Omega$ be a continuous and compact operator. Then $\mathcal{F}$ admits at least one fixed point in Ω.*

To establish our results, we rely on the subsequent lemma.

Lemma 2.7. *Problem (1) is equivalent to the following integral formulation:*
$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q(G_z(s)) \frac{ds}{s} \\ &+ \frac{\log t}{(1-K)\,\Gamma(\alpha)} \Bigg[\int_1^e u(\theta) \bigg(\int_1^\theta \big(\log \frac{\theta}{s}\big)^{\alpha-1} \phi_q(G_z(s)) \frac{ds}{s} \bigg) d\theta \\ &- \int_1^e \big(\log \frac{e}{s}\big)^{\alpha-1} \phi_q(G_z(s)) \frac{ds}{s} \Bigg], \end{aligned} \tag{2.2}$$
where
$$G_z(s) = \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \frac{1}{\tau} \big[f(\tau, z(\tau)) + \int_1^\tau k(\tau, r, z(r)dr \big] d\tau, \tag{2.3}$$
$$K = \int_1^e u(\theta) \log(\theta) d\theta, \quad \textit{with } |K| < 1.$$

Proof. Assuming $z(t)$ satisfies the first equation of (1), Lemma 2.3 yields
$$\phi_p(\mathcal{D}^\alpha z(t)) = \mathcal{I}^\beta f(t, z(t)) + \mathcal{I}^\beta \bigg(\int_1^\tau k(\tau, r, z(r)) dr \bigg) + c_0,$$
which implies that
$$\mathcal{D}^\alpha z(t) = \phi_q \bigg[\mathcal{I}^\beta f(t, z(t)) + \mathcal{I}^\beta \bigg(\int_1^\tau k(\tau, r, z(r)) dr \bigg) + c_0 \bigg]. \tag{2.4}$$
By applying $\mathcal{I}^\alpha$ to equation (2.4) and invoking Lemma 2.3, we obtain
$$z(t) = \mathcal{I}^\alpha \phi_q \bigg[\mathcal{I}^\beta f(t, z(t)) + \mathcal{I}^\beta \bigg(\int_1^\tau k(\tau, r, z(r)) dr \bigg) + c_0 \bigg] + c_1 + c_2 \log t.$$
By the boundary value conditions $z(1) = 0$, $\mathcal{D}^\alpha z(1) = 0$, we get $c_0 = c_1 = 0$.
$$z(t) = \mathcal{I}^\alpha \phi_q \bigg[\mathcal{I}^\beta f(t, z(t)) + \mathcal{I}^\beta \bigg(\int_1^\tau k(\tau, r, z(r)) dr \bigg) \bigg] + c_2 \log t. \tag{2.5}$$

Considering the condition $z(e) = \int_1^e u(\theta)z(\theta)d\theta$ we can obtain that

$$z(e) = \mathcal{I}^\alpha \phi_q\Big[\mathcal{I}^\beta f(e,z(e)) + \mathcal{I}^\beta\Big(\int_1^e k(e,r,z(r))dr\Big)\Big] + c_2 = \int_1^e u(\theta)z(\theta)d\theta$$
$$= \int_1^e u(\theta)\mathcal{I}^\alpha \phi_q\Big[\mathcal{I}^\beta f(s,z(s)) + \mathcal{I}^\beta\Big(\int_1^s k(s,r,z(r))dr\Big)\Big] + c_2\, u(\theta)\log\theta\; d\theta.$$

Therefore

$$c_2 = \frac{1}{(1-K)\,\Gamma(\alpha)}\Big[\int_1^e u(\theta)\Big(\int_1^\theta \big(\log\frac{\theta}{s}\big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}\Big)d\theta - \int_1^e \big(\log\frac{e}{s}\big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}\Big].$$

Substituting c_2 in (2.5) leads to the following form of the solution to (1):

$$z(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}$$
$$+ \frac{\log t}{(1-K)\,\Gamma(\alpha)}\Big[\int_1^e u(\theta)\Big(\int_1^\theta \big(\log\frac{\theta}{s}\big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}\Big)d\theta - \int_1^e \big(\log\frac{e}{s}\big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}\Big],$$

where $G_z(s)$ is as in (2.3). The proof is completed. □

3. Main Results

Let $I = [1,e]$ and let the Banach space $\Sigma = C(I,\mathbb{R})$ be equipped with the norm $\|z\|_\infty = \sup\{|z(t)| \;:\; t \in I\}$.
Define an operator $\mathcal{F} : \Sigma \to \Sigma$ by

$$\mathcal{F}z(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}$$
$$+ \frac{\log t}{(1-K)\,\Gamma(\alpha)}\Big[\int_1^e u(\theta)\Big(\int_1^\theta \big(\log\frac{\theta}{s}\big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}\Big)d\theta - \int_1^e \big(\log\frac{e}{s}\big)^{\alpha-1}\phi_q(G_z(s))\frac{ds}{s}\Big].$$

Evidently, any fixed point of $\mathcal{F}$ represents a solution to problem (1).

For obtaining our results, we introduce two groups of hypotheses. The first group (A1)-(A2) ensures the compactness and boundedness required to apply Schauder's fixed point theorem for the existence of solutions. In contrast, The second group (B1)-(B3) consists of Lipschitz-type assumptions, allowing the use of Banach's contraction principle to deduce uniqueness.

- (A1) $f : I \times \mathbb{R} \to \mathbb{R}$ and $k : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous,
- (A2) There exist non-negative real numbers $a_i, b_i,\ (i = 1,2)$ such that for all $t \in I$ and $x \in \mathbb{R}$

$$|f(t,x)| \le a_1 + a_2|x|^{p-1}, \quad |k(t,s,x)| \le b_1 + b_2|x|^{p-1},$$

- (B1) There exists $L_1 > 0$, such that for all $t \in I$ and $x, y \in \mathbb{R}$
$$|f(t,x) - f(t,y)| \leq L_1|x-y|,$$
- (B2) There exists $L_2 > 0$, such that for all and $x, y \in \mathbb{R}$
$$|k(t,s,x) - k(t,s,y)| \leq L_2|x-y|, \quad \forall\, 1 \leq s \leq t \leq e,$$
- (B3) For $1 < p < 2$, there exist ξ, χ in $C(I, \mathbb{R}^+)$ such that $\forall (t,x) \in I \times \mathbb{R}$
$$|f(t,x(t)| \leq \xi(t), \qquad \int_1^t |k(t,s,x(s))|ds \leq \chi(t), \quad (1 \leq s \leq t \leq e).$$

3.1. **Existence result via Schauder's fixed point theorem.** For convenience, let's define the auxiliary constants as follows:
$$A_0 = \frac{a_1 + b_1(e-1)}{\Gamma(\beta+1)}, \quad A_1 = \frac{a_2 + b_2(e-1)}{\Gamma(\beta+1)}, \quad B_0 = \frac{2^{q-2}A_0^{q-1}}{\Gamma(\alpha+1)},$$
$$B_1 = \frac{2^{q-2}A_1^{q-1}}{\Gamma(\alpha+1)}, \quad R = 1 + \frac{1 + \int_1^e |u(\theta)|d\theta}{|1-K|}.$$
Consider a closed bounded convex subset $\Omega_r = \{x \in C(I,\mathbb{R}) : \|x\|_\infty \leq r\}$, where
$$r \geq \frac{B_0 R}{1 - B_1 R}.$$

Theorem 3.1. *If assumptions (A1)–(A2) hold, then the problem (1) admits at least one solution provided*
$$B_1 R < 1. \tag{3.1}$$

Proof. Notice that $\mathcal{F}$ is continuous, in view of the hypothesis of continuity of f, k and ϕ_q. Now, we need to show that $\mathcal{F}(\Omega_r) \subset \Omega_r$ and the set $\mathcal{F}(\Omega_r)$ is a uniformly bounded. From $(A2)$, for all $z \in \Omega_r$ we have
$$\begin{aligned}
|G_z(s)| &\leq \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \frac{1}{\tau} \left[|f(\tau, z(\tau))| + \int_1^\tau |k(\tau, r, z(r)|dr\right] d\tau \\
&\leq \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \left[(a_1 + a_2|z|^{p-1}) + (b_1 + b_2|z|^{p-1})(e-1)\right] \frac{d\tau}{\tau} \\
&\leq \frac{(a_1 + a_2 r^{p-1}) + (b_1 + b_2 r^{p-1})(e-1)}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \frac{d\tau}{\tau}.
\end{aligned}$$
Using
$$\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \frac{d\tau}{\tau} = \frac{(\log s)^\beta}{\beta\,\Gamma(\beta)} \leq \frac{1}{\Gamma(\beta+1)},$$
we obtain
$$|G_z(s)| \leq \frac{[a_1 + b_1(e-1)] + [a_2 + b_2(e-1)]\, r^{p-1}}{\Gamma(\beta+1)} =: A_0 + A_1 r^{p-1}.$$

Therefore, for all $t \in I$,

$$
\begin{aligned}
|\mathcal{F}z(t)| \le{}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \left| \phi_q(A_0 + A_1 r^{p-1}) \right| \frac{ds}{s} \\
&+ \frac{|\log t|}{|1-K|\Gamma(\alpha)} \bigg[\int_1^e |u(\theta)| \int_1^\theta \big(\log \frac{\theta}{s} \big)^{\alpha-1} \left| \phi_q(A_0 + A_1 r^{p-1}) \right| \frac{ds}{s} \, d\theta \\
&+ \int_1^e \big(\log \frac{e}{s} \big)^{\alpha-1} \left| \phi_q(A_0 + A_1 r^{p-1}) \right| \frac{ds}{s} \bigg].
\end{aligned}
$$

From the definition of the p-Laplacian operator, we have

$$
\begin{aligned}
|\mathcal{F}z(t)| \le{}& \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s} \big)^{\alpha-1} \left| A_0 + A_1 r^{p-1} \right|^{q-1} \frac{ds}{s} \\
&+ \frac{|\log t|}{|1-K|\Gamma(\alpha)} \bigg[\int_1^e |u(\theta)| \int_1^\theta \big(\log \frac{\theta}{s} \big)^{\alpha-1} \left| A_0 + A_1 r^{p-1} \right|^{q-1} \frac{ds}{s} \, d\theta \\
&+ \int_1^e \big(\log \frac{e}{s} \big)^{\alpha-1} \left| A_0 + A_1 r^{p-1} \right|^{q-1} \frac{ds}{s} \bigg].
\end{aligned}
$$

Using $\frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s} \big)^{\alpha-1} \frac{ds}{s} \le \frac{1}{\Gamma(\alpha+1)}$ and $|\log t| \le 1$, we get

$$
\begin{aligned}
|\mathcal{F}z(t)| \le{}& \frac{1}{\Gamma(\alpha+1)} \left| A_0 + A_1 r^{p-1} \right|^{q-1} \\
&+ \frac{1}{|1-K|\Gamma(\alpha+1)} \bigg[\int_1^e |u(\theta)| \left| A_0 + A_1 r^{p-1} \right|^{q-1} d\theta + \left| A_0 + A_1 r^{p-1} \right|^{q-1} \bigg].
\end{aligned}
$$

Using the inequality $(u+v)^n \le 2^{n-1}(u^n + v^n)$ for $u, v \ge 0$, we obtain

$$
(A_0 + A_1 r^{p-1})^{q-1} \le 2^{q-2} \big(A_0^{q-1} + A_1^{q-1} r^{(p-1)(q-1)} \big).
$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $(p-1)(q-1) = 1$, hence

$$
(A_0 + A_1 r^{p-1})^{q-1} \le 2^{q-2} \big(A_0^{q-1} + A_1^{q-1} r \big).
$$

Therefore

$$
|\mathcal{F}z(t)| \le \frac{2^{q-2} \left(|A_0|^{q-1} + |A_1|^{q-1} r \right)}{\Gamma(\alpha+1)} \left(1 + \frac{1 + \int_1^\theta |u(\theta)| d\theta}{|1-K|} \right) =: (B_0 + B_1 r) R.
$$

Since $B_1 R < 1$ and $B_0 R \le (1 - B_1 R) r$, we obtain $|\mathcal{F}z(t)| \le r$, which means that $\mathcal{F}(\Omega_r) \subset \Omega_r$ and the set $\mathcal{F}(\Omega_r)$ is a uniformly bounded.

Next, we establish the equicontinuity of $\mathcal{F}(\Omega_r)$. Let denote the notation

$$
\Phi(r) := \sup_{|u| \le A_0 + A_1 r^{p-1}} |\varphi_q(u)|.
$$

For $t_1, t_2 \in I$ such that $t_1 < t_2$ and for $z \in \Omega_r$, we get

$$\begin{aligned}\mathcal{F}z(t_2) - \mathcal{F}z(t_1) &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \big(\log \frac{t_2}{s}\big)^{\alpha-1} \phi_q(G_z(s)) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \Big[\big(\log \frac{t_2}{s}\big)^{\alpha-1} - \big(\log \frac{t_1}{s}\big)^{\alpha-1} \Big] \phi_q(G_z(s)) \frac{ds}{s} \\ &+ \frac{\log t_2 - \log t_1}{(1-K)\,\Gamma(\alpha)} \Big[\int_1^e u(\theta) \Big(\int_1^\theta \big(\log \frac{\theta}{s}\big)^{\alpha-1} \phi_q(G_z(s)) \frac{ds}{s} \Big) d\theta \\ &- \int_1^e \big(\log \frac{e}{s}\big)^{\alpha-1} \phi_q(G_z(s)) \frac{ds}{s} \Big].\end{aligned}$$

Hence, by taking absolute values and using the boundedness of $\phi_q(G_z)$, we obtain

$$|\mathcal{F}z(t_2) - \mathcal{F}z(t_1)| \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{\Phi(r)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s},$$

$$I_2 = \frac{\Phi(r)}{\Gamma(\alpha)} \int_1^{t_1} \left| \left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right| \frac{ds}{s},$$

$$I_3 = \frac{|\log t_2 - \log t_1| \Phi(r)}{|1-K|\Gamma(\alpha)} \left[\int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{ds}{s} + \int_1^e u(\theta) \int_1^\theta \left(\log \frac{\theta}{s}\right)^{\alpha-1} \frac{ds}{s} d\theta \right].$$

Estimate of I_1. Since the function $s \mapsto (\log(t_2/s))^{\alpha-1}$ is positive on $[t_1, t_2]$, we obtain

$$I_1 \leq \frac{\Phi(r)}{\Gamma(\alpha+1)} \left(\log \frac{t_2}{t_1}\right)^\alpha.$$

Estimate of I_2. The mapping $x \mapsto x^{\alpha-1}$ is locally Lipschitz on $(0, \infty)$ for $\alpha > 1$, hence

$$I_2 \leq \frac{\Phi(r)}{\Gamma(\alpha+1)} \left[(\log t_2)^\alpha - (\log t_1)^\alpha + \left(\log \frac{t_2}{t_1}\right)^\alpha \right].$$

Estimate of I_3. Since $|\log t_2 - \log t_1| \to 0$, as $t_2 \to t_1$ and all integrals in brackets are finite constants, we have

$$I_3 \leq C_0 |\log t_2 - \log t_1|,$$

for some constant $C_0 > 0$ independent of $z \in \Omega_r$.

Combining the estimates for I_1, I_2, and I_3 yields

$$|\mathcal{F}z(t_2) - \mathcal{F}z(t_1)| \leq C_1 \left(\log \frac{t_2}{t_1}\right)^\alpha + C_2 \left[(\log t_2)^\alpha - (\log t_1)^\alpha\right] + C_3 |\log t_2 - \log t_1|,$$

for some positive constants C_1, C_2, C_3 independent of $z \in \Omega_r$.

Since each term on the right-hand side tends to zero as $t_1 \to t_2$, we conclude that the convergence is uniform in $z \in \Omega_r$. Hence $\mathcal{F}(\Omega_r)$ is equicontinuous. According to the Arzelà-Ascoli theorem, $\mathcal{F}$ is a compact operator. Consequently, applying Lemma (2.6) implies the existence of at least one fixed point of $\mathcal{F}$, which corresponds to a solution of the problem (1) on I.

□

3.2. **Existence and uniqueness result via Banach's fixed point theorem.** For the convenience of readers, denote as follows:

$$M = \frac{\|\xi\|_\infty + \|\chi\|_\infty}{\Gamma(\beta+1)}, \quad \eta = \frac{(q-1)M^{q-2}R\big(L_1 + L_2(e-1)\big)}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

Theorem 3.2. *Let $1 < p < 2$. Assume that (B1)-(B3) hold. If $\eta < 1$, then the problem (1) admits a unique solution.*

Proof. From (B3) we have

$$\begin{aligned}
|G_z(s)| &\leq \frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} \frac{1}{\tau}\left[|f(\tau, z(\tau))| + \int_1^\tau |k(\tau, r, z(r)|dr\right] d\tau \\
&\leq \frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} (\xi(t) + \chi(t))\frac{d\tau}{\tau} \\
&\leq \frac{\|\xi\|_\infty + \|\chi\|_\infty}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} \frac{d\tau}{\tau} \qquad (3.2)\\
&\leq \frac{\|\xi\|_\infty + \|\chi\|_\infty}{\Gamma(\beta+1)} = M.
\end{aligned}$$

For each $x, y \in \mathbb{R}$ and $t \in I$, we get

$$\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}y(t)| \leq{}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |\phi_q(G_x(s)) - \phi_q(G_y(s))| \frac{ds}{s} \\
&+ \frac{|\log t|}{|1-K|\Gamma(\alpha)}\left[\int_1^e \big(\log\frac{e}{s}\big)^{\alpha-1} |\phi_q(G_x(s)) - \phi_q(G_y(s))| \frac{ds}{s}\right. \\
&\left.+ \int_1^e u(\theta) \int_1^\theta \big(\log\frac{\theta}{s}\big)^{\alpha-1} |\phi_q(G_x(s)) - \phi_q(G_y(s))| \frac{ds}{s} d\theta\right].
\end{aligned}$$

Since $1 < p < 2$, we can see $q > 2$, and therefore the operator φ_q falls under case (2) of Lemma 2.4. Hence, using the inequality $|\log t| \leq 1$ together with the estimates in (3.2) and the bound (2.1) of Lemma 2.4, we obtain

$$\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}y(t)| \leq{}& \frac{(q-1)M^{q-2}}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |G_x(s) - G_y(s)| \frac{ds}{s} \\
&+ \frac{(q-1)M^{q-2}}{|1-K|\Gamma(\alpha)}\left[\int_1^e \big(\log\frac{e}{s}\big)^{\alpha-1} |G_x(s) - G_y(s)| \frac{ds}{s}\right. \\
&\left.+ \int_1^e u(\theta) \int_1^\theta \big(\log\frac{\theta}{s}\big)^{\alpha-1} |G_x(s) - G_y(s)| \frac{ds}{s} d\theta\right].
\end{aligned}$$

Using (B1)-(B2),

$$\begin{aligned}|G_x(s)-G_y(s)| &\le \frac{1}{\Gamma(\beta)}\int_1^s \Big(\log\frac{s}{\tau}\Big)^{\beta-1}\frac{1}{\tau}\Big[|f(\tau,x(\tau))-f(\tau,y(\tau))| \\ &\quad+\int_1^\tau |k(\tau,r,x(r)-k(\tau,r,y(\tau)|dr\Big]d\tau \\ &\le \frac{1}{\Gamma(\beta)}\int_1^s \Big(\log\frac{s}{\tau}\Big)^{\beta-1}\frac{1}{\tau}\big[L_1+L_2\int_1^\tau dr\big]\,\|x-y\|_\infty\,d\tau \\ &\le \frac{1}{\Gamma(\beta)}\int_1^s \Big(\log\frac{s}{\tau}\Big)^{\beta-1}\big(L_1+L_2(e-1)\big)\,\|x-y\|_\infty\,\frac{d\tau}{\tau} \\ &\le \frac{\big(L_1+L_2(e-1)\big)}{\Gamma(\beta+1)}\,\|x-y\|_\infty\,.\end{aligned}$$

Then

$$\begin{aligned}|\mathcal{F}x(t)-\mathcal{F}y(t)| &\le \frac{(q-1)M^{q-2}\big(L_1+L_2(e-1)\big)}{\Gamma(\alpha)\Gamma(\beta+1)}\,\|x-y\|_\infty\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{ds}{s} \\ &\quad+\frac{(q-1)M^{q-2}\big(L_1+L_2(e-1)\big)}{|1-K|\Gamma(\alpha)\Gamma(\beta+1)}\,\|x-y\|_\infty\Big[\int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{ds}{s} \\ &\quad+\int_1^e u(\theta)\int_1^\theta\big(\log\frac{\theta}{s}\big)^{\alpha-1}\frac{ds}{s}d\theta\Big] \\ &\le \frac{(q-1)M^{q-2}\big(L_1+L_2(e-1)\big)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,\|x-y\|_\infty\left(1+\frac{1+\int_1^\theta|u(\theta)|d\theta}{|1-K|}\right).\end{aligned}$$

Thus

$$|\mathcal{F}x(t)-\mathcal{F}y(t)|\le\eta\,\|x-y\|_\infty\,.$$

Since $\eta<1$, we can see that $\mathcal{F}$ is a contraction. As a result of Lemma 2.5, $\mathcal{F}$ admits a unique fixed point which is the unique solution to problem (1). □

4. Applications

In this section, we provide two examples to illustrate our results.

4.1. **Example 1.** We consider the Caputo–Hadamard fractional boundary value problem on $I=[1,e]$

$$\begin{cases} D^{0.7}\big(\varphi_{1.5}(D^{1.5}z(t))\big)=f(t,z(t))+\displaystyle\int_1^t k(t,s,z(s))\,ds, & t\in(1,e),\\ z(1)=0,\quad D^{1.5}z(1)=0,\quad z(e)=\displaystyle\int_1^e u(\theta)\,z(\theta)\,d\theta. \end{cases}$$

Here $p=1.5$ (so $q=3$), $\alpha=1.5$, $\beta=0.7$, and

$$\begin{aligned} f(t,z)&=0.1\arctan t+0.3e^{-t}+\frac{e^{-t}}{5+t^2}|z|^{0.5},\\ k(t,s,z)&=0.2\sin(ts)+0.1\,e^{-(t+s)}|z|^{0.5},\quad u(\theta)=0.1\,e^{-\theta}.\end{aligned}$$

It is clear that f and k are continuous and $u \in L^1([1,e],\mathbb{R})$.
Taking uniform suprema on I (and $1 \le s \le t \le e$), we obtain

$$a_1 = \sup_{t\in[1,e]} |0.1\arctan t + 0.3e^{-t}| \approx 0.18890, \quad a_2 = \sup_{t\in[1,e]} |\frac{e^{-t}}{5+t^2}| \approx 0.06131,$$

$$b_1 = \sup_{1\le s\le t\le e} |0.2\cos(ts)| = 0.2, \quad b_2 = \sup_{1\le s\le t\le e} |0.1e^{-(t+s)}| \approx 0.01353.$$

$$A_0 = \frac{a_1 + b_1(e-1)}{\Gamma(1.7)} \approx 0.59603, \qquad A_1 = \frac{a_2 + b_2(e-1)}{\Gamma(1.7)} \approx 0.09465,$$

$$B_0 = \frac{2^{q-1}A_0^{q-1}}{\Gamma(2.5)} \approx 0.53447, \qquad B_1 = \frac{2^{q-1}A_1^{q-1}}{\Gamma(2.5)} \approx 0.01348,$$

$$K = \int_1^e u(\theta)\ln\theta\, d\theta \approx 0.01347, \qquad \int_1^e |u(\theta)|\, d\theta = 0.1(e^{-1} - e^{-e}) \approx 0.03019.$$

Therefore,

$$R = 1 + \frac{1 + \int_1^e |u(\theta)|\, d\theta}{|1-K|} \approx 1 + \frac{1+0.03019}{|1-0.01347|} \approx 2.04425.$$

The condition (3.1) of Theorem 3.1 requires

$$B_1 R \approx 0.01348 \times 2.04425 \approx 0.02755 < 1,$$

which is satisfied. Hence, we can choose

$$r \ge \frac{B_0 R}{1 - B_1 R} \approx \frac{0.53447 \times 2.04425}{1 - 0.02755} \approx 1.1236.$$

Consequently, by Theorem 3.1, the problem (1) has at least a solution.

4.2. **Example 2.** We consider the Caputo–Hadamard fractional boundary value problem on $I = [1,e]$

$$\begin{cases} D^{0.6}\big(\varphi_{1.5}(D^{1.8}z(t))\big) = f(t,z(t)) + \int_1^t k(t,s,z(s))\, ds, \quad t \in (1,e), \\ z(1) = 0, \quad D^{1.8}z(1) = 0, \quad z(e) = \int_1^e u(\theta)\, z(\theta)\, d\theta, \end{cases}$$

where $p = 1.5$ (hence $q = 3$), $\alpha = 1.8$, $\beta = 0.6$, and

$$f(t,z) = \frac{0.3\cos z + 0.7}{t^3+2}, \qquad k(t,s,z) = 0.1\sin z + \frac{0.04}{s+1}, \qquad u(\theta) = 0.04\,\theta e^{-\theta}.$$

For $(t,s,z) \in I \times I \times \mathbb{R}$, we have

$$L_1 = \sup_{t\in I} \frac{0.3}{t^3+2} = 0.1, \qquad L_2 = \sup_{s\in I} 0.1 = 0.1,$$

$$\|\xi\|_\infty = \sup_{t\in I} \frac{1}{t^3+2} = 0.3333, \quad \|\chi\|_\infty = \max_{t\in I} \left[0.1(t-1) + 0.04\ln\frac{t+1}{2}\right] \approx 0.1966.$$

Then

$$M = \frac{\|\xi\|_\infty + \|\chi\|_\infty}{\Gamma(1.6)} \approx \frac{0.33333 + 0.1966}{0.92214} \approx 0.5746.$$

For the nonlocal term, with $u(\theta) = 0.04\,\theta e^{-\theta}$,

$$K = \int_1^e u(\theta) \ln\theta \, d\theta \approx 0.0048, \qquad \int_1^e |u(\theta)|\, d\theta = 0.1(e^{-1} - e^{-e}) \approx 0.01948.$$

Therefore

$$R \;=\; 1 + \frac{1 + \displaystyle\int_1^e |u(\theta)|\, d\theta}{|1-K|} \approx 1 + \frac{1 + 0.01948}{1 - 0.0048} \approx 2.0247,$$

and the contraction constant

$$\eta = \frac{(q-1)M^{q-2}R\big(L_1 + L_2(e-1)\big)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \approx \frac{2 \cdot 0.5746 \cdot 2.0247 \cdot 0.2718}{1.6765 \cdot 0.92214} \approx 0.41 < 1.$$

Hence $\mathcal{F}$ is a contraction and the problem (1) admits a unique solution.

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