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PONTRYAGIN'S APPROACH TO THE RISK-SENSITIVE CONTROL PROBLEM FOR FRACTIONAL BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates a class of optimal control problems for backward stochastic systems driven by fractional Brownian motion with Hurst parameter varying between 0 and 1, and the cost functional is risk-sensitive. Using variational inequality techniques, we derive optimality conditions under the assumption of a convex set of admissible controls. We first address the risk-neutral case to establish existence of an optimal solution. The framework extends classical control theory to fractional backward stochastic systems. A linear–quadratic stochastic example is provided to illustrate the applicability of the results.

Keywords. Backward stochastic differential equation, fractional Brownian motion, Pontryagin maximum principle, stochastic control, Risk-sensitive optimization
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1. Introduction

In this study, our objective is to derive both conditions for optimality within a risk-sensitive control framework. This objective is accomplished through the use of the Pontryagin-type stochastic maximum principle (SMP). The system under consideration is modeled by a backward stochastic differential equation (BSDE) driven by fractional Brownian motion (fBm) characterized by a Hurst parameter $H \in (0, 1)$.

In a related contribution, Hu and Peng investigated nonlinear BSDEs associated with an fBm in the case where the Hurst parameter exceeds one half, employing tools from Malliavin calculus together with partial differential equation (PDE) techniques. Their work established both the existence and uniqueness of the corresponding solutions, as reported in [13]. Comparable findings were also presented in [14]. Djehiche et al. developed an SMP for a particular class of risk-sensitive mean-field control problems, in their publication [6]. Also Lim and

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Zhou [17] explored the necessary and sufficient conditions for risk-sensitive control problems whose dynamics are described by a stochastic differential equation (SDE).

Building upon these results, Shi and Wu extended the framework to nonlinear forward SDEs incorporating jump processes, under the assumption that the admissible control set has the property of convexity, as presented in [19]. In a broader context, these authors also demonstrated applications of the derived principles to financial models in [20]. Pontryagin-type SMP formulations for risk-sensitive control can be found in references [3] and [4], where the former focuses on systems governed by BSDEs and the latter on those driven by SDEs, with particular attention to singular control scenarios.

For more general formulations of the risk-sensitive stochastic maximum principle, one may refer to the study by Khallout and Chala [15], which investigates forward–backward SDEs. Moreover, in the context of backward doubly SDE, relevant contributions can be found in the work of Hafayed and Chala [8]. Both of these studies consider convex sets of admissible controls.

Our analysis is motivated by the methodological framework introduced by [6]. In their formulation, the drift, diffusion, and initial cost functions are not solely contingent on the state and control variables but also take into account the means of the state process. Within the risk-sensitive control setting, the system dynamics are represented by a nonlinear BSDE driven by an fBm $dy_t^{\upsilon} = -l\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + z_t^{\upsilon} dW_t^H$, $y_T^{\upsilon} = \xi$.

The optimization framework seeks to minimize an initial cost functional that incorporates risk sensitivity, $\mathcal{J}^{\theta}(\upsilon) = \mathbb{E}\left[\exp\theta\left(\int_0^T g\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + f\left(y_0^{\upsilon}\right)\right)\right]$.

A control u is deemed optimal if it constitutes a solution to $\mathcal{J}^{\theta}(u) = \inf_{\upsilon \in \mathcal{U}} \mathcal{J}^{\theta}(\upsilon)$.

The paper successfully resolves the existence of an optimal solution for achieving the objectives at hand, and it establishes both necessary and sufficient optimality conditions for this model. The approach taken involves a stepwise transformation of the risk-sensitive control problem. Initially, it is reformulated in terms of an augmented state process and a terminal payoff problem. This transformation takes into account the convex nature of the set of controls. Subsequently, the intermediate first-order adjoint processes are simplified to a more straightforward form. The optimality conditions are then established, employing the logarithmic transform as demonstrated in Lemma 4.1.

More precisely, following the approach presented by Lim and Zhou in [17], it is shown that one may employ a general square-integrable martingale to transform the pair (p_1, q_1) into the adjoint process $(\widetilde{p}_1(t), 0)$, where $\widetilde{p}_1(t)$ remains a square-integrable martingale. This implies that $\widetilde{p}_1(t)$ is equal to $\widetilde{p}_1(T)$, which is also equal to the constant $\mathbb{E}[\widetilde{p}_1(T)]$. It's important to note that this generic martingale is not directly related to the adjoint process $\overrightarrow{p}(t)$ as discussed in [17], but it is an integral part of the adjoint equation associated with the risk-sensitive SMP, as indicated in Theorem 3.3 bellow.

The remainder of this paper is organized as follows: In Section 2, we present a detailed formulation of the problem, introduce the risk-sensitive model, and state the standing assumptions of our analysis. Section 3 is dedicated to the

analysis of our system of BSDE governed by an fBm and elucidates why we opted to transform the system into forward BSDEs. In Section 4, we present our first primary outcome, which comprises the necessary conditions associated with the risk-sensitive control model, together with several preliminary lemmas. The subsequent Section 5 contains the proofs of these results and introduces a novel transformation technique for the adjoint processes. Section 6 then presents our second main contribution, concerning the sufficient optimality conditions for the risk-sensitive performance criterion. Finally, Section 7 concludes the paper by illustrating the theoretical findings through a risk-sensitive backward stochastic linear–quadratic example, for which an explicit feedback expression of the optimal control is derived.

2. Formulation of the Problem

Let U be nonempty convex subset of the real numbers, and we consider a strictly positive real number T, we define W^H as one dimensional fBm with Hurst parameter H takes values from 0 to 1, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the class of $\mathbb{P}-$null sets of $\mathcal{F}$ as $\mathcal{N}$, we provide the space $(\Omega, \mathcal{F}, \mathbb{P})$ by the natural filtration of the fBm W^H, $\mathcal{F}_t^H = \sigma\left(W_s^H, 0 \leq s \leq t\right) \vee \mathcal{N}$ for each $t \in [0,T]$.

Let $\mathcal{P}^2\left([0,T], \mathbb{R}\right)$ the set of one-dimensional continuous random $\mathcal{F}_t^H$-adapted processes $\{\varphi_t, t \in [0,T]\}$ which satisfy $\|\varphi\|_{\mathcal{P}^2}^2 = \mathbb{E}\left[\sup_{t\in[0,T]} |\varphi_t|^2\right] < \infty$.

In the same way we denote by $\mathcal{N}^2\left([0,T], \mathbb{R}\right)$ the set of one dimensional jointly $\mathcal{F}_t^H$-adapted random processes which satisfy

$\|\varphi\|_{\mathcal{N}^2}^2 = \mathbb{E}\left[\int_0^T \left(Hs^{2H-1}\right)^2 |\varphi_s|^2 \, ds\right] < \infty$.

It is known that an fBm $W^H = \left(W_t^H : 0 \leq t \leq T\right)$ with Hurst index $H \in (0,1)$, is a continuous and centered Gaussian process with covariance function $\mathbb{E}\left[W_t^H W_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$, for all $s, t \in [0,T]$.

We remind the reader that when $H = \frac{1}{2}$, the fractional Brownian motion reduces to a standard Brownian motion.

Theorem 2.1. [2] *Let H be in the interval $(0,1)$. Assume that $\Pi(s,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to $C^{1,2}(\mathbb{R} \times \mathbb{R})$, and that the random variables*

$$\Pi\left(t, W_t^H\right), \quad \int_0^t \frac{\partial \Pi}{\partial s}\left(s, W_s^H\right) ds \text{ and } \int_0^t \frac{\partial^2 \Pi}{\partial x^2}\left(s, W_s^H\right) s^{2H-1} ds,$$

are square integrable for every t, then

$$\begin{aligned}\Pi\left(t, W_t^H\right) &= \Pi(0,0) + \int_0^t \frac{\partial \Pi}{\partial s}\left(s, W_s^H\right) ds + \int_0^t \frac{\partial \Pi}{\partial x}\left(s, W_s^H\right) dW_s^H \\ &\quad + H \int_0^t \frac{\partial^2 \Pi}{\partial x^2}\left(s, W_s^H\right) s^{2H-1} ds.\end{aligned}$$

After some concepts have been given about the fBm, we consider the following BSDE

$$\begin{cases} dy_t^{\upsilon} &= -l\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + z_t^{\upsilon} dW_t^H, \\ y_T^{\upsilon} &= \xi,\ t \in [0,T], \end{cases} \tag{2.1}$$

where $l : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$, and the terminal condition ξ is an $\mathcal{F}_t^H-$ measurable random variable such that $\mathbb{E}|\xi|^2 < \infty$, and υ is an admissible control.

Definition 2.2. We define an admissible control as any process adapted to the filtration $\mathcal{F}_t^H$ and taking values in the set U satisfying the condition $\mathbb{E}\left[\sup_{t\in[0,T]} |\upsilon_t|^2\right] < \infty$. Let $\mathcal{U}$ denote the set of all admissible controls, assumed to be convex.

The minimization criterion, corresponding to the initial risk-sensitive cost, is defined as follows

$$\mathcal{J}^{\theta}(\upsilon) = \mathbb{E}\left[\exp\theta\left(\int_0^T g\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + f\left(y_0^{\upsilon}\right)\right)\right], \tag{2.2}$$

where θ is the risk-sensitive index, $f : \mathbb{R} \to \mathbb{R}$, $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$. Within the framework of the control problem, we seek to minimize the cost functional $\mathcal{J}^{\theta}$ over all admissible controls. If $u \in \mathcal{U}$ represents an optimal control, it satisfies

$$\mathcal{J}(u) = \inf_{\upsilon\in\mathcal{U}} \mathcal{J}(\upsilon). \tag{2.3}$$

To ensure the well-posedness of the problem, these assumptions are required.

Assumption 2.3. -The functions l and g are continuously differentiable in (y, z, u).

-The function f is continuously differentiable, and there exists some $c > 0$, such that for all $y \in \mathbb{R} : |f(y)| \leq c(1 + |y|)$.

-There exists $C > 0$; such that, all the derivatives of l and g are bounded by $C(1 + |y| + |z| + |\upsilon|)$.

Assumption 2.4. The second derivatives of the Hamiltonian $\mathcal{H}$ with respect to (y, z) are bounded.

Proposition 2.5. *The system* (2.1) *admits a unique solution .*

Proof. See the paper [1]. □

Any control satisfying the system $\{(2.1), (2.2), (2.3)\}$ is defined as an optimal control. The objective of this study is to establish a necessary optimality condition for such control, formulated within the framework of the risk-sensitive stochastic maximum principle.

3. On the Risk-Neutral Stochastic Maximum Principle

In this research, the standard approach to solving the problem is not feasible due to the presence of the exponential function in the expression of the cost functional $\mathcal{J}^{\theta}$. To overcome this challenge, a supplementary state process x_t^{υ} has been introduced, which is the solution of the following forward SDE:
$dx_t^{\upsilon} = g\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt$, with the initial condition $x_0^{\upsilon} = 0$.

The backward-type control problem can be cast in an equivalent form as

$$\begin{cases} \inf\limits_{\upsilon\in\mathcal{U}} \mathbb{E}\left[\exp\theta\left(x_T^{\upsilon}+f\left(y_0^{\upsilon}\right)\right)\right]=\inf\limits_{\upsilon\in\mathcal{U}} \mathbb{E}\left[\Delta(x_T^{\upsilon},y_0^{\upsilon})\right], \\ \text{subject to} \\ \begin{array}{rcl} dx_t^{\upsilon} & = & g\left(t,y_t^{\upsilon},z_t^{\upsilon},\upsilon_t\right)dt,\ x_0^{\upsilon}=0, \\ dy_t^{\upsilon} & = & -l\left(t,y_t^{\upsilon},z_t^{\upsilon},\upsilon_t\right)dt+z_t^{\upsilon}dW_t^H,\quad y_T^{\upsilon}=\xi. \end{array} \end{cases} \tag{3.1}$$

We put

$$B_T^{\theta}=\exp\left[\theta\left(\int_0^T g\left(t,y_t^{\upsilon},z_t^{\upsilon},\upsilon_t\right)dt+f\left(y_0^{\upsilon}\right)\right)\right], \tag{3.2}$$

and define $\Gamma_T=\int_0^T g\left(t,y_t^{\upsilon},z_t^{\upsilon},\upsilon_t\right)dt+f\left(y_0^{\upsilon}\right)$, the risk-sensitive loss functional is given by

$$\Gamma_{\theta}=\frac{1}{\theta}\log\mathbb{E}\left[\exp\left\{\theta\Gamma_T\right\}\right].$$

When the risk-sensitive index θ is small, the loss functional Γ_{θ} can be expanded as $\mathbb{E}\left(\Gamma_T\right)+\frac{\theta}{2}Var\left(\Gamma_T\right)+O\left(\theta^2\right)$, where, $Var(\Gamma_T)$ denotes the variance of Γ_T. If θ is less than 0, the variance of Γ_T, as a measure of risk, enhances the performance Γ_{θ}, in which case the optimizer is referred to as a risk seeker. Whereas if θ is more than 0, the variance of Γ_T worsens the performance Γ_{θ}, and the optimizer is labeled as risk averse. For more detailed information, please refer to the paper [21].

Notation 3.1. Throughout this paper, we will use the following notations.

For $\Upsilon\in\{l,\ g\}$, $\tau\in\{\upsilon,u\}$, $\zeta\in\{x,y,z,\upsilon,u\}$ we put

$$\Upsilon\left(t\right)\ =\ \Upsilon\left(t,y_t,z_t,\upsilon_t\right),\ \Upsilon^{\tau}\left(t\right)=\Upsilon\left(t,y_t,z_t,\tau_t\right),\ \Upsilon_{\zeta}\left(t\right)=\frac{\partial\Upsilon}{\partial\zeta}\left(t,y_t,z_t,\upsilon_t\right).$$

$$\begin{cases} \widetilde{\mathcal{H}}^{\theta}\left(t\right) & = & \widetilde{\mathcal{H}}^{\theta}\left(t,x_t,y_t,z_t,\upsilon_t,\overrightarrow{p}\left(t\right),\overrightarrow{q}(t)\right), \\ \widetilde{\mathcal{H}}_{\zeta}^{\theta}\left(t\right) & = & \frac{\partial\widetilde{\mathcal{H}}^{\theta}}{\partial\zeta}\left(t,x_t,y_t,z_t,\upsilon_t,\overrightarrow{p}\left(t\right),\overrightarrow{q}(t)\right). \end{cases}$$

and

$$\begin{cases} \mathcal{H}^{\theta}\left(t\right) & = & \mathcal{H}^{\theta}\left(t,y_t,z_t,\upsilon_t,\widetilde{p}_2\left(t\right),\widetilde{q}_2(t),I^{\theta}\left(t\right),k\left(t\right)\right), \\ \mathcal{H}_{\zeta}^{\theta}\left(t\right) & = & \frac{\partial\mathcal{H}^{\theta}}{\partial\zeta}\left(t,y_t,z_t,\upsilon_t,\widetilde{p}_2\left(t\right),\widetilde{q}_2(t),I^{\theta}\left(t\right),k\left(t\right)\right). \end{cases}$$

If we make the assumption that Assumption 2.3 is valid, we can determine the adjoint equation by applying the forward-backward type control version of the SMP for risk-neutral situations (as outlined in Theorem 3.3 below) to the augmented state dynamics (x,y,z).

Lemma 3.2. *There exists a unique pair of $\mathcal{F}^H$-adapted processes $((p_1,q_1),(p_2,q_2))$, solving the following system of forward-backward SDE.*

$$\begin{cases} d\overrightarrow{p}\left(t\right)=\begin{pmatrix} dp_1\left(t\right) \\ dp_2\left(t\right) \end{pmatrix} & = & \begin{pmatrix} 0 & 0 \\ g_y\left(t\right) & l_y\left(t\right) \end{pmatrix}\begin{pmatrix} p_1\left(t\right) \\ p_2\left(t\right) \end{pmatrix}dt \\ & & +\ \begin{pmatrix} q_1\left(t\right) \\ \frac{\widetilde{\mathcal{H}}_z^{\theta}(t)}{Ht^{2H-1}} \end{pmatrix}dW_t^H, \\ p_1\left(T\right)=\theta B_T^{\theta}, & & p_2\left(0\right)=\theta B_T^{\theta}f_y\left(y_0^u\right), \end{cases} \tag{3.3}$$

with $\mathbb{E}\left[\sum\limits_{i=1}^{2}\sup_{t\in[0,T]}\left|p_i\left(t\right)\right|^2+\int_0^T\left(Ht^{2H-1}\right)^2\left|q_1\left(t\right)\right|^2dt\right]<\infty.$

Proof. In this context, we assume that $\widetilde{\mathcal{H}}^{\theta}$ represents the Hamiltonian associated with the optimal state dynamics (x, y, z) and the pair of adjoint processes $(\overrightarrow{p}(t), \overrightarrow{q}(t))$, where $\widetilde{\mathcal{H}}^{\theta}(t)$ is defined as $g(t)\, p_1(t) + l(t)\, p_2(t)$.

We have

$$\begin{cases} dp_1(t) &= \widetilde{\mathcal{H}}^{\theta}_x(t)\, dt + q_1(t)\, dW^H_t,\ p_1(T) = \Delta_x(x^{\upsilon}_T, y^{\upsilon}_0), \\ dp_2(t) &= \widetilde{\mathcal{H}}^{\theta}_y(t)\, dt + \frac{1}{Ht^{2H-1}} \widetilde{\mathcal{H}}^{\theta}_z(t)\, dW^H_t,\ p_2(0) = \Delta_y(x^{\upsilon}_T, y^{\upsilon}_0), \end{cases}$$

That implies

$$\begin{cases} dp_1(t) &= 0dt + q_1(t)\, dW^H_t, \\ dp_2(t) &= (g_y(t)\, p_1(t) + l_y(t)\, p_2(t))\, dt \\ &+ \frac{1}{Ht^{2H-1}} (g_z(t)\, p_1(t) + l_z(t)\, p_2(t))\, dW^H_t, \\ p_1(T) &= \theta B^{\theta}_T,\ p_2(0) = \theta f_y(y^{\upsilon}_0)\, B^{\theta}_T. \end{cases}$$

Which is equivalent the system (3.3). □

The Theorem below provides the risk-neutral version of the stochastic maximum principle in a forward-backward formulation.

Theorem 3.3. *Assuming that Assumption 2.3 holds, when (x, y, z) represents an optimal solution to the risk-neutral control problem (3.1), there exist pairs of $\mathcal{F}^H_t-$adapted processes (p_1, q_1) and (p_2, q_2) that fulfill (3.3), as follows:*

$$\widetilde{\mathcal{H}}^{\theta}_{\upsilon}(t)\, (\upsilon_t - u_t) \geq 0, \tag{3.4}$$

for all $u \in \mathcal{U}$, almost every $t \in [0, T]$ and $\mathbb{P}$-almost surely.

Proof. See [9]. □

4. Results Statement

In this part, we present the primary result of this work. However, prior to that, we introduce several lemmas that will assist us in our analysis. To begin, we aim to establish the connection between exponential utility and the backward quadratic stochastic equation with a Hurst parameter H. It is crucial to express the expected exponential utility in the following manner.

$$e^{\theta Y^{\theta}_t} = \widetilde{\mathbb{E}}\left[B^{\theta}_{t,T} \mid \mathcal{F}^H_t\right] = \widetilde{\mathbb{E}}\left[\exp\theta\left(\int_t^T g(s)\, ds + f(y^{\upsilon}_0)\right) \mid \mathcal{F}^H_t\right], \tag{4.1}$$

Here, $\widetilde{\mathbb{E}}$ denotes the quasi-conditional expectation introduced by [10]. For further insights into the optimization of expected exponential utility, readers can refer to the papers [5] and [12].

Lemma 4.1. *The expected exponential utility (4.1) possesses a necessary and sufficient condition, which is expressed through the following backward quadratic stochastic equation.*

$$\left\{ dY^{\theta}_t = -\left(g(t) + \frac{\theta H t^{2H-1}}{2} |k(t)|^2\right) dt + k(t)\, dW^H_t,\ Y^{\theta}_T = f\left(y^{\upsilon}_0\right), \right. \tag{4.2}$$

where $\mathbb{E}\left[\int_0^T \left(Ht^{2H-1}\right)^2 |k(t)|^2\, dt\right] < \infty$.

Lemma 4.2. *Suppose that Assumption 2.3 holds, then*

$$\mathbb{E}\left[\sup\left|Y_t^\theta\right|\right] \leq C_T. \tag{4.3}$$

In particular, I^θ solves the following linear BSDE

$$dI_t^\theta = \theta k(t) I_t^\theta dW_t^H, \quad I_T^\theta = B_T^\theta. \tag{4.4}$$

Then, the process defined on $\left(\Omega, \mathcal{F}, \mathcal{F}_t^H, \mathbb{P}\right)$ for every $0 \leq t \leq T$, by

$$\frac{I_t^\theta}{I_0^\theta} = \exp\left(\theta \int_0^t k(s)\, dW_s^H - \frac{\theta^2}{2} \int_0^t H s^{2H-1} |k(s)|^2\, ds\right), \tag{4.5}$$

is a uniformly bounded $\mathcal{F}_t^H$-quasi-martingale.

In the subsequent section, we state and demonstrate the essential optimality conditions for the system governed by SDE with a risk-sensitive performance objective. Subsequently, we present the Risk-Sensitive SMP theorem.

Proposition 4.3. *The risk-sensitive for the adjoint equation satisfied by $(\widetilde{p_2}, \widetilde{q_2})$ and $\left(I^\theta, k\right)$ becomes*

$$\begin{cases} d\widetilde{p_2}(t) &= \mathcal{H}_y^\theta(t)\, dt + \frac{1}{Ht^{2H-1}} \mathcal{H}_z^\theta(t)\, dW_t^{H,\theta}, \quad \widetilde{p_2}(0) = f_y\left(y_0^u\right). \\ dI_t^\theta &= \theta k(t) I_t^\theta dW_t^H, \quad I_T^\theta = B_T^\theta. \end{cases} \tag{4.6}$$

The solution $\left(\widetilde{p}, \widetilde{q}, I^\theta, k\right)$ of the system (4.6) is unique, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{p}(t)|^2 + \sup_{t\in[0,T]} \left|I^\theta(t)\right|^2 + \int_0^T \left(Ht^{2H-1}\right)^2 \left(|\widetilde{q}(t)|^2 + |k(t)|^2\right) dt\right] < \infty, \tag{4.7}$$

where

$$\mathcal{H}^\theta(t) = \left(k(t) - z_t \theta k(t) Ht^{2H-1}\right) \widetilde{p_2}(t) + g(t). \tag{4.8}$$

Theorem 4.4. *(Risk-sensitive SMP) Assume that Assumption 2.3 holds, if (y, z, u) is an optimal solution to the risk-sensitive control problem $\{(2.1), (2.2), (2.3)\}$, then there exist pairs of $\mathcal{F}_t^H-$adapted processes $(\widetilde{p_2}, \widetilde{q_2})$ and $\left(I^\theta, k\right)$ that satisfy (4.6) and (4.7), such that*

$$\mathcal{H}_\upsilon^\theta(t)(\upsilon_t - u_t) \geq 0, \tag{4.9}$$

for all $\upsilon \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}$-almost surely, where the Hamiltonian $\widetilde{\mathcal{H}}^\theta$ associated in (3.1) is given by

$$\widetilde{\mathcal{H}}^\theta(t, x_t, y_t, z_t, \upsilon_t, \overrightarrow{p}(t), \overrightarrow{q}(t)) = \theta I_t^\theta \mathcal{H}^\theta\left(t, y_t, z_t, \upsilon_t, \begin{pmatrix} \widetilde{p_2}(t) \\ \widetilde{q_2}(t) \end{pmatrix}, I_t^\theta, k_t\right), \tag{4.10}$$

where $\mathcal{H}^\theta$ is given by (4.8).

5. Proofs

This portion of the work is dedicated to substantiating the lemmas introduced in the preceding paragraph.

5.1. **Proof of lemma 4.1.** i) From the formula (4.1) we can write

$$\begin{aligned} e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)} &= \widetilde{\mathbb{E}}\left[\exp\theta\left(\int_0^T g(s)\,ds + f\left(y_0^\upsilon\right)\right) \mid \mathcal{F}_t^H\right] \\ &= \widetilde{\mathbb{E}}\left[B_T^\theta \mid \mathcal{F}_t^H\right], \end{aligned}$$

by the fractional Clark-Ocone formula [11], there exists a unique $\mathcal{F}_t^H$-adapted square integrable process M such that

$$e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)} = \mathbb{E}\left[B_T^\theta\right] + \int_0^t M_s dW_s^H. \tag{5.1}$$

We applying Itô formula to (5.1), get

$$\begin{aligned} d\left(e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}\right) &= \left[\theta dY_t^\theta + \theta g(t)\,dt + \frac{1}{2}\theta^2\left\langle dY_t^\theta, dY_t^\theta\right\rangle\right] e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)} \\ &= M_t dW_t^H, \end{aligned}$$

which implies

$$dY_t^\theta + g(t)\,dt + \frac{1}{2}\theta\left\langle dY_t^\theta, dY_t^\theta\right\rangle = \frac{M_t}{\theta} e^{-\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)} dW_t^H. \tag{5.2}$$

Hence the second term is an $\mathcal{F}_t^H$-quasi-martingale, then

$$\begin{aligned} \left\langle dY_t^\theta, dY_t^\theta\right\rangle &= \left(\frac{M_t}{\theta} e^{-\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)} dW_t^H\right)^2 \\ &= Ht^{2H-1}\left|k(t)\right|^2 dt, \end{aligned}$$

where $k(t) := \frac{M_t}{\theta} e^{-\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}$. Then (5.2) becomes becomes

$$dY_t^\theta + g(t)\,dt + \frac{\theta}{2} Ht^{2H-1}\left|k(t)\right|^2 dt = k(t)\,dW_t^H,$$

finally we get

$$\begin{cases} dY_t^\theta = -\left(g(t) + \frac{\theta H t^{2H-1}}{2}\left|k(t)\right|^2\right) dt + k(t)\,dW_t^H, \\ Y_T^\theta = f\left(y_0^\upsilon\right). \end{cases}$$

ii) We will prove the opposite, by applying Itô to $e^{\theta Y_t^\theta}$, and from (4.2) we get

$$d\left(e^{\theta Y_t^\theta}\right) = -\theta e^{\theta Y_t^\theta} g(t)\,dt + \theta e^{\theta Y_t^\theta} k(t)\,dW_t^H. \tag{5.3}$$

On the other side we have

$$d\left(e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}\right) = e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}\theta h(t)\,dt + e^{\theta\int_0^t g(s)ds} d\left(e^{\theta Y_t^\theta}\right),$$

we replacing (5.3) into above equality we get

$$d\left(e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}\right) = e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}\theta k(t)\,dW_t^H,$$

but we have

$$\int_t^T d\left(e^{\theta\left(Y_s^\theta+\int_0^s g(u)du\right)}\right)=e^{\theta\left(Y_T^\theta+\int_0^T g(s)ds\right)}-e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)},$$

that implies

$$e^{\theta\left(Y_T^\theta+\int_0^T g(s)ds\right)}=e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}+\int_t^T e^{\theta\left(Y_s^\theta+\int_0^s g(u)du\right)}\theta k\left(s\right)dW_s^H,$$

by passing mathematical quasi-conditional expectation

$$\widetilde{\mathbb{E}}\left[e^{\theta\left(Y_T^\theta+\int_0^T g(s)ds\right)}\mid\mathcal{F}_t^H\right]=\widetilde{\mathbb{E}}\left[e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}\mid\mathcal{F}_t^H\right]=e^{\theta\left(Y_t^\theta+\int_0^t g(s)ds\right)}.$$

Since $Y_T^\theta=f\left(y_0^v\right)$, finally we get

$$e^{\theta Y_t^\theta}=\widetilde{\mathbb{E}}\left[e^{\theta\left(f\left(y_0^v\right)+\int_t^T g(s)ds\right)}\mid\mathcal{F}_t^H\right]=\widetilde{\mathbb{E}}\left[B_{t,T}^\theta\mid\mathcal{F}_t^H\right],$$

which is equal the form (4.1).

5.2. **Proof of lemma** 4.2**.** We adopt a similar methodology as employed in [6] and introduce a transformation of the adjoint processes (p_1,q_1) and (p_2,q_2) into a new pair by excluding the first component (p_1,q_1). This allows us to formulate the SMP solely in terms of the new process, which comprises only the last two adjoint processes.

At first, we remark that $dp_1(t)=q_1(t)\,dW_t^H$ and $p_1(T)=\theta B_T^\theta$, the explicit solution of this fractional BSDE (see the paper[1]) is

$$p_1(t)=\theta\widetilde{\mathbb{E}}\left[B_T^\theta\mid\mathcal{F}_t^H\right]=\theta I_t^\theta, \tag{5.4}$$

where $I_t^\theta:=\widetilde{\mathbb{E}}\left[B_T^\theta\mid\mathcal{F}_t^H\right]$, $0\le t\le T$.

If we take a close-up look of (5.4), we can determine a transformation of $(\overrightarrow{p},\overrightarrow{q})$ into an adjoint process $(\widetilde{p},\widetilde{q})$, where $\widetilde{p_1}(t)=\frac{1}{\theta I_t^\theta}p_1(t)=1$; that mean

$$\widetilde{p}(t)=\begin{pmatrix}\widetilde{p_1}(t)\\ \widetilde{p_2}(t)\end{pmatrix}:=\frac{1}{\theta I_t^\theta}\overrightarrow{p}(t),\ 0\le t\le T. \tag{5.5}$$

By using (3.3) and (5.5), we have $\widetilde{p_1}(T)=1$ and $\widetilde{p_2}(0)=f_y\left(y_0^v\right)$.

To explore the characteristics of this novel process $(\widetilde{p}(t),\widetilde{q}(t))$, it is imperative to demonstrate the following crucial properties of the quasi-martingale I^θ.

As established in Lemma 4.1, the process Y^θ constitutes the first component of the $\mathcal{F}_t^H$-adapted pair of processes $\left(Y^\theta,k\right)$, serving as the unique solution to the quadratic backward stochastic differential equation (4.2).

Now, we will present Lemma 4.2 proof's

Proof. First, we prove (4.3), we have g and f are bounded by constant $C>0$, and since the exponential function is positive, then from (3.2) we get

$$0\le e^{-(1+T)\theta C}\le B_T^\theta\le e^{(1+T)\theta C}. \tag{5.6}$$

Therefore I^θ is uniformly bounded $\mathcal{F}_t^H$-quasi-martingale satisfying

$$0\le e^{-(1+T)\theta C}\le I_t^\theta\le e^{(1+T)\theta C},\ 0\le t\le T. \tag{5.7}$$

The sufficient conditions of the logarithmic transform established in [7], (Proposition 3.1) can be applied in the quasi-martingale I^θ as follows

$$I_t^\theta = \exp\theta\left(\int_0^t g(s)\,ds + Y_t^\theta\right), \tag{5.8}$$

for every $0 \le t \le T$, and $I_0^\theta = \exp\left(\theta Y_0^\theta\right) = \mathbb{E}\left[B_T^\theta\right]$.

It is very easy to see from (5.7) and the boundedness of g that $\mathbb{E}\left[\sup\left|Y_t^\theta\right|\right] \le C_T$, where, C_T is a positive constant that depends only on T and the boundedness of k and f.

Second, we prove (4.4), by applying Itô to (5.8), then from (4.2) we obtain

$$\begin{aligned} dI_t^\theta &= \theta g(t) I_t^\theta dt + \theta I_t^\theta dY_t^\theta + \frac{\theta^2}{2} I_t^\theta \left\langle dY_t^\theta, dY_t^\theta \right\rangle \\ &= \theta I_t^\theta k(t)\, dW_t^H. \end{aligned}$$

Finally, we can prove (4.5), by replacing formula of Y_t^θ in (5.8), we get

$$\begin{aligned} I_t^\theta &= \exp\left\{\theta\int_0^t g(s)\,ds - \int_0^t \theta\left(g(s) + \tfrac{\theta}{2} H s^{2H-1}|k(s)|^2\right)ds \right. \\ &+ \left. \theta\int_0^t k(s)\,dW_s^H\right\} \\ &= \exp\left\{-\tfrac{\theta^2}{2}\int_0^t H s^{2H-1}|k(s)|^2\,ds + \theta\int_0^t k(s)\,dW_s^H\right\}. \end{aligned}$$

Then

$$d\left(\ln I_t^\theta\right) = d\left(-\frac{\theta^2}{2}\int_0^t H s^{2H-1}|k(s)|^2\,ds + \theta\int_0^t k(s)\,dW_s^H\right).$$

By introducing the integral and from properties Logarithmic function, we obtain

$$\ln\frac{I_t^\theta}{I_0^\theta} = -\frac{\theta^2}{2}\int_0^t H s^{2H-1}|k(s)|^2\,ds + \theta\int_0^t k(s)\,dW_s^H.$$

At last, we have

$$\frac{I_t^\theta}{I_0^\theta} = \exp\left(\theta\int_0^t k(s)\,dW_s^H - \frac{\theta^2}{2}\int_0^t H s^{2H-1}|k(s)|^2\,ds\right).$$

In view of (4.4), the last expression is a uniformly bounded $\mathcal{F}_t^H$-quasi-martingale. □

5.3. **Proof of proposition 4.3.**

Proof. Under the Assumption 2.4, the system (4.6) has a unique solution. Now we identify the processes $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that

$$d\widetilde{p}(t) = \widetilde{\alpha}(t)\,dt + \widetilde{\beta}(t)\,dW_t^H. \tag{5.9}$$

By applying integration by parts formula to the process $\overrightarrow{p}(t) = \theta I^\theta(t)\,\widetilde{p}(t)$, and using the expression of I_t^θ in (4.4), we arrive

$$d\widetilde{p}(t) = \frac{1}{\theta I_t^\theta} d\overrightarrow{p}(t) - \theta\widetilde{p}(t)\,k(t)\,dW_t^H - \theta k(t)\,\widetilde{\beta}(t)\,Ht^{2H-1}dt.$$

According to the system (3.3) , and from the relation (5.5) , we get

$$\begin{aligned} d\widetilde{p}(t) &= \left[\begin{pmatrix} 0 & 0 \\ g_y(t) & l_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} - \theta k(t)\,\widetilde{\beta}(t)\,Ht^{2H-1}\right]dt \\ &+ \left[\begin{pmatrix} \widetilde{q_1}(t) \\ \frac{1}{Ht^{2H-1}}\left(g_z(t)\,\widetilde{p_1}(t) + l_z(t)\,\widetilde{p_2}(t)\right) \end{pmatrix} - \theta\widetilde{p}(t)\,k(t)\right]dW_t^H. \end{aligned} \tag{5.10}$$

By identifying the coefficients between (5.9) and (5.10), we get the drift term

$$\widetilde{\alpha}(t) = \begin{pmatrix} 0 & 0 \\ g_y(t) & l_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} - \theta k(t)\,\widetilde{\beta}(t)\,Ht^{2H-1},$$

and the diffusion term of the process $\widetilde{p}(t)$

$$\widetilde{\beta}(t) = \begin{pmatrix} \widetilde{q_1}(t) \\ \frac{1}{Ht^{2H-1}}\left(g_z(t)\,\widetilde{p_1}(t) + l_z(t)\,\widetilde{p_2}(t)\right) \end{pmatrix} - \theta\widetilde{p}(t)\,k(t). \tag{5.11}$$

Finally, we obtain

$$d\widetilde{p}(t) = \left\{\begin{pmatrix} 0 & 0 \\ g_y(t) & l_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} - \theta k(t)\,\widetilde{\beta}(t)\,Ht^{2H-1}\right\}dt + \widetilde{\beta}(t)\,dW_t^H.$$

It is easily confirmed that

$$\begin{aligned} d\widetilde{p_1}(t) &= -\theta k(t)\,\widetilde{\beta_1}(t)\,Ht^{2H-1}dt + \widetilde{\beta_1}(t)\,dW_t^H \\ &= \widetilde{\beta_1}(t)\left[dW_t^H - \theta k(t)\,Ht^{2H-1}dt\right],\ \widetilde{p_1}(T) = 1. \end{aligned}$$

Bearing in mind (4.5) , the fractional Girsanov's theorem (see [16]) permits us to write

$$d\widetilde{p_1}(t) = \widetilde{\beta_1}(t)\,dW_t^{H,\theta},\ \widetilde{p_1}(T) = 1\ \ \mathbb{P}^\theta - a.s,$$

where

$$dW_t^{H,\theta} = dW_t^H - \theta k(t)\,Ht^{2H-1}dt, \tag{5.12}$$

is a $\mathbb{P}^\theta$ fBm, and

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \exp\left(\theta\int_0^t k(s)\,Hs^{2H-1}dW_s^H - \frac{\theta^2}{2}\int_0^t \|k(s)\|_{\mathcal{N}^2}^2\,ds\right)$$

As stated by in (4.5) and (5.6), the measures associated with the probability space $\mathbb{P}^\theta$ and $\mathbb{P}$ are equivalent. Given that $\widetilde{p_1}(t) = \frac{1}{\theta I_t^\theta}p_1(t)$, is square integrable, we get that $\widetilde{p_1}(t) = \widetilde{\mathbb{E}}^{\mathbb{P}^\theta}\left[\widetilde{p_1}(T) \mid \mathcal{F}_t^H\right] = 1$, and it follows that the process $\widetilde{q_1}(t)$ is a finite quadratic variation, such that $\mathbb{E}\left[\int_0^T |\widetilde{q_1}(t)|^2\,dt\right] = 0$.This implies that, for almost every $0 \le t \le T$, $\widetilde{q_1}(t) = 0$, $\mathbb{P}^\theta$ and $\mathbb{P} - a.s.$ we have

$$d\widetilde{p}(t) = \begin{pmatrix} 0 & 0 \\ g_y(t) & l_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix}dt + \widetilde{\beta}(t)\,dW_t^{H,\theta}. \tag{5.13}$$

By replacing the relation(5.11) into (5.13), we obtain

$$\begin{aligned} d\widetilde{p}(t) &= \begin{pmatrix} 0 & 0 \\ g_y(t) & l_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix}dt \\ &+ \left\{\begin{pmatrix} \widetilde{q_1}(t) \\ \frac{1}{Ht^{2H-1}}\left(g_z(t)\,\widetilde{p_1}(t) + l_z(t)\,\widetilde{p_2}(t)\right) \end{pmatrix} - \theta\widetilde{p}(t)\,k(t)\right\}dW_t^{H,\theta}. \end{aligned} \tag{5.14}$$

Therefore, the second component $\widetilde{p_2}(t)$ given in (5.14) has the form

$$\begin{aligned} d\widetilde{p_2}(t) &= [g_y(t)\widetilde{p_1}(t) + l_y(t)\widetilde{p_2}(t)]\,dt \\ &+ \tfrac{1}{Ht^{2H-1}}\left[g_z(t)\widetilde{p_1}(t) + l_z(t)\widetilde{p_2}(t) - Ht^{2H-1}\theta\widetilde{p_2}(t)k(t)\right]dW_t^{H,\theta}. \end{aligned}$$

The main risk-sensitive for the second adjoint equation satisfied by $(\widetilde{p_2}, \widetilde{q_2})$ and $\left(I^\theta, k\right)$ becomes

$$\begin{cases} d\widetilde{p_2}(t) &= \mathcal{H}_y^\theta(t)\,dt + \frac{1}{Ht^{2H-1}}\mathcal{H}_z^\theta(t)\,dW_t^{H,\theta}, \\ dI_t^\theta &= \theta k(t) I_t^\theta dW_t^H, \\ I_T^\theta &= B_T^\theta,\ \widetilde{p_2}(0) = f_y(y_0^u). \end{cases}$$

The solution $\left(\widetilde{p}, \widetilde{q}, I^\theta, k\right)$ of the system (4.6) is unique, such that (4.7) holds, and the Hamiltonian associated with the optimal state dynamics (y, z) and adjoint processes $\left(\widetilde{p_2}, \widetilde{q_2}, I^\theta, k\right)$ given by (4.8).

The proof is completed. □

To obtain the risk-sensitive SMP stated in Theorem 4.4, expressed via the adjoint processes $(\widetilde{p_2}, \widetilde{q_2})$ and $\left(I^\theta, k\right)$, solving (4.6), we observe that the positivity of I^θ, allows the variational inequality (3.4) can be expressed as $\mathcal{H}_v^\theta(t) \geq 0$, for all $v \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}$-almost surely.

6. Risk-sensitive sufficient optimality conditions

In this section, our focus is on investigating the circumstances under which the necessary optimality condition (3.4) transforms into a sufficient condition for optimality.

Theorem 6.1. *If we assume that the Hamiltonian $\widetilde{\mathcal{H}}^\theta$ is convex for all (y, z, u), and for any $v \in \mathcal{U}$, the process $y_T^v = \xi$ is a one-dimensional $\mathcal{F}_t^H-$measurable random variable with $\mathbb{E}|\xi|^2 < \infty$, then u constitutes an optimal solution to the control problem $\{(2.1), (2.2), (2.3)\}$ if and only if it complies with the necessary optimality condition (3.4).*

Proof. Let $v, u \in \mathcal{U}$ (u be the optimal), we have

$$\mathcal{J}^\theta(v) - \mathcal{J}^\theta(u) = \mathbb{E}\left[\exp\theta\left(x_T^v + f(y_0^v)\right) - \exp\theta\left(x_T^u + f(y_0^u)\right)\right],$$

by a Taylor's expansion of exponential function,then by (3.2), and according to (3.3), we get

$$\begin{aligned} \mathcal{J}^\theta(v) - \mathcal{J}^\theta(u) &= \mathbb{E}\left[\theta f_y(y_0^u)(y_0^v - y_0^u)\exp\theta\left(x_T^u + f(y_0^u)\right)\right] \\ &+ \mathbb{E}\left[\theta\left(x_T^v - x_T^u\right)\exp\theta\left(x_T^u + f(y_0^u)\right)\right] \\ &= \mathbb{E}\left[p_2(0)(y_0^v - y_0^u)\right] + \mathbb{E}\left[p_1(T)(x_T^v - x_T^u)\right]. \end{aligned} \tag{6.1}$$

By applying integration by parts formula to $p_1(t)(x_t^v - x_t^u)$,

$$\mathbb{E}\left[p_1(T)(x_T^v - x_T^u)\right] = \mathbb{E}\left[\int_0^T p_1(t)\left(g^v(t) - g^u(t)\right)dt\right], \tag{6.2}$$

Similarly on $p_2(t)(y_t^\upsilon - y_t^u)$

$$\begin{aligned}\mathbb{E}\left[p_2(0)(y_0^\upsilon - y_0^u)\right] &= \mathbb{E}\left[-\int_0^T (y_t^\upsilon - y_t^u)\,\widetilde{\mathcal{H}}_y^\theta(t)\,dt\right] \\ &+ \mathbb{E}\left[\int_0^T p_2(t)\left(l^\upsilon(t) - l^u(t)\right)dt\right] \\ &- \mathbb{E}\left[\int_0^T \widetilde{\mathcal{H}}_z^\theta(t)\,(z_t^\upsilon - z_t^u)\,dt\right].\end{aligned} \tag{6.3}$$

By replacing (6.2) and (6.3) into (6.1), we get

$$\begin{aligned}\mathcal{J}^\theta(\upsilon) - \mathcal{J}^\theta(u) &= \mathbb{E}\Big[\int_0^T \widetilde{\mathcal{H}}^\theta\left(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, \upsilon_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) \\ &- \widetilde{\mathcal{H}}^\theta\left(t, x_t^u, y_t^u, z_t^u, , u_t\,\overrightarrow{p}(t), \overrightarrow{q}(t)\right)dt\Big] \\ &- \mathbb{E}\left[\int_0^T (y_t^\upsilon - y_t^u)\,\widetilde{\mathcal{H}}_y^\theta(t)\,dt\right] - \mathbb{E}\left[\int_0^T \widetilde{\mathcal{H}}_z^\theta(t)\,(z_t^\upsilon - z_t^u)\,dt\right].\end{aligned} \tag{6.4}$$

With using the Hamiltonian convexity at (y, z, υ) in the above inequality and as a direct result from (3.4), we arrive at

$$\mathcal{J}^\theta(\upsilon) - \mathcal{J}^\theta(u) \geq 0.$$

Hence the result. □

Remark 6.2. Based on (4.10), we can observe that there exists a connection between the Hamiltonian associated with the risk-neutral framework and the Hamiltonian associated with the risk-sensitive framework. In fact, we can express this relationship as follows:

$$\begin{aligned}&\mathcal{J}^\theta(\upsilon) - \mathcal{J}^\theta(u) \\ \geq\ &\mathbb{E}\left[\int_0^T \theta I_t^\theta \mathcal{H}_\upsilon^\theta\left(t, y_t^u, z_t^u, u_t, \binom{\widetilde{p_2}(t)}{\widetilde{q_2}(t)}, I_t^\theta, k(t)\right)(\upsilon_t - u_t)\,dt\right] \\ \geq\ &0.\end{aligned}$$

Since $\theta I_t^\theta > 0$, the last inequality implies $\mathcal{J}^\theta(\upsilon) - \mathcal{J}^\theta(u) \geq 0$.

7. Application

We provide an example of a risk-sensitive backward stochastic linear quadratic problem and present the explicit optimal control in feedback form. We also illustrate our main results, including the Risk-Sensitive SMP, as outlined in Theorem 4.4. We begin by considering the following state dynamics.

$$dy_t = -(\beta y_t + \upsilon_t)\,dt + z_t dW_t^H,\ y_T^\upsilon = \epsilon. \tag{7.1}$$

Our risk-sensitive cost functional is structured in the following manner

$$\mathcal{J}^\theta(\upsilon) = \mathbb{E}\left[\exp\theta\left(-\frac{1}{2}\int_0^T \left(\upsilon_t^2 + y_t^2\right)dt + \frac{1}{2}y_0^2\right)\right], \tag{7.2}$$

where: β is real constant and $\theta > 0$.

Our aim is to minimize over $\mathcal{U}$, (7.2) subject to (7.1) by choosing υ which satisfies the following equality

$$\mathcal{J}(u) = \inf_{\upsilon \in \mathcal{U}} \mathcal{J}(\upsilon). \tag{7.3}$$

Consequently, we can utilize Theorem 4.4 to address our linear-quadratic risk-sensitive stochastic optimal control problem $\{(7.1),(7.2),(7.3)\}$. The Hamiltonian function, as defined in (4.8), is given by

$$\mathcal{H}^{\theta}(t) := \left(\beta y_t + \upsilon_t - z_t \theta k(t) H t^{2H-1}\right) \widetilde{p}_2(t) - \tfrac{1}{2}\left(\upsilon_t^2 + y_t^2\right),$$

minimizing the Hamiltonian over $\mathcal{U}$ we obtain

$$u_t = \widetilde{p}_2(t). \tag{7.4}$$

Then, (7.1) becomes

$$dy_t^u = -\left(\beta y_t^u + \widetilde{p}_2(t)\right) dt + z_t^u dW_t^H, \quad y_T^u = \epsilon. \tag{7.5}$$

In light of (5.12), we can express the optimal adjoint equation, originally stated in (4.6), as follows

$$d\widetilde{p}_2(t) = \left\{-y_t^u + \left(\beta + \theta^2 k^2(t) H t^{2H-1}\right) \widetilde{p}_2(t)\right\} dt - \theta k(t) \widetilde{p}_2(t) dW_t^H. \tag{7.6}$$

By utilizing both (7.5) and (7.6), we have formulated a system known as the fully coupled forward-backward system.

$$\begin{cases} dy_t^u &= -\left(\beta y_t^u + \widetilde{p}_2(t)\right) dt + z_t^u dW_t^H, \\ d\widetilde{p}_2(t) &= \left\{-y_t^u + \left(\beta + \theta^2 k^2(t) H t^{2H-1}\right) \widetilde{p}_2(t)\right\} dt \\ &- \theta k(t) \widetilde{p}_2(t) dW_t^H, \\ y_T^u &= \epsilon, \ \widetilde{p}_2(0) = y_0^u. \end{cases} \tag{7.7}$$

The solution to this system, which is challenging to find explicitly, is addressed by employing a technique similar to the one employed by Yong and Zhou, as described in [22]. To do so, we contemplate the following linear form:

$$\widetilde{p}_2(t) = \chi(t) y_t^u + \varphi(t), \tag{7.8}$$

with χ and φ are deterministic functions. Then applying Itô's formula on (7.8), we obtain

$$d\widetilde{p}_2(t) = \left(\dot{\chi}_t y_t^u - \beta \chi_t y_t^u - \chi_t \widetilde{p}_2(t) + \dot{\varphi}_t\right) dt + \chi_t z_t^u dW_t^H, \tag{7.9}$$

where $\dot{\varphi}$ and $\dot{\chi}_t$ are the derivative of φ and χ with respect to t.

Putting (7.8) into (7.9)

$$d\widetilde{p}_2(t) = \left(\left(\dot{\chi}_t - \beta \chi_t - \chi_t^2\right) y_t^u - \chi_t \varphi_t + \dot{\varphi}_t\right) dt + \chi_t z_t^u dW_t^H. \tag{7.10}$$

On the other hand, after substituting (7.8) into (7.6), we arrive at

$$\begin{array}{rcl} d\widetilde{p}_2(t) &=& \left\{\left(\left(\beta + \theta^2 k^2(t) H t^{2H-1}\right) \chi_t - 1\right) y_t^u \right. \\ &+& \left.\left(\beta + \theta^2 k^2(t) H t^{2H-1}\right) \varphi_t\right\} dt - \theta k(t) \left(\chi_t y_t + \varphi_t\right) dW_t^H. \end{array} \tag{7.11}$$

By identification drift terms between (7.10) and (7.11), we obtain the following Riccati and ordinary differential equations, respectively.

$$\dot{\chi}_t - \chi_t^2 - 2\chi_t \left(\beta + \frac{1}{2}\theta^2 k^2(t) H t^{2H-1}\right) + 1 = 0, \ \chi_0 = 1. \tag{7.12}$$

$$\dot{\varphi}_t - \left(\chi_t + \beta + \theta^2 k^2(t) H t^{2H-1}\right) \varphi_t = 0, \ \varphi_0 = 0. \tag{7.13}$$

By using the same identification, we get $\chi_t z_t = -\theta k(t)(\chi_t y_t + \varphi_t)$, which implies

$$\varphi_t = -\frac{1}{\theta k(t)}\chi_t z_t - \chi_t y_t. \tag{7.14}$$

Finally, from (7.4) and by substituting (7.14) into (7.8) we have

$$u_t(z_t) = \widetilde{p_2}(t) = -\frac{1}{\theta k(t)}\chi_t z_t, \tag{7.15}$$

where $\chi(t)$ is determined by (7.12).

Theorem 7.1. *We assume that (χ_t, φ_t) are the solutions of system (7.12) and (7.13), then the optimal control of our linear-quadratic risk-sensitive stochastic optimal control problem $\{(7.1), (7.2), (7.3)\}$ takes on the state feedback form as depicted in (7.15).*

7.1. **Explicit Solution of the Riccati Equation.** In this part we will derive the explicit solution of the Riccati differential equation given in (7.12). For simplicity we set

$$M = -\left(\beta + \frac{1}{2}\theta^2 k^2(t) H t^{2H-1}\right). \tag{7.16}$$

Therefore we have $\dot{\chi}_t = \chi_t^2 - 2\chi_t M - 1$. It is easily to find the discriminant $\Delta = 4(M^2+1)$, And through it we can find two solutions $\varphi_1 = M - \sqrt{M^2+1}$ and $\varphi_2 = M + \sqrt{M^2+1}$.

Then we get

$$dt = \frac{1}{\chi_t^2 - 2M\chi_t - 1} d\chi_t.$$

Integrating from 0 to t, taking into consideration $\chi_0 = 1$, then after some simplification, we arrive at last

$$\chi_t = \frac{\left(M + \sqrt{M^2+1}\right) - \left(M - \sqrt{M^2+1}\right) B}{1 - B}. \tag{7.17}$$

where

$$B = \left|\frac{1 - \left(M + \sqrt{M^2+1}\right)}{1 - \left(M - \sqrt{M^2+1}\right)}\right| \exp\left(2t\sqrt{M^2+1}\right). \tag{7.18}$$

Corollary 7.2. *The Riccati equation (7.12) admits an explicit solution, which is presented in (7.17), where the coefficients M and B are given by (7.16) and (7.18), respectively.*

8. Conclusion

This paper examines a backward stochastic differential system driven by a fractional Brownian motion. Two main results are established: the necessary optimality conditions (Theorem 4.4) and the sufficient optimality conditions (Theorem 6.1). The analysis relies on the convexity of the Hamiltonian and employs an exponential utility cost, following the framework of [4]. A link between the expected exponential utility and a quadratic backward stochastic differential equation is established in Lemma 4.1. We conclude this paper by some Linear Quadratic

application with risk sensitive performance type, where the feedback state of the optimal control is shown after solving Riccati equation.

In contrast to prior studies, such as [3] and [8], which primarily addressed the special case $H = 1/2$, the present work extends the analysis to the full range of Hurst parameters $H \in (0, 1)$. Furthermore, unlike our previous study [9], which focused on a risk-neutral framework, the current paper investigates a risk-sensitive setting, building upon the risk-neutral results as a foundation for addressing the more general problem.

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