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ON COUPLED LANGEVIN-TYPE FRACTIONAL DIFFERENTIAL SYSTEMS WITH MIXED ERDÉLYI–KOBER AND CAPUTO DERIVATIVES

FURKAN ERKAN[1], NUKET AYKUT HAMAL[2*] and SOTIRIS K. NTOUYAS[3]

Abstract. This manuscript investigates the existence and uniqueness of solutions for a coupled system of Langevin-type fractional differential equations involving Erdélyi-Kober and Caputo fractional derivatives, subject to non-separated boundary conditions. Uniqueness is established using the Banach contraction principle, while existence results are derived via the Leray–Schauder alternative and Krasnosel'skii's fixed point theorem. To illustrate the validity and applicability of the theoretical findings, several examples are provided.

Keywords. Coupled systems, Erdélyi-Kober fractional derivative, Caputo fractional derivative, existence and uniqueness, fixed point theory
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1. Introduction

Extending the classical concepts of differentiation and integration to non-integer (fractional) orders has led to the development of fractional calculus, a mathematical framework that offers greater flexibility in modeling complex dynamical systems. Unlike traditional integer-order derivatives, fractional derivatives provide tools for analyzing systems with memory, hereditary properties, and spatial heterogeneity by allowing differentiation and integration of arbitrary (real or complex) order. This generalization has proven especially valuable in disciplines such as physics, engineering, biology, control theory, and finance, where classical models often fall short in capturing real-world behaviors. As such, a thorough understanding of both the theoretical underpinnings and practical implementations of fractional calculus is essential for advancing research in both fundamental and applied sciences. Comprehensive treatments of the subject can be found in the monographs by Samko et al. [18], Gorenflo and Mainardi [9], Podlubny [15], Hilfer [10], and Kilbas et al. [12].

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* Corresponding author

Classical Langevin equations, originally formulated to describe Brownian motion, are limited in their capacity to model anomalous diffusion, long-range memory, and delayed feedback mechanisms—phenomena commonly observed in complex physical, biological, and financial systems. To overcome these limitations, researchers have introduced fractional Langevin equations and their coupled counterparts, incorporating various definitions of fractional derivatives. These models extend the classical Langevin framework to account for nonlocal temporal behavior and enhanced memory effects.

For instance, the work in [20] establishes the existence and uniqueness of solutions for a coupled system of Langevin-type equations involving Riemann–Liouville and Hadamard fractional derivatives, subject to fractional integral boundary conditions. Several related studies have investigated coupled systems of Langevin fractional differential equations with various types of fractional operators and boundary conditions; see [3, 5, 6, 13, 16] for further developments.

The analysis of fractional Langevin equations remains a vibrant and evolving research area, driven by its wide-ranging applications in statistical mechanics, materials science, biological modeling, and quantitative finance. It continues to foster significant theoretical advancements while offering powerful tools for modeling real-world systems exhibiting nonlocal and memory-dependent dynamics.

On the other hand, coupled systems of fractional differential equations have garnered significant attention due to their natural emergence in various complex phenomena across science and engineering. Such systems are essential in modeling interactions between multiple dynamic processes that exhibit memory and hereditary properties. Numerous studies have explored the existence, uniqueness, and qualitative behavior of solutions to these systems; see, for example, [1, 4, 11, 21, 22] for related developments.

Recently, in [17], the authors combined the Caputo and Erdélyi–Kober fractional derivatives to investigate a boundary value problem with nonlocal boundary conditions of the form:

$$\begin{cases} {}^{E}D^{\xi,\gamma}_{\sigma}({}^{C}D^{\vartheta}\nu)(t) = g(t,\nu(t)),\ t\in[0,b], \\ \lambda_1\,\nu(0)+\lambda_2\,\nu(b) = f_1(\nu), \\ \delta_1\,{}^{C}D^{\vartheta-\sigma(1+\xi)}\nu(0)+\delta_2\,{}^{C}D^{\vartheta-\sigma(1+\xi)}\nu(b) = f_2(\nu), \end{cases} \tag{1.1}$$

in which $0<\vartheta,\gamma<1,\ \sigma>0,\ \xi>-1$ with $\vartheta>\sigma(1+\xi),\ g\in C([0,b],\mathbb{R})$, $\lambda_i,\delta_i,\in\mathbb{R},\ i=1,2$ and $f_i: C([0,b],\mathbb{R})\longrightarrow\mathbb{R},\ i=1,2$.

In the present paper, we analyze the coupled system of fractional Langevin-type equations with mixed Erdélyi–Kober and Caputo fractional derivatives of the form:

$$\begin{cases} {}^{E}D^{\xi_1,\gamma_1}_{\sigma_1}({}^{C}D^{\vartheta_1}+\kappa_1)p_1(t) = \varphi_1(t,p_1(t),p_2(t)), \quad t\in[0,b], \\ {}^{E}D^{\xi_2,\gamma_2}_{\sigma_2}({}^{C}D^{\vartheta_2}+\kappa_2)p_2(t) = \varphi_2(t,p_1(t),p_2(t)), \quad t\in[0,b], \\ \lambda_1\,p_1(0)+\mu_1\,{}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)}p_2(0)=0,\ \lambda_2\,p_2(0)+\mu_2\,{}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)}p_1(0)=0, \\ \delta_1\,p_1(b)+\zeta_1\,{}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)}p_2(b)=0,\ \delta_2\,p_2(b)+\zeta_2\,{}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)}p_1(b)=0, \end{cases} \tag{1.2}$$

where the differential operators ${}^{E}D_{\sigma_i}^{\xi_i,\gamma_i}$ denote the Erdélyi–Kober fractional derivatives of order $0 < \gamma_i < 1$, with parameters $\sigma_i > 0$, $\xi_i > -1$ for $i = 1, 2$ and ${}^{C}D^{\vartheta_i}$ denote the Caputo fractional derivatives of order $0 < \vartheta_i < 1$ and $\vartheta_i > \sigma_i(1 + \xi_i)$, $\kappa_i, \lambda_i, \mu_i, \delta_i, \zeta_i \in \mathbb{R}$ such that $|\lambda_i| + |\mu_i| \neq 0$, $|\delta_i| + |\zeta_i| \neq 0$ for $i = 1, 2$ and the nonlinear functions $\varphi_i : [0, b] \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are continuous.

To the best of our knowledge, this work is the first to investigate coupled systems involving Langevin-type fractional differential equations with mixed Erdélyi-Kober and Caputo derivatives. The results presented here are novel and represent a substantial contribution to the growing body of literature on coupled systems of fractional differential equations, particularly those governed by non-standard fractional operators.

The remainder of this paper is organized as follows: Section 2 reviews the necessary preliminaries of fractional calculus and introduces the notations, definitions, and fundamental concepts used throughout the paper. Section 3 is devoted to establishing the existence and uniqueness of solutions. Uniqueness is proven via Banach's contraction mapping principle, while the existence is demonstrated using the Leray–Schauder alternative and Krasnosel'skii's fixed point theorem. In Section 4, we provide illustrative examples to demonstrate the applicability and effectiveness of the theoretical results.

2. Preliminaries

In this section, we introduce key definitions and supporting lemmas that form the foundation for the theoretical results developed in the subsequent sections.

2.1. **Erdélyi–Kober fractional derivative.**

Definition 2.1. [12, 18]. The Erdélyi–Kober fractional integral of order $\gamma > 0$ with $\sigma > 0$ and $\xi \in \mathbb{R}$ of a continuous function $p : (0, \infty) \to \mathbb{R}$ is defined by

$$ {}^{E}I_{\sigma}^{\xi,\gamma} p(t) = \frac{\sigma t^{-\sigma(\gamma+\xi)}}{\Gamma(\gamma)} \int_0^t s^{\sigma\xi+\sigma-1}(t^{\sigma} - s^{\sigma})^{\gamma-1} p(s) ds, $$

provided the right side is point-wise defined on $\mathbb{R}^+$.

Definition 2.2. [12, 18]. The Erdélyi–Kober fractional derivative ${}^{E}D_{\sigma}^{\xi,\gamma}$ of order $\gamma > 0$, with $n - 1 < \gamma \leq n$, $\sigma > 0$, $\xi \in \mathbb{R}$, is defined by

$$ {}^{E}D_{\sigma}^{\xi,\gamma} p(t) = t^{-\sigma\xi} \left(\frac{1}{\sigma t^{\sigma-1}} \frac{d}{dt} \right)^n t^{\sigma(n+\xi)} \left({}^{E}I_{\sigma}^{\xi+\gamma,n-\gamma} p \right)(t). $$

Lemma 2.3. [14]. *Let $n - 1 < \gamma < n$, $n \in \mathbb{N}$, $\vartheta > -\sigma(\xi + 1)$ and $p \in C_{\vartheta}^{\infty}[a, b]$. Then we get*

$$ {}^{E}I_{\sigma}^{\xi,\gamma} \left({}^{E}D_{\sigma}^{\xi,\gamma} p \right)(t) = p(t) - \sum_{i=0}^{n-1} c_i t^{-\sigma(1+\xi+i)}, $$

where

$$ c_i = \frac{\Gamma(n-i)}{\Gamma(\gamma-i)} \lim_{t\to 0} t^{\sigma(1+\xi+i)} \prod_{j=1+i}^{n-1} \left(1 + \xi + j + \frac{1}{\sigma} t \frac{d}{dt} \right) \left({}^{E}I_{\sigma}^{\xi+\gamma,n-\gamma} p \right)(t). $$

Lemma 2.4. [2]. *Let $\gamma, \sigma, \mu > 0$ and $\xi > -\Big(\dfrac{\mu+\sigma}{\sigma}\Big)$. Then we have*

$$^{E}I_{\sigma}^{\xi,\gamma} t^{\mu} = \frac{t^{\mu}\,\Gamma\left(\xi + \dfrac{\mu}{\sigma} + 1\right)}{\Gamma\left(\xi + \dfrac{\mu}{\sigma} + \gamma + 1\right)}.$$

2.2. **Caputo fractional derivative.**

Definition 2.5. [12]. The Riemann-Liouville fractional integral of order $\vartheta > 0$ for $p \in L^1[a,b]$ is defined by

$$I_{a^+}^{\vartheta} p(t) = \frac{1}{\Gamma(\vartheta)} \int_a^t (t-s)^{\vartheta-1} p(s) ds.$$

Definition 2.6. [12]. For an at least n-times differentiable function $p : [a,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $n-1 < \vartheta < n$ is defined by

$$^{C}D_{a^+}^{\vartheta} p(t) = \frac{1}{\Gamma(n-\vartheta)} \int_a^t (t-s)^{n-\vartheta-1} \left(\frac{d}{ds}\right)^n p(s) ds,$$

where $n = [\vartheta] + 1$, with $[\vartheta]$ representing the integer part of the real number ϑ.

Lemma 2.7. [12]. *For $\vartheta > 0$, the general solution of the Caputo fractional differential equation is given by*

$$I_{a^+}^{\vartheta}\left({}^{C}D_{a^+}^{\vartheta} p(t)\right) = p(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} \left[\left(\frac{d}{dt}\right)^j p(t)\right]_{t=a}, \quad n = [\vartheta]+1.$$

Lemma 2.8. [12]. *Let $\vartheta \in \mathbb{R}^+$. If $p \in C^1[a,b]$, then ${}^{C}D_{a^+}^{\vartheta} I_{a^+}^{\vartheta} p(t) = p(t)$.*

Lemma 2.9. [12]. *Let $\vartheta_1, \vartheta_2, \in \mathbb{R}^+$ with $\vartheta_2 > \vartheta_1$. Then,*

$$^{C}D_{a^+}^{\vartheta_1} I_{a^+}^{\vartheta_2} p(t) = I^{\vartheta_2-\vartheta_1} p(t).$$

Lemma 2.10. [12]. *Let $\vartheta \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$. Then,*

(i) $I_{a^+}^{\vartheta} (t-a)^{\mu-1} = \dfrac{\Gamma(\mu)}{\Gamma(\mu+\vartheta)} (t-a)^{\mu+\vartheta-1}$;

(ii) ${}^{C}D_{a^+}^{\vartheta} (t-a)^{\mu-1} = \dfrac{\Gamma(\mu)}{\Gamma(\mu-\vartheta)} (t-a)^{\mu-\vartheta-1}$.

2.3. **An auxiliary lemma.** The lemma below provides the essential framework for reformulating problem (1.2) as an equivalent integral equation, which facilitates the application of fixed point theorems.

Lemma 2.11. *Let $\hbar_1, \hbar_2 : [0,b] \to \mathbb{R}$ be continuous functions. Then the solution of the following linear system:*

$$\begin{cases} {}^{E}D_{\sigma_1}^{\xi_1,\gamma_1}({}^{C}D^{\vartheta_1} + \kappa_1) p_1(t) = \hbar_1(t), \quad t \in [0,b], \\ {}^{E}D_{\sigma_2}^{\xi_2,\gamma_2}({}^{C}D^{\vartheta_2} + \kappa_2) p_2(t) = \hbar_2(t), \quad t \in [0,b], \\ \lambda_1 p_1(0) + \mu_1 {}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)} p_2(0) = 0, \ \lambda_2 p_2(0) + \mu_2 {}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)} p_1(0) = 0, \\ \delta_1 p_1(b) + \zeta_1 {}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)} p_2(b) = 0, \ \delta_2 p_2(b) + \zeta_2 {}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)} p_1(b) = 0, \end{cases} \tag{2.1}$$

is equivalent to the integral equations

$$\begin{aligned} p_1(t) = \Omega_1 \left\{ \frac{\Phi_2\Lambda_1(b) - \Psi_1\Lambda_2(b)}{\Delta} \right\} t^{\vartheta_1-\sigma_1(1+\xi_1)} \\ + I^{\vartheta_1}\left({}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}\hbar_1(t)\right) - \kappa_1 I^{\vartheta_1} p_1(t) - \Theta_1 \left\{ \frac{\Phi_1\Lambda_2(b) - \Psi_2\Lambda_1(b)}{\Delta} \right\}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} p_2(t) = \Omega_2 \left\{ \frac{\Phi_1\Lambda_2(b) - \Psi_2\Lambda_1(b)}{\Delta} \right\} t^{\vartheta_2-\sigma_2(1+\xi_2)} \\ + I^{\vartheta_2}\left({}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}\hbar_2(t)\right) - \kappa_2 I^{\vartheta_2} p_2(t) - \Theta_2 \left\{ \frac{\Phi_2\Lambda_1(b) - \Psi_1\Lambda_2(b)}{\Delta} \right\}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Lambda_i(b) &= \delta_i\Big(\kappa_i I^{\vartheta_i} p_i(b) - I^{\vartheta_i}\Big({}^{E}I^{\xi_i,\gamma_i}_{\sigma_i}\hbar_i(b)\Big)\Big) \\ &\quad + \zeta_i\left(\kappa_{3-i} I^{\sigma_{3-i}(1+\xi_{3-i})} p_{3-i}(b) - I^{\sigma_{3-i}(1+\xi_{3-i})}\Big({}^{E}I^{\xi_{3-i},\gamma_{3-i}}_{\sigma_{3-i}}\hbar_{3-i}(b)\Big)\right), \\ \Omega_i &= \frac{\Gamma\left(1-\sigma_i(1+\xi_i)\right)}{\Gamma\left(\vartheta_i - \sigma_i(1+\xi_i)+1\right)}, \quad \Phi_i = \delta_i\Omega_i b^{\vartheta_i-\sigma_i(1+\xi_i)}, \\ \Theta_i &= \frac{\mu_i}{\lambda_i}\Gamma\left(1-\sigma_{3-i}(1+\xi_{3-i})\right), \quad \Psi_i = \Theta_i\left[\frac{\zeta_i\lambda_i}{\mu_i} - \delta_i\right], \end{aligned} \tag{2.4}$$

and it is assumed that $\Delta := \Phi_1\Phi_2 - \Psi_1\Psi_2 \neq 0$ *and* $\lambda_i, \mu_i \neq 0$, $i = 1, 2$.

Proof. Let (p_1, p_2) be a solution of the system (2.1). Applying the Erdélyi–Kober integral operators ${}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}$ and ${}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}$ on both sides of the first and second equations in (2.1), respectively, and using Lemma 2.3, we obtain for $t \in [0, b]$,

$${}^{C}D^{\vartheta_1} p_1(t) + \kappa_1\, p_1(t) = c_0 t^{-\sigma_1(1+\xi_1)} + {}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}\hbar_1(t), \tag{2.5}$$

$${}^{C}D^{\vartheta_2} p_2(t) + \kappa_2\, p_2(t) = d_0 t^{-\sigma_2(1+\xi_2)} + {}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}\hbar_2(t). \tag{2.6}$$

Now, by applying the Riemann-Liouville integral operators I^{ϑ_1} and I^{ϑ_2} on both sides of equations (2.5) and (2.6), respectively, and applying Lemma 2.7, we obtain

$$p_1(t) = c_0 \frac{\Gamma\left(1-\sigma_1(1+\xi_1)\right)}{\Gamma\left(\vartheta_1 - \sigma_1(1+\xi_1)+1\right)} t^{\vartheta_1-\sigma_1(1+\xi_1)} + I^{\vartheta_1}\left({}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}\hbar_1(t)\right) - \kappa_1 I^{\vartheta_1} p_1(t) + c_1, \tag{2.7}$$

$$p_2(t) = d_0 \frac{\Gamma\left(1-\sigma_2(1+\xi_2)\right)}{\Gamma\left(\vartheta_2 - \sigma_2(1+\xi_2)+1\right)} t^{\vartheta_2-\sigma_2(1+\xi_2)} + I^{\vartheta_2}\left({}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}\hbar_2(t)\right) - \kappa_2 I^{\vartheta_2} p_2(t) + d_1. \tag{2.8}$$

By applying Lemmas 2.9 and 2.10 to the fractional derivatives in the boundary conditions, we obtain

$${}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)} p_1(t) = c_0\,\Gamma\left(1-\sigma_1(1+\xi_1)\right) + I^{\sigma_1(1+\xi_1)}\left({}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}\hbar_1(t)\right) - \kappa_1 I^{\sigma_1(1+\xi_1)} p_1(t), \tag{2.9}$$

$$
{}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)}p_2(t)=d_0\,\Gamma\left(1-\sigma_2(1+\xi_2)\right)+I^{\sigma_2(1+\xi_2)}\left({}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}\hbar_2(t)\right)-\kappa_2 I^{\sigma_2(1+\xi_2)}p_2(t). \tag{2.10}
$$

By applying the boundary conditions $\lambda_1\,p_1(0)+\mu_1\,{}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)}p_2(0)=0$ and $\lambda_2\,p_2(0)+\mu_2\,{}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)}p_1(0)=0$, we obtain

$$
c_1\lambda_1+d_0\mu_1\Gamma\left(1-\sigma_2(1+\xi_2)\right)=0, \tag{2.11}
$$

$$
d_1\lambda_2+c_0\mu_2\Gamma\left(1-\sigma_1(1+\xi_1)\right)=0, \tag{2.12}
$$

while, by applying the boundary conditions $\delta_1\,p_1(b)+\zeta_1\,{}^{C}D^{\vartheta_2-\sigma_2(1+\xi_2)}p_2(b)=0$ and $\delta_2\,p_2(b)+\zeta_2\,{}^{C}D^{\vartheta_1-\sigma_1(1+\xi_1)}p_1(b)=0$, we obtain

$$
\begin{aligned}
&c_0\delta_1\frac{\Gamma\left(1-\sigma_1(1+\xi_1)\right)}{\Gamma\left(\vartheta_1-\sigma_1(1+\xi_1)+1\right)}b^{\vartheta_1-\sigma_1(1+\xi_1)}+c_1\delta_1+d_0\zeta_1\Gamma\left(1-\sigma_2(1+\xi_2)\right)\\
&=\delta_1\left(\kappa_1 I^{\vartheta_1}p_1(b)-I^{\vartheta_1}\left({}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}\hbar_1(b)\right)\right)+\zeta_1\left(\kappa_2 I^{\sigma_2(1+\xi_2)}p_2(b)-I^{\sigma_2(1+\xi_2)}\left({}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}\hbar_2(b)\right)\right),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
&d_0\delta_2\frac{\Gamma\left(1-\sigma_2(1+\xi_2)\right)}{\Gamma\left(\vartheta_2-\sigma_2(1+\xi_2)+1\right)}b^{\vartheta_2-\sigma_2(1+\xi_2)}+d_1\delta_2+c_0\zeta_2\Gamma\left(1-\sigma_1(1+\xi_1)\right)\\
&=\delta_2\left(\kappa_2 I^{\vartheta_2}p_2(b)-I^{\vartheta_2}\left({}^{E}I^{\xi_2,\gamma_2}_{\sigma_2}\hbar_2(b)\right)\right)+\zeta_2\left(\kappa_1 I^{\sigma_1(1+\xi_1)}p_1(b)-I^{\sigma_1(1+\xi_1)}\left({}^{E}I^{\xi_1,\gamma_1}_{\sigma_1}\hbar_1(b)\right)\right).
\end{aligned} \tag{2.14}
$$

Solving the system of equation (2.11) and (2.12) for c_1 and d_1, and replacing to equations (2.13) and (2.14), in view of notations (2.4), we obtain the following linear system

$$
\begin{cases}
c_0\Phi_1+d_0\Psi_1=\Lambda_1(b),\\
c_0\Psi_2+d_0\Phi_2=\Lambda_2(b).
\end{cases} \tag{2.15}
$$

Solving the system (2.15) for c_0 and d_0, we find that

$$
c_0=\frac{\Phi_2\Lambda_1(b)-\Psi_1\Lambda_2(b)}{\Delta},
$$

$$
d_0=\frac{\Phi_1\Lambda_2(b)-\Psi_2\Lambda_1(b)}{\Delta}.
$$

Substituting the values of c_0 and d_0 in (2.11) and (2.12), respectively, we get

$$
c_1=-\Theta_1\left\{\frac{\Phi_1\Lambda_2(b)-\Psi_2\Lambda_1(b)}{\Delta}\right\},
$$

$$
d_1=-\Theta_2\left\{\frac{\Phi_2\Lambda_1(b)-\Psi_1\Lambda_2(b)}{\Delta}\right\}.
$$

Replacing c_0, c_1, d_0 and d_1 in (2.7) and (2.8), we get the solutions (2.2) and (2.3).

Conversely, by applying the Caputo fractional derivatives of orders ϑ_1 and ϑ_2 to (2.2) and (2.3), respectively, and using Lemma 2.8 along with the property that the Caputo derivative of a constant is zero, we obtain

$$ {}^{C}D^{\vartheta_1}p_1(t)+\kappa_1 p_1(t)=\left[\frac{\Phi_2\Lambda_1(b)-\Psi_1\Lambda_2(b)}{\Delta}\right]t^{-\sigma_1(1+\xi_1)}+{}^{E}I_{\sigma_1}^{\xi_1,\gamma_1}\hbar_1(t), \tag{2.16}$$

$$ {}^{C}D^{\vartheta_2}p_2(t)+\kappa_2 p_2(t)=\left[\frac{\Phi_1\Lambda_2(b)-\Psi_2\Lambda_1(b)}{\Delta}\right]t^{-\sigma_2(1+\xi_2)}+{}^{E}I_{\sigma_2}^{\xi_2,\gamma_2}\hbar_2(t). \tag{2.17}$$

Then, by applying the Erdélyi–Kober fractional derivatives of orders γ_1 and γ_2 to (2.16) and (2.17), respectively, we obtain

$$\begin{cases} {}^{E}D_{\sigma_1}^{\xi_1,\gamma_1}({}^{C}D^{\vartheta_1}+\kappa_1)p_1(t)=\hbar_1(t), & t\in[0,b],\\ {}^{E}D_{\sigma_2}^{\xi_2,\gamma_2}({}^{C}D^{\vartheta_2}+\kappa_2)p_2(t)=\hbar_2(t), & t\in[0,b]. \end{cases}$$

To verify that (p_1,p_2) satisfies the boundary conditions in lines 3 and 4 of problem (2.1), we first evaluate (2.2) and (2.3) at $t=0$ and $t=b$ to obtain the first conditions in those lines. Subsequently, to derive the second conditions in the same lines, we apply the Caputo fractional derivatives of orders $\vartheta_1-\sigma_1(1+\xi_1)$ and $\vartheta_2-\sigma_2(1+\xi_2)$ to (2.2) and (2.3), respectively, at the points $t=0$ and $t=b$. By carrying out direct computation, we verify that these second conditions are also satisfied. The proof is completed. □

3. Main Results

Let $C([0,b],\mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on $[0,b]$, equipped with the norm $\|p_1\|_{X_1}=\sup\{|p_1(t)|:t\in[0,b]\}$. The product space $\left(X_1\times X_2,\|\cdot\|_{X_1\times X_2}\right)$ is also a Banach space with the norm $\|(p_1,p_2)\|_{X_1\times X_2}=\|p_1\|_{X_1}+\|p_2\|_{X_2}$ for $(p_1,p_2)\in X_1\times X_2$.

According to Lemma 2.11, we define an operator $T:X_1\times X_2\to X_1\times X_2$ by

$$T(p_1,p_2)(t)=\begin{pmatrix} T_1(p_1,p_2)(t)\\ T_2(p_1,p_2)(t)\end{pmatrix},$$

where

$$\begin{aligned} T_1(p_1,p_2)(t)=\Omega_1&\left\{\frac{\Phi_2\hat\Lambda_1(b)-\Psi_1\hat\Lambda_2(b)}{\Delta}\right\}t^{\vartheta_1-\sigma_1(1+\xi_1)}\\ &+I^{\vartheta_1}\left({}^{E}I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(t,p_1(t),p_2(t))\right)\\ &-\kappa_1 I^{\vartheta_1}p_1(t)-\Theta_1\left\{\frac{\Phi_1\hat\Lambda_2(b)-\Psi_2\hat\Lambda_1(b)}{\Delta}\right\},\quad t\in[0,b], \end{aligned}\tag{3.1}$$

and

$$\begin{aligned} T_2(p_1,p_2)(t)=\Omega_2&\left\{\frac{\Phi_1\hat\Lambda_2(b)-\Psi_2\hat\Lambda_1(b)}{\Delta}\right\}t^{\vartheta_2-\sigma_2(1+\xi_2)}\\ &+I^{\vartheta_2}\left({}^{E}I_{\sigma_2}^{\xi_2,\gamma_2}\varphi_2(t,p_1(t),p_2(t))\right) \end{aligned}$$

$$- \kappa_2 I^{\vartheta_2} p_2(t) - \Theta_2 \left\{ \frac{\Phi_2 \hat{\Lambda}_1(b) - \Psi_1 \hat{\Lambda}_2(b)}{\Delta} \right\}, \quad t \in [0,b], \tag{3.2}$$

where

$$\hat{\Lambda}_i(b) = \delta_i \Big(\kappa_i I^{\vartheta_i} p_i(b) - I^{\vartheta_i} \Big({}^E I^{\xi_i,\gamma_i}_{\sigma_i} \varphi_i(b, p_1(b), p_2(b)) \Big) \Big)$$
$$+ \zeta_i \left(\kappa_{3-i} I^{\sigma_{3-i}(1+\xi_{3-i})} p_{3-i}(b) - I^{\sigma_{3-i}(1+\xi_{3-i})} \Big({}^E I^{\xi_{3-i},\gamma_{3-i}}_{\sigma_{3-i}} \varphi_{3-i}(b, p_1(b), p_2(b)) \Big) \right).$$

For simplicity, we denote

$$\begin{aligned}
\Upsilon_1 &= |\kappa_1| \left(\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)} (\Xi_1 |\delta_1| + 1) + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_2 \right), \\
\Upsilon_2 &= |\kappa_1| \left(|\delta_1| \frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)} \Xi_3 + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_4 \right), \\
\Upsilon_3 &= |\kappa_2| \left(|\delta_2| \frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)} \Xi_2 + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)} \Xi_1 \right), \\
\Upsilon_4 &= |\kappa_2| \left(\frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)} (\Xi_4 |\delta_2| + 1) + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)} \Xi_3 \right), \\
\Upsilon_5 &= \frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)} (\Xi_1 |\delta_1| + 1) + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_2, \\
\Upsilon_6 &= |\delta_1| \frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)} \Xi_3 + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_4, \\
\Upsilon_7 &= |\delta_2| \frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)} \Xi_2 + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)} \Xi_1, \\
\Upsilon_8 &= \frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)} (\Xi_4 |\delta_2| + 1) + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)} \Xi_3,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\Xi_1 &= \frac{1}{|\Delta|} \Big(|\Omega_1| |\Phi_2| b^{\vartheta_1 - \sigma_1(1+\xi_1)} + |\Theta_1| |\Psi_2| \Big), \\
\Xi_2 &= \frac{1}{|\Delta|} \Big(|\Omega_1| |\Psi_1| b^{\vartheta_1 - \sigma_1(1+\xi_1)} + |\Theta_1| |\Phi_1| \Big), \\
\Xi_3 &= \frac{1}{|\Delta|} \Big(|\Omega_2| |\Psi_2| b^{\vartheta_2 - \sigma_2(1+\xi_2)} + |\Theta_2| |\Phi_2| \Big), \\
\Xi_4 &= \frac{1}{|\Delta|} \Big(|\Omega_2| |\Phi_1| b^{\vartheta_2 - \sigma_2(1+\xi_2)} + |\Theta_2| |\Psi_1| \Big),
\end{aligned} \tag{3.4}$$

and

$$Q_i = \Upsilon_{2i-1} + \Upsilon_{2i}, \quad i = 1,2,3,4. \tag{3.5}$$

Our first result addresses the existence and uniqueness of solutions to problem (1.2). The proof is based on the application of Banach's contraction mapping principle [7], a fundamental tool in nonlinear analysis.

Theorem 3.1. *Let $\varphi_1, \varphi_2 : [0,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous functions and $\Delta \neq 0$. Assume that the following condition is satisfied:*

($\mathcal{H}_1$) *There exist constants $\mathcal{L}_1, \mathcal{L}_2 > 0$ such that, for all $t \in [0,b]$ and $p_i, q_i \in \mathbb{R}$, $i = 1,2$,*

$$|\varphi_1(t,p_1,p_2) - \varphi_1(t,q_1,q_2)| \leq \mathcal{L}_1 \left(|p_1 - q_1| + |p_2 - q_2|\right),$$
$$|\varphi_2(t,p_1,p_2) - \varphi_2(t,q_1,q_2)| \leq \mathcal{L}_2 \left(|p_1 - q_1| + |p_2 - q_2|\right).$$

Then the coupled system (1.2) has a unique solution on $[0,b]$ provided that

$$Q_1 + Q_2 + \mathcal{L}_1 Q_3 + \mathcal{L}_2 Q_4 < 1, \tag{3.6}$$

where Q_i, $i = 1,2,3,4$ are given by (3.5).

Proof. Firstly, we show that $T(B_{r_1}) \subset B_{r_1}$, where

$$B_{r_1} = \{(p_1,p_2) \in X_1 \times X_2 : \|(p_1,p_2)\|_{X_1\times X_2} \leq r_1\}$$

is a closed and bounded ball with

$$r_1 \geq \frac{\mathcal{M}_1 Q_3 + \mathcal{M}_2 Q_4}{1 - \left[Q_1 + Q_2 + \mathcal{L}_1 Q_3 + \mathcal{L}_2 Q_4\right]},$$

where

$$\mathcal{M}_1 = \sup_{t\in[0,b]} \{|\varphi_1(t,0,0)| : t \in [0,b]\} < \infty$$

and

$$\mathcal{M}_2 = \sup_{t\in[0,b]} \{|\varphi_2(t,0,0)| : t \in [0,b]\} < \infty,$$

by the assumption ($\mathcal{H}_1$), it follows that

$$\begin{aligned} |\varphi_1(t,p_1(t),p_2(t))| &\leq |\varphi_1(t,p_1(t),p_2(t)) - \varphi_1(t,0,0)| + |\varphi_1(t,0,0)| \\ &\leq \mathcal{L}_1(\|p_1\|_{X_1} + \|p_2\|_{X_2}) + \mathcal{M}_1 \leq \mathcal{L}_1 r_1 + \mathcal{M}_1, \\ |\varphi_2(t,p_1(t),p_2(t))| &\leq |\varphi_2(t,p_1(t),p_2(t)) - \varphi_2(t,0,0)| + |\varphi_2(t,0,0)| \\ &\leq \mathcal{L}_2(\|p_1\|_{X_1} + \|p_2\|_{X_2}) + \mathcal{M}_2 \leq \mathcal{L}_2 r_1 + \mathcal{M}_2. \end{aligned}$$

Using the notations in (3.4), for each $(p_1,p_2) \in B_r$, we obtain

$$\begin{aligned} &|T_1\left(p_1,p_2\right)(t)| \\ &\leq |\Omega_1| \left\{ \frac{|\Phi_2| \left|\hat{\Lambda}_1(b)\right| + |\Psi_1| \left|\hat{\Lambda}_2(b)\right|}{|\Delta|} \right\} b^{\vartheta_1 - \sigma_1(1+\xi_1)} \\ &\quad + \left| I^{\vartheta_1}\left({}^E I_{\sigma_1}^{\xi_1,\gamma_1} \varphi_1(t,p_1(t),p_2(t))\right)\right| + |\kappa_1| \left|I^{\vartheta_1} p_1(t)\right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\Theta_1\right|\left\{\frac{\left|\Phi_1\right|\left|\hat{\Lambda}_2(b)\right|+\left|\Psi_2\right|\left|\hat{\Lambda}_1(b)\right|}{|\Delta|}\right\}, \\
=&\frac{1}{|\Delta|}\Big(\left|\Omega_1\right|\left|\Phi_2\right| b^{\vartheta_1-\sigma_1\left(1+\xi_1\right)}+\left|\Theta_1\right|\left|\Psi_2\right|\Big)\left|\hat{\Lambda}_1(b)\right| \\
&+\frac{1}{|\Delta|}\Big(\left|\Omega_1\right|\left|\Psi_1\right| b^{\vartheta_1-\sigma_1\left(1+\xi_1\right)}+\left|\Theta_1\right|\left|\Phi_1\right|\Big)\left|\hat{\Lambda}_2(b)\right| \\
&+\left|I^{\vartheta_1}\left({ }^E I_{\sigma_1}^{\xi_1, \gamma_1} \varphi_1\left(t, p_1(t), p_2(t)\right)\right)\right|+\left|\kappa_1\right|\left|I^{\vartheta_1} p_1(t)\right| \\
=&\,\Xi_1\left|\hat{\Lambda}_1(b)\right|+\Xi_2\left|\hat{\Lambda}_2(b)\right|+\left|I^{\vartheta_1}\left({ }^E I_{\sigma_1}^{\xi_1, \gamma_1} \varphi_1\left(t, p_1(t), p_2(t)\right)\right)\right|+\left|\kappa_1\right|\left|I^{\vartheta_1} p_1(t)\right|,
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\left|\hat{\Lambda}_1(b)\right| \leq &\left|\delta_1\right|\Big(\left|\kappa_1\right|\left|I^{\vartheta_1} p_1(b)\right|+\left|I^{\vartheta_1}\left({ }^E I_{\sigma_1}^{\xi_1, \gamma_1} \varphi_1\left(b, p_1(b), p_2(b)\right)\right)\right|\Big) \\
&+\left|\zeta_1\right|\left(\left|\kappa_2\right|\left|I^{\sigma_2\left(1+\xi_2\right)} p_2(b)\right|+\left|I^{\sigma_2\left(1+\xi_2\right)}\left({ }^E I_{\sigma_2}^{\xi_2, \gamma_2} \varphi_2\left(b, p_1(b), p_2(b)\right)\right)\right|\right) \\
\leq &\left|\delta_1\right|\left(\left|\kappa_1\right| \frac{b^{\vartheta_1}}{\Gamma\left(\vartheta_1+1\right)}\left\|p_1\right\|_{X_1}+\frac{b^{\vartheta_1} \Gamma\left(\xi_1+1\right)}{\Gamma\left(\xi_1+\gamma_1+1\right) \Gamma\left(\vartheta_1+1\right)}\left\|\varphi_1\right\|\right) \\
&+\left|\zeta_1\right|\left(\left|\kappa_2\right| \frac{b^{\sigma_2\left(1+\xi_2\right)}}{\Gamma\left(\sigma_2\left(1+\xi_2\right)+1\right)}\left\|p_2\right\|_{X_2}\right. \\
&\left.+\frac{b^{\sigma_2\left(1+\xi_2\right)} \Gamma\left(\xi_2+1\right)}{\Gamma\left(\xi_2+\gamma_2+1\right) \Gamma\left(\sigma_2\left(1+\xi_2\right)+1\right)}\left\|\varphi_2\right\|\right),
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\left|\hat{\Lambda}_2(b)\right| \leq &\left|\delta_2\right|\Big(\left|\kappa_2\right|\left|I^{\vartheta_2} p_2(b)\right|+\left|I^{\vartheta_2}\left({ }^E I_{\sigma_2}^{\xi_2, \gamma_2} \varphi_2\left(b, p_1(b), p_2(b)\right)\right)\right|\Big) \\
&+\left|\zeta_2\right|\left(\left|\kappa_1\right|\left|I^{\sigma_1\left(1+\xi_1\right)} p_1(b)\right|+\left|I^{\sigma_1\left(1+\xi_1\right)}\left({ }^E I_{\sigma_1}^{\xi_1, \gamma_1} \varphi_1\left(b, p_1(b), p_2(b)\right)\right)\right|\right) \\
\leq &\left|\delta_2\right|\left(\left|\kappa_2\right| \frac{b^{\vartheta_2}}{\Gamma\left(\vartheta_2+1\right)}\left\|p_2\right\|_{X_2}+\frac{b^{\vartheta_2} \Gamma\left(\xi_2+1\right)}{\Gamma\left(\xi_2+\gamma_2+1\right) \Gamma\left(\vartheta_2+1\right)}\left\|\varphi_2\right\|\right) \\
&+\left|\zeta_2\right|\left(\left|\kappa_1\right| \frac{b^{\sigma_1\left(1+\xi_1\right)}}{\Gamma\left(\sigma_1\left(1+\xi_1\right)+1\right)}\left\|p_1\right\|_{X_1}\right. \\
&\left.+\frac{b^{\sigma_1\left(1+\xi_1\right)} \Gamma\left(\xi_1+1\right)}{\Gamma\left(\xi_1+\gamma_1+1\right) \Gamma\left(\sigma_1\left(1+\xi_1\right)+1\right)}\left\|\varphi_1\right\|\right).
\end{aligned}
\tag{3.9}
$$

Inserting (3.8) and (3.9) into (3.7), together with the notations in (3.3), we obtain

$$
\left|T_1\left(p_1, p_2\right)(t)\right|
$$

$$
\begin{aligned}
&\leq |\kappa_1| \left(\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)} \left(\Xi_1 |\delta_1| + 1\right) + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_2 \right) \|p_1\|_{X_1} \\
&\quad + |\kappa_2| \left(|\delta_2| \frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)} \Xi_2 + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)} \Xi_1 \right) \|p_2\|_{X_2} \\
&\quad + \left(\frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)} \left(\Xi_1 |\delta_1| + 1\right) \right. \\
&\quad \left. + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_2 \right) \Big(\mathcal{L}_1 r_1 + \mathcal{M}_1 \Big) \\
&\quad + \left(|\delta_2| \frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)} \Xi_2 \right. \\
&\quad \left. + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)} \Xi_1 \right) \Big(\mathcal{L}_2 r_1 + \mathcal{M}_2 \Big).
\end{aligned}
$$

Hence,

$$
\|T_1(p_1,p_2)\|_{X_1} \leq \Upsilon_1 \|p_1\|_{X_1} + \Upsilon_3 \|p_2\|_{X_2} + \Upsilon_5 \Big(\mathcal{L}_1 r_1 + \mathcal{M}_1 \Big) + \Upsilon_7 \Big(\mathcal{L}_2 r_1 + \mathcal{M}_2 \Big). \tag{3.10}
$$

In a similar way of computation, we get

$$
\|T_2(p_1,p_2)\|_{X_2} \leq \Upsilon_2 \|p_1\|_{X_1} + \Upsilon_4 \|p_2\|_{X_2} + \Upsilon_6 \Big(\mathcal{L}_1 r_1 + \mathcal{M}_1 \Big) + \Upsilon_8 \Big(\mathcal{L}_2 r_1 + \mathcal{M}_2 \Big). \tag{3.11}
$$

From the two inequalities (3.10) and (3.11) above and using the notations (3.5), we can conclude that

$$
\begin{aligned}
&\|T(p_1,p_2)\|_{X_1\times X_2} \\
&= \|T_1(p_1,p_2)\|_{X_1} + \|T_2(p_1,p_2)\|_{X_2} \\
&\leq Q_1 \|p_1\|_{X_1} + Q_2 \|p_2\|_{X_2} + Q_3 (\mathcal{L}_1 r + \mathcal{M}_1) + Q_4 (\mathcal{L}_2 r + \mathcal{M}_2) \\
&\leq (Q_1+Q_2)(\|p_1\|_{X_1} + \|p_2\|_{X_2}) + r\big(\mathcal{L}_1 Q_3 + \mathcal{L}_2 Q_4\big) + \mathcal{M}_1 Q_3 + \mathcal{M}_2 Q_4 \\
&\leq r_1\big(Q_1 + Q_2 + \mathcal{L}_1 Q_3 + \mathcal{L}_2 Q_4\big) + \mathcal{M}_1 Q_3 + \mathcal{M}_2 Q_4 \\
&\leq r_1
\end{aligned}
$$

which yields that $T(B_{r_1}) \subseteq B_{r_1}$. Next, we will show that the operator T is a contraction.

For each $(p_1,p_2),(q_1,q_2) \in B_r$ and for any $t \in [0,b]$, we have

$$
\begin{aligned}
&|T_1(p_1,p_2)(t) - T_1(q_1,q_2)(t)| \\
&\leq |\kappa_1| \left(\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)} \left(\Xi_1 |\delta_1| + 1\right) + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)} \Xi_2 \right) \|p_1 - q_1\|_{X_1}
\end{aligned}
$$

$$
\begin{aligned}
&+|\kappa_2|\left(|\delta_2|\frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)}\Xi_2+|\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)}\Xi_1\right)\|p_2-q_2\|_{X_2}\\
&+\left(\frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)}\left(\Xi_1|\delta_1|+1\right)\right.\\
&\left.+|\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)}\Xi_2\right)\mathcal{L}_1\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big)\\
&+\left(|\delta_2|\frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)}\Xi_2\right.\\
&\left.+|\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)}\Xi_1\right)\mathcal{L}_2\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|T_1(p_1,p_2)-T_1(q_1,q_2)\|_{X_1}\le{}&\Upsilon_1\|p_1-q_1\|_{X_1}+\Upsilon_3\|p_2-q_2\|_{X_2}\\
&+\mathcal{L}_1\Upsilon_5\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big)\\
&+\mathcal{L}_2\Upsilon_7\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big).
\end{aligned}\tag{3.12}
$$

Similarly, one can find that

$$
\begin{aligned}
\|T_2(p_1,p_2)-T_2(q_1,q_2)\|_{X_2}\le{}&\Upsilon_2\|p_1-q_1\|_{X_1}+\Upsilon_4\|p_2-q_2\|_{X_2}\\
&+\mathcal{L}_1\Upsilon_6\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big)\\
&+\mathcal{L}_2\Upsilon_8\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big).
\end{aligned}\tag{3.13}
$$

From (3.12) and (3.13), we deduce that

$$
\begin{aligned}
&|T(p_1,p_1)(t)-T(q_1,q_2)(t)|\\
&\le Q_1\|p_1-q_1\|_{X_1}+Q_2\|p_2-q_2\|_{X_2}+\mathcal{L}_1Q_3\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big)\\
&\quad+\mathcal{L}_2Q_4\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big)\\
&\le\Big(Q_1+Q_2+\mathcal{L}_1Q_3+\mathcal{L}_2Q_4\Big)\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big).
\end{aligned}
$$

Consequently, it follows that

$$
\begin{aligned}
\|T(p_1,p_2)-T(q_1,q_2)\|_{X_1\times X_2}\le{}&\Big(Q_1+Q_2+\mathcal{L}_1Q_3+\mathcal{L}_2Q_4\Big)\\
&\times\Big(\|p_1-q_1\|_{X_1}+\|p_2-q_2\|_{X_2}\Big),
\end{aligned}
$$

which, in view of condition (3.6), implies that the operator T is a contraction. Hence, by Banach's contraction mapping principle, the operator T has a unique fixed point. Therefore, the system (1.2) has a unique solution on $[0,b]$. This completes the proof. $\square$

In the following result, the existence of solutions to system (1.2) is established using the Leray–Schauder alternative [8].

Theorem 3.2. *Let $\varphi_1, \varphi_2 : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions and $\Delta \neq 0$. Assume that the following condition is satisfied:*

($\mathcal{H}_2$) *There exist constants $u_i, \nu_i \geq 0$ $(i = 1, 2)$ and $u_0 > 0$, $\nu_0 > 0$ such that,*

$$|\varphi_1(t, p_1, p_2)| \leq u_0 + u_1 |p_1| + u_2 |p_2| ,$$
$$|\varphi_2(t, p_1, p_2)| \leq \nu_0 + \nu_1 |p_1| + \nu_2 |p_2| .$$

Then the coupled system (1.2) has at least one solution on $[0, b]$ provided that

$$Q_1 + Q_3 u_1 + Q_4 \nu_1 < 1 \quad \text{and} \quad Q_2 + Q_3 u_2 + Q_4 \nu_2 < 1, \tag{3.14}$$

where Q_i, $i = 1, 2, 3, 4$ are given by (3.5).

Proof. First of all, we show that the operator $T : X_1 \times X_2 \to X_1 \times X_2$ is completely continuous. Note that the continuity of the functions φ_1 and φ_2 implies that the operator T is continuous. Let $B_{r_2} \subseteq X_1 \times X_2$ be a bounded set, where $B_{r_2} = \{(p_1, p_2) \in X_1 \times X_2 : \|(p_1, p_2)\|_{X_1 \times X_2} \leq r_2\}$. For each $(p_1, p_2) \in B_{r_2}$, we obtain

$$\begin{aligned}
&|T_1 (p_1, p_2) (t)| \\
&\leq |\kappa_1| \left(\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1 + 1)} (\Xi_1 |\delta_1| + 1) + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1 + \xi_1) + 1)} \Xi_2 \right) \|p_1\|_{X_1} \\
&\quad + |\kappa_2| \left(|\delta_2| \frac{b^{\vartheta_2}}{\Gamma(\vartheta_2 + 1)} \Xi_2 + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1 + \xi_2) + 1)} \Xi_1 \right) \|p_2\|_{X_2} \\
&\quad + \left(\frac{b^{\vartheta_1} \Gamma(\xi_1 + 1)}{\Gamma(\xi_1 + \gamma_1 + 1)\Gamma(\vartheta_1 + 1)} (\Xi_1 |\delta_1| + 1) \right. \\
&\quad \left. + |\zeta_2| \frac{b^{\sigma_1(1+\xi_1)} \Gamma(\xi_1 + 1)}{\Gamma(\xi_1 + \gamma_1 + 1)\Gamma(\sigma_1(1 + \xi_1) + 1)} \Xi_2 \right) \left(u_0 + u_1 \|p_1\|_{X_1} + u_2 \|p_2\|_{X_2} \right) \\
&\quad + \left(|\delta_2| \frac{b^{\vartheta_2} \Gamma(\xi_2 + 1)}{\Gamma(\xi_2 + \gamma_2 + 1)\Gamma(\vartheta_2 + 1)} \Xi_2 \right. \\
&\quad \left. + |\zeta_1| \frac{b^{\sigma_2(1+\xi_2)} \Gamma(\xi_2 + 1)}{\Gamma(\xi_2 + \gamma_2 + 1)\Gamma(\sigma_2(1 + \xi_2) + 1)} \Xi_1 \right) \left(\nu_0 + \nu_1 \|p_1\|_{X_1} + \nu_2 \|p_2\|_{X_2} \right).
\end{aligned}$$

which leads to

$$\begin{aligned}
\|T_1(p_1, p_2)\|_{X_1} \leq& \left(\Upsilon_1 + \Upsilon_5 u_1 + \Upsilon_7 \nu_1 \right) \|p_1\|_{X_1} + \left(\Upsilon_3 + \Upsilon_5 u_2 + \Upsilon_7 \nu_2 \right) \|p_2\|_{X_2} \\
&+ \Upsilon_5 u_0 + \Upsilon_7 \nu_0.
\end{aligned}$$

In the same way, we have

$$\|T_2(p_1, p_2)\|_{X_2} \leq \left(\Upsilon_2 + \Upsilon_6 u_1 + \Upsilon_8 \nu_1 \right) \|p_1\|_{X_1} + \left(\Upsilon_4 + \Upsilon_6 u_2 + \Upsilon_8 \nu_2 \right) \|p_2\|_{X_2}$$

$$+ \Upsilon_6 u_0 + \Upsilon_8 \nu_0.$$

Hence,

$$\begin{aligned}
&\|T(p_1,p_2)\|_{X_1\times X_2} \\
&= \|T_1(p_1,p_2)\|_{X_1} + \|T_2(p_1,p_2)\|_{X_2} \\
&\le (Q_1 + Q_3 u_1 + Q_4\nu_1)\|p_1\|_{X_1} + (Q_2 + Q_3 u_2 + Q_4 \nu_2)\|p_2\|_{X_2} + Q_3 u_0 + Q_4 \nu_0 \\
&\le (Q_1 + Q_2 + Q_3(u_1+u_2) + Q_4(\nu_1+\nu_2))r_2 + Q_3 u_0 + Q_4\nu_0,
\end{aligned}$$

which implies that $T(B_{r_2})$ is uniformly bounded.

Now we show that the operator T is equicontinuous. Let $t_1, t_2 \in [0,b]$ with $t_1 < t_2$. Then, we obtain

$$\begin{aligned}
&|T_1(p_1,p_2)(t_2) - T_1(p_1,p_2)(t_1)| \\
&\le |\Omega_1| \left\{ \frac{|\Phi_2|\left|\hat{\Lambda}_1(b)\right| + |\Psi_1|\left|\hat{\Lambda}_2(b)\right|}{|\Delta|} \right\} \left| t_2^{\vartheta_1-\sigma_1(1+\xi_1)} - t_1^{\vartheta_1-\sigma_1(1+\xi_1)} \right| \\
&\quad + \left| I^{\vartheta_1}\left({}^E I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(t_2,p_1(t_2),p_2(t_2))\right) - I^{\vartheta_1}\left({}^E I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(t_1,p_1(t_1),p_2(t_1))\right)\right| \\
&\quad + |\kappa_1|\left| I^{\vartheta_1}p_1(t_2) - I^{\vartheta_1}p_1(t_1)\right| \\
&\le \frac{|\Omega_1|}{|\Delta|}\Bigg\{\Bigg[|\kappa_1|\left(|\delta_1|\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)}|\Phi_2| + |\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)}|\Psi_1|\right) \\
&\quad + |\kappa_2|\left(|\delta_2|\frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)}|\Psi_1| + |\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)}|\Phi_2|\right) \\
&\quad + (u_1+u_2)\Bigg(|\delta_1|\frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)}|\Phi_2| \\
&\quad + |\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)}|\Psi_1|\Bigg) \\
&\quad + (\nu_1+\nu_2)\Bigg(|\delta_2|\frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)}|\Psi_1| \\
&\quad + |\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)}|\Phi_2|\Bigg)\Bigg]\Bigg\} r_2\left(t_2^{\vartheta_1-\sigma_1(1+\xi_1)} - t_1^{\vartheta_1-\sigma_1(1+\xi_1)}\right) \\
&\quad + \frac{1}{\Gamma(\vartheta_1+1)}\left\{\left(|\kappa_1| + (u_1+u_2)\frac{\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma+1)}\right)r_2 + u_0\frac{\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)}\right\} \\
&\quad \times\left[2(t_2-t_1)^{\vartheta_1} + (t_2^{\vartheta_1} - t_1^{\vartheta_1})\right],
\end{aligned}$$

which implies that

$$|T_1(p_1,p_2)(t_2)-T_1(p_1,p_2)(t_1)|\to 0, \quad as \quad t_1\to t_2,$$

independently of $(p_1,p_2)\in B_{r_2}$. Analogously, we obtain

$$\begin{aligned}
&|T_2(p_1,p_2)(t_2)-T_2(p_1,p_2)(t_1)|\\
&\le \frac{|\Omega_2|}{|\Delta|}\Bigg\{\Bigg[|\kappa_1|\left(|\delta_1|\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)}|\Psi_2|+|\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)}|\Phi_1|\right)\\
&\quad+|\kappa_2|\left(|\delta_2|\frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)}|\Phi_1|+|\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)}|\Psi_2|\right)\\
&\quad+(u_1+u_2)\Bigg(|\delta_1|\frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)}|\Psi_2|\\
&\quad+|\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)}|\Phi_1|\Bigg)\\
&\quad+(\nu_1+\nu_2)\Bigg(|\delta_2|\frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)}|\Phi_1|\\
&\quad+|\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)}|\Psi_2|\Bigg)\Bigg]\Bigg\}r_2\left(t_2^{\vartheta_2-\sigma_2(1+\xi_2)}-t_1^{\vartheta_2-\sigma_2(1+\xi_2)}\right)\\
&\quad+\frac{1}{\Gamma(\vartheta_2+1)}\left\{\left(|\kappa_2|+(\nu_1+\nu_2)\frac{\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)}\right)r_2+\nu_0\frac{\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)}\right\}\\
&\quad\times\left[2(t_2-t_1)^{\vartheta_2}+(t_2^{\vartheta_2}-t_1^{\vartheta_2})\right],
\end{aligned}$$

which implies that

$$|T_2(p_1,p_2)(t_2)-T_2(p_1,p_2)(t_1)|\to 0, \quad \text{as} \quad t_1\to t_2,$$

independently of $(p_1,p_2)\in B_r$. Consequently, the set $T(B_{r_2})$ is equicontinuous. Hence, by the Arzelá–Ascoli theorem, $T(B_{r_2})$ is relatively compact. It follows that the operator T is completely continuous.

Finally, we show the boundedness of the set

$$S=\{(p_1,p_2)\in X_1\times X_2:(p_1,p_2)=\eta\, T(p_1,p_2),\ 0<\eta<1\}.$$

Let any $(p_1,p_2)\in S$, then $(p_1,p_2)=\eta\, T(p_1,p_2)$. We have, for all $t\in[0,b]$,

$$p_1(t)=\eta\, T_1(p_1,p_2)(t), \quad p_2(t)=\eta\, T_2(p_1,p_2)(t).$$

Then, we get

$$\begin{aligned}
\|p_1\|_{X_1}\le{}&\left(\Upsilon_1+\Upsilon_5u_1+\Upsilon_7\nu_1\right)\|p_1\|_{X_1}+\left(\Upsilon_3+\Upsilon_5u_2+\Upsilon_7\nu_2\right)\|p_2\|_{X_2}\\
&+\Upsilon_5u_0+\Upsilon_7\nu_0,
\end{aligned}$$

$$\|p_2\|_{X_2} \leq \Big(\Upsilon_2 + \Upsilon_6 u_1 + \Upsilon_8 \nu_1\Big) \|p_1\|_{X_1} + \Big(\Upsilon_4 + \Upsilon_6 u_2 + \Upsilon_8 \nu_2\Big) \|p_2\|_{X_2} + \Upsilon_6 u_0 + \Upsilon_8 \nu_0.$$

From the above inequalities, we get

$$\|p_1\|_{X_1} + \|p_2\|_{X_2} \leq (Q_1 + Q_3 u_1 + Q_4 \nu_1) \|p_1\|_{X_1} + (Q_2 + Q_3 u_2 + Q_4 \nu_2) \|p_2\|_{X_2} + Q_3 u_0 + Q_4 \nu_0,$$

which implies that

$$\|(p_1, p_2)\|_{X_1 \times X_2} \leq \frac{Q_3 u_0 + Q_4 \nu_0}{Q^*},$$

where

$$Q^* = \min\Big\{1 - \big[Q_1 + Q_3 u_1 + Q_4 \nu_1\big], 1 - \big[Q_2 + Q_3 u_2 + Q_4 \nu_2\big]\Big\}.$$

Consequently, the set S is bounded. Therefore, by applying the Leray–Schauder alternative, we conclude that the operator T has at least one fixed point, which is a solution of the system (1.2) on $[0, b]$. The proof is completed. □

To conclude, the existence of solutions is also demonstrated by applying Krasnosel'skii's fixed point theorem [19], under suitable conditions.

Theorem 3.3. *Let $\varphi_1, \varphi_2 : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions and $\Delta \neq 0$. Assume that the following condition is satisfied:*

($\mathcal{H}_3$) *There exist nonnegative functions $f_1, f_2 \in C([0, b], \mathbb{R}^+)$ such that, for all $t \in [0, b]$ and $p_i \in \mathbb{R}$, $i = 1, 2$,*

$$|\varphi_i(t, p_1, p_2)| \leq f_i(t), \quad i = 1, 2.$$

Then the coupled system (1.2) has at least one solution on $[0, b]$ provided that

$$Q_1 + Q_2 < 1, \tag{3.15}$$

where Q_i $(i = 1, 2)$ are given by (3.5).

Proof. Firstly, we separate the operators T_1 and T_2, defined by (3.1) and (3.2), respectively, as follows

$$\begin{aligned} T_1 (p_1, p_2)(t) &= T_{1,1} (p_1, p_2)(t) + T_{1,2} (p_1, p_2)(t), \\ T_2 (p_1, p_2)(t) &= T_{2,1} (p_1, p_2)(t) + T_{2,2} (p_1, p_2)(t), \end{aligned}$$

where

$$\begin{aligned} T_{1,1} (p_1, p_2)(t) = &-\frac{1}{\Delta}\Big(\Omega_1 \Phi_2 t^{\vartheta_1 - \sigma_1(1+\xi_1)} + \Theta_1 \Psi_2\Big) \\ &\times \Bigg[\delta_1\Big(I^{\vartheta_1}\Big({}^E I^{\xi_1, \gamma_1}_{\sigma_1} \varphi_1(b, p_1(b), p_2(b))\Big)\Big) \\ &+ \zeta_1 \Big(I^{\sigma_2(1+\xi_2)}\Big({}^E I^{\xi_2, \gamma_2}_{\sigma_2} \varphi_2(b, p_1(b), p_2(b))\Big)\Big)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Delta}\Big(\Omega_1\Psi_1 t^{\vartheta_1-\sigma_1(1+\xi_1)}+\Theta_1\Phi_1\Big)\\
&\times\Bigg[\delta_2\Big(I^{\vartheta_2}\Big({}^{E}I_{\sigma_2}^{\xi_2,\gamma_2}\varphi_2(b,p_1(b),p_2(b))\Big)\Big)\\
&+\zeta_2\left(I^{\sigma_1(1+\xi_1)}\Big({}^{E}I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(b,p_1(b),p_2(b))\Big)\right)\Bigg]\\
&+I^{\vartheta_1}\Big({}^{E}I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(t,p_1(t),p_2(t))\Big),\ t\in[0,b],
\end{aligned}
$$

$$
\begin{aligned}
T_{1,2}\left(p_1,p_2\right)(t)=&\ \frac{1}{\Delta}\Big(\Omega_1\Phi_2 t^{\vartheta_1-\sigma_1(1+\xi_1)}+\Theta_1\Psi_2\Big)\\
&\times\left[\delta_1\kappa_1 I^{\vartheta_1}p_1(b)+\zeta_1\kappa_2 I^{\sigma_2(1+\xi_2)}p_2(b)\right]\\
&-\frac{1}{\Delta}\Big(\Omega_1\Psi_1 t^{\vartheta_1-\sigma_1(1+\xi_1)}+\Theta_1\Phi_1\Big)\\
&\times\left[\delta_2\kappa_2 I^{\vartheta_2}p_2(b)+\zeta_2\kappa_1 I^{\sigma_1(1+\xi_1)}p_1(b)\right]-\kappa_1 I^{\vartheta_1}p_1(t),\ t\in[0,b],
\end{aligned}
$$

$$
\begin{aligned}
T_{2,1}\left(p_1,p_2\right)(t)=&\ \frac{1}{\Delta}\Big(\Omega_2\Psi_2 t^{\vartheta_2-\sigma_2(1+\xi_2)}+\Theta_2\Phi_2\Big)\\
&\times\Bigg[\delta_1\Big(I^{\vartheta_1}\Big({}^{E}I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(b,p_1(b),p_2(b))\Big)\Big)\\
&+\zeta_1\left(I^{\sigma_2(1+\xi_2)}\Big({}^{E}I_{\sigma_2}^{\xi_2,\gamma_2}\varphi_2(b,p_1(b),p_2(b))\Big)\right)\Bigg]\\
&-\frac{1}{\Delta}\Big(\Omega_2\Phi_1 t^{\vartheta_2-\sigma_2(1+\xi_2)}+\Theta_2\Psi_1\Big)\\
&\times\Bigg[\delta_2\Big(I^{\vartheta_2}\Big({}^{E}I_{\sigma_2}^{\xi_2,\gamma_2}\varphi_2(b,p_1(b),p_2(b))\Big)\Big)\\
&+\zeta_2\left(I^{\sigma_1(1+\xi_1)}\Big({}^{E}I_{\sigma_1}^{\xi_1,\gamma_1}\varphi_1(b,p_1(b),p_2(b))\Big)\right)\Bigg]\\
&+I^{\vartheta_2}\Big({}^{E}I_{\sigma_2}^{\xi_2,\gamma_2}\varphi_2(t,p_1(t),p_2(t))\Big),\ t\in[0,b],
\end{aligned}
$$

$$
\begin{aligned}
T_{2,2}\left(p_1,p_2\right)(t)=&-\frac{1}{\Delta}\Big(\Omega_2\Psi_2 t^{\vartheta_2-\sigma_2(1+\xi_2)}+\Theta_2\Phi_2\Big)\\
&\times\left[\delta_1\kappa_1 I^{\vartheta_1}p_1(b)+\zeta_1\kappa_2 I^{\sigma_2(1+\xi_2)}p_2(b)\right]
\end{aligned}
$$

$$+\frac{1}{\Delta}\Big(\Omega_2\Phi_1 t^{\vartheta_2-\sigma_2(1+\xi_2)}+\Theta_2\Psi_1\Big)$$
$$\times\left[\delta_2\kappa_2 I^{\vartheta_2}p_2(b)+\zeta_2\kappa_1 I^{\sigma_1(1+\xi_1)}p_1(b)\right]-\kappa_2 I^{\vartheta_2}p_2(t),\ t\in[0,b].$$

Obviously,

$$T=T_{1,1}+T_{1,2}+T_{2,1}+T_{2,2}.$$

Let us introduce the closed ball

$$B_{r_3}=\{(p_1,p_2)\in X_1\times X_2:\|(p_1,p_2)\|_{X_1\times X_2}\le r_3\}$$

with

$$r_3\ge\frac{Q_3\|f_1\|_{X_1}+Q_4\|f_2\|_{X_2}}{1-\big[Q_1+Q_2\big]}$$

where Q_3 and Q_4 are given by (3.5) and

$$\|f_1\|_{X_1}=\sup_{t\in[0,b]}|f_1(t)|<\infty\quad and\quad \|f_2\|_{X_2}=\sup_{t\in[0,b]}|f_2(t)|<\infty.$$

For any $(p_1,p_2),(q_1,q_2)\in B_{r_3}$, we get

$$|T_{1,1}(p_1,p_2)(t)+T_{1,2}(q_1,q_2)(t)|\le\Upsilon_1\|p_1\|_{X_1}+\Upsilon_3\|p_2\|_{X_2}+\Upsilon_5\|f_1\|_{X_1}+\Upsilon_7\|f_2\|_{X_2}.$$

Similarly, we have

$$|T_{2,1}(p_1,p_2)(t)+T_{2,2}(q_1,q_2)(t)|\le\Upsilon_2\|p_1\|_{X_1}+\Upsilon_4\|p_2\|_{X_2}+\Upsilon_6\|f_1\|_{X_1}+\Upsilon_8\|f_2\|_{X_2}.$$

Thus, we obtain

$$\begin{aligned}\|T_1(p_1,p_2)+T_2(q_1,q_2)\|_{X_1\times X_2}&\le Q_1\|p_1\|_{X_1}+Q_2\|p_2\|_{X_2}+Q_3\|f_1\|_{X_1}+Q_4\|f_2\|_{X_2}\\&\le(Q_1+Q_2)r_3+\|f_1\|_{X_1}+Q_4\|f_2\|_{X_2}\\&\le r_3,\end{aligned}$$

which implies that $T_1(p_1,p_2)+T_2(p_2,q_2)\in B_{r_3}$.

Now, it is proven that the operator $(T_{1,2},T_{2,2})$ is a contraction mapping. For (p_1,p_2), $(q_1,q_2)\in B_{r_3}$ and for any $t\in[0,b]$, we have

$$\begin{aligned}&|T_{1,2}(p_1,p_2)(t)-T_{1,2}(q_1,q_2)(t)|\\&\le\Xi_1\Big[|\delta_1||\kappa_1|I^{\vartheta_1}|p_1(b)-q_1(b)|+|\zeta_1||\kappa_2|I^{\sigma_2(1+\xi_2)}|p_2(b)-q_2(b)|\Big]\\&\quad+\Xi_2\Big[|\delta_2||\kappa_2|I^{\vartheta_2}|p_2(b)-q_2(b)|+|\zeta_2||\kappa_1|I^{\sigma_1(1+\xi_1)}|p_1(b)-q_1(b)|\Big]\\&\quad+|\kappa_1|I^{\vartheta_1}|p_1(t)-q_1(t)|\\&\le|\kappa_1|\left(\frac{b^{\vartheta_1}}{\Gamma(\vartheta_1+1)}(\Xi_1|\delta_1|+1)+|\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}}{\Gamma(\sigma_1(1+\xi_1)+1)}\Xi_2\right)\|p_1-q_1\|_{X_1}\\&\quad+|\kappa_2|\left(|\delta_2|\frac{b^{\vartheta_2}}{\Gamma(\vartheta_2+1)}\Xi_2+|\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}}{\Gamma(\sigma_2(1+\xi_2)+1)}\Xi_1\right)\|p_2-q_2\|_{X_2}\\&=\Upsilon_1\|p_1-q_1\|_{X_1}+\Upsilon_2\|p_2-q_2\|_{X_2}\end{aligned}$$

and consequently, we obtain

$$\left\|T_{1,2}(p_1,p_2)-T_{1,2}(q_1,q_2)\right\|_{X_1} \leq \Upsilon_1 \left\|p_1-q_1\right\|_{X_1} + \Upsilon_3 \left\|p_2-q_2\right\|_{X_2}. \tag{3.16}$$

Similarly, we can find that

$$\left\|T_{2,2}(p_1,p_2)-T_{2,2}(q_1,q_2)\right\|_{X_2} \leq \Upsilon_2 \left\|p_1-q_1\right\|_{X_1} + \Upsilon_4 \left\|p_2-q_2\right\|_{X_2}. \tag{3.17}$$

From the inequalities (3.16) and (3.17), we conclude that

$$\begin{aligned}\left\|(T_{1,2},T_{2,2})(p_1,p_2)-(T_{1,2},T_{2,2})(q_1,q_2)\right\|_{X_1\times X_2} &\leq (Q_1+Q_2)\\ &\quad \times \Big(\left\|p_1-q_1\right\|_{X_1} + \left\|p_2-q_2\right\|_{X_2} \Big),\end{aligned}$$

which means that $(T_{1,2},T_{2,2})$ is a contraction.

Next, we show that the operator $(T_{1,1},T_{2,1})$ is uniformly bounded on B_{r_3}. Continuity of φ_1 and φ_2 implies that the operator $(T_{1,1},T_{2,1})$ is continuous. For any $(p_1,p_2)\in B_{r_3}$, we have

$$\|T_{1,1}(p_1,p_2)\|_{X_1} \leq \Upsilon_5\|f_1\|_{X_1} + \Upsilon_7\|f_2\|_{X_2},$$

and

$$\|T_{2,1}(p_1,p_2)\|_{X_2} \leq \Upsilon_6\|f_1\|_{X_1} + \Upsilon_8\|f_2\|_{X_2}.$$

Consequently, we obtain

$$\|(T_{1,1},T_{2,1})\|_{X_1\times X_2} \leq Q_3\|f_1\|_{X_1} + Q_4\|f_2\|_{X_2}.$$

Hence, $(T_{1,1},T_{2,1})$ is uniformly bounded. Lastly, we will prove the equicontinuity of $(T_{1,1},T_{2,1})$. For $t_1,t_2\in[0,b]$ with $t_1<t_2$ and $(p_1,p_2)\in B_{r_3}$, we have

$$\begin{aligned}
&|T_{1,1}(p_1,p_2)(t_2)-T_{1,1}(p_1,p_2)(t_1)|\\
&\leq \frac{|\Omega_1|}{|\Delta|}\Bigg[\|f_1\|_{X_1}\Bigg(|\delta_1|\frac{b^{\vartheta_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)}|\Phi_2|\\
&\quad + |\zeta_2|\frac{b^{\sigma_1(1+\xi_1)}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\sigma_1(1+\xi_1)+1)}|\Psi_1|\Bigg)\\
&\quad + \|f_2\|_{X_2}\Bigg(|\delta_2|\frac{b^{\vartheta_2}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\vartheta_2+1)}|\Psi_1|\\
&\quad + |\zeta_1|\frac{b^{\sigma_2(1+\xi_2)}\Gamma(\xi_2+1)}{\Gamma(\xi_2+\gamma_2+1)\Gamma(\sigma_2(1+\xi_2)+1)}|\Phi_2|\Bigg)\Bigg]\left(t_2^{\vartheta_1-\sigma_1(1+\xi_1)}-t_1^{\vartheta_1-\sigma_1(1+\xi_1)}\right)\\
&\quad + \frac{\|f_1\|_{X_1}\Gamma(\xi_1+1)}{\Gamma(\xi_1+\gamma_1+1)\Gamma(\vartheta_1+1)}\Big[2(t_2-t_1)^{\vartheta_1}+(t_2^{\vartheta_1}-t_1^{\vartheta_1})\Big],
\end{aligned}$$

which implies that

$$|T_{1,1}(T_{1,1},T_{2,1})(t_2)-T_{1,1}(p_1,p_2)(t_1)|\to 0,$$

as $t_1\to t_2$ independently of $(p_1,p_2)\in B_{r_3}$. Similarly, we obtain

$$|T_{2,1}(p_1,p_2)(t_2)-T_{2,1}(p_1,p_2)(t_1)|\to 0,$$

as $t_1 \to t_2$ independently of $(p_1, p_2) \in B_{r_3}$. Consequently, we obtain,

$$|(T_{1,1}, T_{2,1})(p_1, p_2)(t_2) - (T_{1,1}, T_{2,1})(p_1, p_2)(t_1)| \to 0$$

as $t_1 \to t_2$ independently of $(p_1, p_2) \in B_{r_3}$ and hence, the operator $(T_{1,1}, T_{2,1})$ is equicontinuous. By applying Arzelá–Ascoli Theorem, we conclude that the operator $(T_{1,1}, T_{2,1})$ is compact on B_{r_3}. Consequently, by applying Krasnosel'skiĭ's fixed point theorem, system (1.2) has at least one solution on $[0, b]$. This completes the proof. $\square$

4. Examples

Consider a coupled system of Langevin fractional differential equations with mixed Erdélyi–Kober and Caputo fractional derivatives given by

$$\begin{cases} {}^{E}D_{\frac{1}{3}}^{-\frac{1}{4},\frac{1}{8}}\left({}^{C}D^{\frac{1}{2}} + \dfrac{1}{20}\right)p_1(t) = \varphi_1(t, p_1(t), p_2(t)), \quad t \in \left[0, \dfrac{1}{7}\right], \\ {}^{E}D_{\frac{1}{6}}^{\frac{1}{4},\frac{3}{4}}\left({}^{C}D^{\frac{1}{4}} + \dfrac{1}{10}\right)p_2(t) = \varphi_2(t, p_1(t), p_2(t)), \quad t \in \left[0, \dfrac{1}{7}\right], \\ 8\ p_1(0) + 2\ {}^{C}D^{\frac{1}{4}}p_2(0) = 0, \quad 64\ p_2(0) + 4\ {}^{C}D^{\frac{1}{24}}p_1(0) = 0, \\ \dfrac{1}{16}\ p_1\left(\dfrac{1}{7}\right) + \dfrac{1}{24}\ {}^{C}D^{\frac{1}{4}}p_2\left(\dfrac{1}{7}\right) = 0, \quad \dfrac{1}{32}\ p_2\left(\dfrac{1}{7}\right) + \dfrac{1}{48}\ {}^{C}D^{\frac{1}{24}}p_1\left(\dfrac{1}{7}\right) = 0, \end{cases} \tag{4.1}$$

where $b = \frac{1}{7}$, $\vartheta_1 = \frac{1}{2}$, $\vartheta_2 = \frac{1}{4}$, $\gamma_1 = \frac{1}{8}$, $\gamma_2 = \frac{3}{4}$, $\xi_1 = -\frac{1}{4}$, $\xi_2 = \frac{1}{4}$, $\sigma_1 = \frac{1}{3}$, $\sigma_2 = \frac{1}{6}$, $\kappa_1 = \frac{1}{20}$, $\kappa_2 = \frac{1}{10}$, $\lambda_1 = 8$, $\lambda_2 = 64$, $\mu_1 = 2$, $\mu_2 = 4$, $\delta_1 = \frac{1}{16}$, $\delta_2 = \frac{1}{32}$, $\zeta_1 = \frac{1}{24}$ and $\zeta_2 = \frac{1}{48}$.

With the given data, it is found that $\Omega_1 \approx 1.35195$, $\Omega_2 \approx 1.20061$, $\Phi_1 \approx 0.05194$, $\Phi_2 \approx 0.03459$, $\Theta_1 \approx 0.29343$, $\Theta_2 \approx 0.07658$, $\Psi_1 \approx 0.03056$, $\Psi_2 \approx 0.02313$, $\Delta \approx 0.00109$, $\Xi_1 \approx 32.58715$, $\Xi_2 \approx 31.51376$, $\Xi_3 \approx 25.90825$, $\Xi_4 \approx 54.89908$, $\Upsilon_1 \approx 0.11029$, $\Upsilon_2 \approx 0.07419$, $\Upsilon_3 \approx 0.16560$, $\Upsilon_4 \approx 0.23196$, $\Upsilon_5 \approx 1.95716$, $\Upsilon_6 \approx 1.64888$, $\Upsilon_7 \approx 1.50104$, $\Upsilon_8 \approx 2.38166$, $Q_1 \approx 0.18448$, $Q_2 \approx 0.39756$, $Q_3 \approx 3.60604$, $Q_4 \approx 3.8827$.

(i) Consider the nonlinear functions φ_1 and φ_2 defined by

$$\varphi_1(t, p_1(t), p_2(t)) = \frac{\cos^4 \pi t}{\sqrt{t + 324}}\left(\frac{{p_1}^2 + |p_1|}{1 + |p_1|}\right) + \frac{e^{-\sin^2 t}}{2(t+3)^2}\tan^{-1}|p_2| + t^2 + \frac{1}{2}, \tag{4.2}$$

$$\varphi_2(t, p_1(t), p_2(t)) = \frac{e^{1-(t+1)^2}}{3(t+7)}\sin|p_1| + \frac{\sin^2 \pi t}{t^3 + 21}\left(\frac{{p_2}^2 + |p_2|}{1 + |p_2|}\right) + \sqrt{t} + 2. \tag{4.3}$$

Note that φ_1 and φ_2 satisfy the Lipschitz condition, since

$$|\varphi_1(t, p_1, p_2) - \varphi_1(t, q_1, q_2)| \le \frac{1}{18}\left(|p_1 - q_1| + |p_2 - q_2|\right),$$

$$|\varphi_2(t, p_1, p_2) - \varphi_2(t, q_1, q_2)| \le \frac{1}{21}\left(|p_1 - q_1| + |p_2 - q_2|\right).$$

By choosing, $\mathcal{L}_1 = \frac{1}{18}$ and $\mathcal{L}_2 = \frac{1}{21}$, we obtain

$$Q_1 + Q_2 + \mathcal{L}_1 Q_3 + \mathcal{L}_2 Q_4 \approx 0.96726 < 1.$$

Thus, all assumptions of Theorem 3.1 are satisfied. Hence, the coupled system of Langevin fractional differential equations (4.1) with φ_1 and φ_2 given by (4.2) and (4.3), respectively, has a unique solution on $\left[0, \frac{1}{7}\right]$.

(ii) Consider the nonlinear functions φ_1 and φ_2 defined by

$$\varphi_1(t, p_1(t), p_2(t)) = \frac{\tan^{-1}|p_2|}{3\pi(t+2)^2} + \frac{e^{-\cos^2 p_2}}{8(t+2)}|p_1| + \frac{1}{\sqrt{64 + \sin^8 p_1}}\left(\frac{{p_2}^2}{1+|p_2|}\right), \tag{4.4}$$

$$\varphi_2(t, p_1(t), p_2(t)) = \frac{\sin|p_1|}{12(t+1)} + \frac{1}{\sqrt{81 + \cos^4 p_2}}\left(\frac{{p_1}^2}{1+|p_1|}\right) + \frac{e^{-(p_1+1)^2}}{4(t^6+3)^2}|p_2|. \tag{4.5}$$

Then, we have

$$|\varphi_1(t, p_1, p_2)| \le \frac{1}{24} + \frac{1}{16}|p_1| + \frac{1}{8}|p_2|,$$

$$|\varphi_2(t, p_1, p_2)| \le \frac{1}{12} + \frac{1}{9}|p_1| + \frac{1}{36}|p_2|.$$

By choosing, $u_0 = \frac{1}{24}$, $u_1 = \frac{1}{16}$, $u_2 = \frac{1}{8}$, $\nu_0 = \frac{1}{12}$, $\nu_1 = \frac{1}{9}$ and $\nu_2 = \frac{1}{36}$, we can find that

$$Q_1 + Q_3 u_1 + Q_4 \nu_1 \approx 0.84126 < 1 \quad \textit{and} \quad Q_2 + Q_3 u_2 + Q_4 \nu_2 \approx 0.956167 < 1,$$

Thus, all assumptions of Theorem 3.2 are satisfied. Hence, the coupled system of Langevin fractional differential equations (4.1) with φ_1 and φ_2 given by (4.4) and (4.5), respectively, have at least one solution on $\left[0, \frac{1}{7}\right]$.

(iii) Consider bounded functions φ_1 and φ_2 given by

$$\varphi_1(t, p_1(t), p_2(t)) = \frac{1}{7(t+1)}\left(\frac{|p_1|}{1+|p_1|}\right) + \sin\left(\sqrt{1+|p_2|}\right) + e^{-t}, \tag{4.6}$$

$$\varphi_2(t, p_1(t), p_2(t)) = \cos\left(\sqrt{1+|p_1|}\right) + \frac{1}{t^2+2}\left(\frac{|p_2|}{1+|p_2|}\right) + \ln(t^4+4) + 1. \tag{4.7}$$

Then, we have

$$|\varphi_1(t, p_1, p_2)| \le \frac{1}{7(t+1)} + e^{-t} + 1 := f_1(t),$$

$$|\varphi_2(t, p_1, p_2)| \le \frac{1}{t^2+2} + \ln(t^4+4) + 2 := f_2(t),$$

which means that the condition $(\mathcal{H}_3)$ in Theorem 3.3 is satisfied. Moreover, the inequality (3.15) is fulfilled. Thus, all assumptions of Theorem

3.3 are satisfied. Hence, the coupled system of Langevin fractional differential equations (4.1) with φ_1 and φ_2 given by (4.6) and (4.7), respectively, has at least one solution on $\left[0, \frac{1}{7}\right]$.

5. Conclusions

In this study, we have investigated a coupled system of Langevin-type fractional differential equations characterized by the presence of mixed Erdélyi-Kober and Caputo fractional derivatives. To lay the groundwork for our main results, we initially derived an auxiliary result concerning the linearized form of the system, which enabled us to reformulate the problem within the framework of fixed-point theory. Building upon this formulation, we rigorously established the existence and uniqueness of solutions by employing a combination of analytical tools, including the Banach contraction principle, the Leray–Schauder alternative, and Krasnosel'skii's fixed point theorem.

To substantiate our theoretical findings, we constructed illustrative numerical examples that demonstrate the applicability and accuracy of the proposed methods. The results obtained under this specific configuration are original and offer meaningful contributions to the growing body of research on nonlocal boundary value problems, especially those involving systems with hybrid fractional derivatives of Erdélyi-Kober and Caputo types.

Looking ahead, we intend to extend our investigation to more generalized boundary value problems for coupled fractional systems, particularly exploring the effects of alternative and more complex boundary conditions in the presence of mixed fractional operators.

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[1] Department of Mathematics, Ege University, Bornova, Izmir, 35100, Türkiye
Email address: 91240000352@ogrenci.ege.edu.tr

[2] Department of Mathematics, Ege University, Bornova, Izmir, 35100, Türkiye
Email address: nuket.aykut@ege.edu.tr

[3] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
Email address: sntouyas@uoi.gr