Gulf Journal of Mathematics ISSN: 2309-4966
Vol 22, Issue 1 (2026) 1-11 https://doi.org/10.56947/gjom.v22i1.3834

A MILD SOLUTION APPROACH TO GENERAL FRACTIONAL EVOLUTION EQUATIONS WITH NONLOCAL CONSTRAINTS

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ABSTRACT. We study the generalized conformable derivative $\mathcal{N}_F^\alpha$ and the associated $\mathcal{N}_F^\alpha$–C_0 semigroups on Banach spaces. Using the time change $G_\alpha(t) = \int_0^t F(s,\alpha)\,ds$, we represent the generalized semigroup as $T_F^\alpha(t) = S(G_\alpha(t))$, where S is a classical C_0–semigroup, and obtain the estimate $\|T_F^\alpha(t)\| \leq Me^{\omega G_\alpha(t)}$. An $\mathcal{N}_F^\alpha$–Laplace transform adapted to the function G_α is introduced together with a suitable convolution property. This allows us to derive a variation–of–constants formula for the nonlinear problem $\mathcal{N}_F^\alpha x(t) = Ax(t) + f(t, x(t))$ subject to a nonlocal initial condition $x(0) = x_0 + g(x)$. Under appropriate compactness and growth assumptions on the semigroup and standard continuity hypotheses on the nonlinear terms, the existence of mild solutions is obtained via Schaefer's fixed point theorem, while uniqueness may be obtained under an additional local Lipschitz condition. As an application, we establish the existence of a mild solution for a semilinear heat equation with a nonlocal initial condition driven by $\mathcal{N}_F^{1/2}$. Several known conformable and classical results are recovered as special cases.

Keywords. Generalized fractional operators, mild solution framework, integro-differential systems, fractional integral transforms.
2020 Mathematics Subject Classification. Primary 45N05, 45D05; Secondary 43A15.

1. Introduction

Fractional models have proved to be a flexible tool for describing memory and hereditary effects in diffusion, viscoelasticity, transport, and many other phenomena; see the monographs [14, 15, 16]. Among the various approaches to fractional differentiation, the *conformable* viewpoint has attracted considerable attention since it preserves several classical differential rules while remaining relatively simple for analysis and computation [9, 20, 10].

Date: Received: Nov 14, 2025; Revised: Dec 12, 2025; Accepted: Dec 24, 2025.
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In this paper, we work within a generalized conformable framework based on a weight function $F(\cdot,\alpha)$ and the associated derivative

$$\mathcal{N}_F^{\alpha} f(t) = \lim_{h\to 0} \frac{f\Big(t + \frac{h}{F(t,\alpha)}\Big) - f(t)}{h}, \qquad 0<\alpha\le 1, \quad F(t,\alpha)\neq 0,$$

which reduces to known conformable operators for particular choices of F and encompasses a broad class of time scalings used in applications (see [9, 20, 10, 3] and the transform-based treatments in [19, 23, 1]).

A central objective of this work is to study abstract evolution equations driven by $\mathcal{N}_F^{\alpha}$ within a semigroup framework. Classical C_0-semigroups provide a robust setting for the analysis of existence and qualitative properties of Cauchy problems [4, 5, 6, 7]. In the present context, the time reparametrization

$$G_\alpha(t) := \int_0^t F(s,\alpha)\, ds$$

establishes a natural link between generalized $\mathcal{N}_F^{\alpha}$–semigroups $\{T(t)\}_{t\ge 0}$ and classical C_0-semigroups $\{S(\rho)\}_{\rho\ge 0}$ through the relation $T(t) = S(G_\alpha(t))$. This correspondence preserves strong continuity and yields growth estimates of the form

$$\|T(t)\| \le M e^{\omega G_\alpha(t)},$$

allowing standard semigroup techniques to be transferred to the $\mathcal{N}_F^{\alpha}$ framework; see [7, 4] and the mapping principles discussed in [8, 11].

We are particularly interested in evolution equations subject to nonlocal initial conditions, where the initial state depends on the unknown trajectory through integral functionals. Such problems arise naturally in control theory, diffusion through heterogeneous media, and population dynamics, and have been widely studied in both classical and fractional contexts; see, for example, [21, 22]. Related functional-analytic tools and operator-theoretic settings, including recent developments in modern function spaces, can be found for instance in [13, 12].

Within this setting, we establish an existence theory for mild solutions of abstract evolution equations of the form

$$\mathcal{N}_F^{\alpha} x(t) = Ax(t) + f(t, x(t)), \qquad x(0) = x_0 + g(x),$$

posed in a Banach space and endowed with nonlocal initial conditions. The analysis relies on a variation-of-constants formula obtained via the time change G_α and on Schaefer's fixed point theorem under suitable compactness and growth assumptions. Issues of uniqueness are not the primary focus of this work and are only briefly addressed under additional restrictive conditions.

As an application, we consider a semilinear heat equation with a nonlocal initial condition driven by $\mathcal{N}_F^{1/2}$. This example is intended to illustrate how the abstract framework applies to a concrete partial differential equation, without aiming at a detailed comparison with other fractional models.

The paper is organized as follows. In Section 2, we recall the definition and basic properties of $\mathcal{N}_F^{\alpha}$, the associated semigroups, and the $\mathcal{N}_F^{\alpha}$–Laplace transform [9, 20, 19]. Section 3 is devoted to the mild solution formulation and the main existence result based on semigroup compactness and Schaefer's fixed point theorem

[7]. In Section 4, we present the application to a nonlocal semilinear heat equation, relying on classical properties of the heat semigroup [7] and compactness assumptions inspired by [21, 22].

We introduce the $\mathcal{N}_F^\alpha$-derivative and emphasize its key features.

2. Preliminaries

In this section, we briefly recall the definition of the generalized conformable derivative $\mathcal{N}_F^\alpha$, the associated semigroups, and the $\mathcal{N}_F^\alpha$–Laplace transform. The presentation is intentionally concise, and we restrict ourselves to the results that are needed in the sequel.

2.1. The generalized conformable derivative.

Definition 2.1 ([3])**.** Let $\alpha \in (0,1]$ and let $f : [0,+\infty) \to \mathbb{R}$. The generalized conformable derivative of order α associated with a weight function $F(\cdot,\alpha)$ is defined by

$$\mathcal{N}_F^\alpha f(t) := \lim_{h\to 0} \frac{f\Big(t + \frac{h}{F(t,\alpha)}\Big) - f(t)}{h},$$

whenever the limit exists, with $F(t,\alpha) \neq 0$ for all $t \geq 0$.

The following elementary properties are standard and follow directly from the definition; see [3].

Proposition 2.2 ([3])**.** *Let $\alpha \in (0,1]$ and assume that f, g are $\mathcal{N}_F^\alpha$-differentiable at some $t > 0$. Then:*

(1) $\mathcal{N}_F^\alpha(af + bg) = a\,\mathcal{N}_F^\alpha(f) + b\,\mathcal{N}_F^\alpha(g)$,
(2) $\mathcal{N}_F^\alpha(\lambda) = 0$ *for all* $\lambda \in \mathbb{R}$,
(3) $\mathcal{N}_F^\alpha(fg) = \mathcal{N}_F^\alpha(f)\,g + f\,\mathcal{N}_F^\alpha(g)$,
(4) *if f is differentiable, then* $\mathcal{N}_F^\alpha(f)(t) = \dfrac{f'(t)}{F(t,\alpha)}$.

2.2. $\mathcal{N}_F^\alpha$ semigroups.

Let $G_\alpha(t) := \int_0^t F(s,\alpha)\,ds$.

Definition 2.3. Let X be a Banach space and $0 < \alpha \leq 1$. A family $\{T(t)\}_{t\geq 0} \subset L(X)$ is called an $\mathcal{N}_F^\alpha$–semigroup if

(1) $T(0) = I$,
(2) $T(G_\alpha^{-1}(t+s)) = T(G_\alpha^{-1}(t))\,T(G_\alpha^{-1}(s))$, for all $t, s \geq 0$.

Definition 2.4. An $\mathcal{N}_F^\alpha$–semigroup $\{T(t)\}_{t\geq 0}$ is called strongly continuous (or a C_0–$\mathcal{N}_F^\alpha$–semigroup) if

$$\lim_{t\to 0^+} \|T(t)x - x\| = 0 \quad \text{for all } x \in X.$$

Its infinitesimal generator A is defined by

$$Ax = (\mathcal{N}_F^\alpha T)(0)x, \quad D(A) = \{x \in X : (\mathcal{N}_F^\alpha T)(0)x \text{ exists}\}.$$

The connection between classical C_0–semigroups and $\mathcal{N}_F^\alpha$–semigroups is summarized below.

Proposition 2.5 ([2]). *Let $(S(t))_{t\geq 0}$ be a classical C_0–semigroup on X. Then*

$$T(t) := S(G_\alpha(t)), \quad t \geq 0,$$

defines a C_0–$\mathcal{N}_F^\alpha$–semigroup. Conversely, every C_0–$\mathcal{N}_F^\alpha$–semigroup arises in this way. Moreover, there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq M e^{\omega G_\alpha(t)}, \quad t \geq 0.$$

2.3. The $\mathcal{N}_F^\alpha$–Laplace transform.

Definition 2.6 ([19]). Let $\alpha \in (0,1]$ and let $f : [0,\infty) \to \mathbb{R}$. The $\mathcal{N}_F^\alpha$–Laplace transform of f is defined by

$$\mathcal{L}_F^\alpha(f)(s) := \int_0^\infty e^{-sG_\alpha(t)}\, f(t)\, F(t,\alpha)\, dt,$$

whenever the integral converges.

Proposition 2.7. *Let f be continuous and $\mathcal{N}_F^\alpha$–differentiable on $(0,\infty)$. Then, for all $s > 0$,*

$$\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha f)(s) = s\,\mathcal{L}_F^\alpha(f)(s) - f(0).$$

Remark 2.8. If $x : [0,\infty) \to \mathbb{R}$ is such that the transforms exist, then

$$\mathcal{L}_F^\alpha(x \circ G_\alpha^{-1})(s) = \mathcal{L}(x)(s),$$

where $\mathcal{L}$ denotes the classical Laplace transform.

3. Main results

Throughout this section, we work in a Banach space X and consider a finite time interval $[0,\tau]$. All functions and operators are assumed to satisfy the hypotheses stated below, which ensure that the integral formulation of the problem is well defined and that compactness arguments can be applied. In particular, suitable measurability and integrability conditions on $F(\cdot,\alpha)$ and $f(\cdot,\cdot)$ are imposed to guarantee the existence of the integrals appearing in the mild solution formula.

Let $\alpha \in (0,1]$, let $F(\cdot,\alpha)$ be measurable and locally integrable on $[0,\tau]$, and define

$$G_\alpha(t) := \int_0^t F(s,\alpha)\, ds.$$

We assume that G_α is continuous and strictly increasing, so that G_α^{-1} exists on $[0, G_\alpha(\tau)]$. Let A be the generator of a classical C_0-semigroup $(S(\theta))_{\theta\geq 0}$ on X, and define the associated $\mathcal{N}_F^\alpha$-semigroup by

$$T(t) := S\big(G_\alpha(t)\big), \qquad t \geq 0.$$

Then $T(0) = I$ and

$$T\big(G_\alpha^{-1}(r+s)\big) = T\big(G_\alpha^{-1}(r)\big)\, T\big(G_\alpha^{-1}(s)\big), \qquad r,s \geq 0.$$

We consider the following Cauchy problem:

$$\begin{cases} \mathcal{N}_F^\alpha x(t) = Ax(t) + f(t,x(t)), \quad t \in [0,\tau], \\ x(0) = x_0 + g(x). \end{cases} \tag{3.1}$$

Lemma 3.1. *Every solution of* (3.1) *can be written as*

$$x(t) = S\big(G_\alpha(t)\big)\,[x_0+g(x)] + \int_0^t F(s,\alpha)\,S\big(G_\alpha(t)-G_\alpha(s)\big)\,f(s,x(s))\,ds, \qquad t\in[0,\tau].$$

Equivalently,

$$x(t) = T(t)\,[x_0+g(x)] \;+\; \int_0^t F(s,\alpha)\,T\Big(G_\alpha^{-1}\big(G_\alpha(t)-G_\alpha(s)\big)\Big)\,f(s,x(s))\,ds.$$

Proof. Consider the time change $\theta = G_\alpha(t)$ and set $y(\theta) := x(G_\alpha^{-1}(\theta))$. The $\mathcal{N}_F^\alpha$-Laplace transform

$$\mathcal{L}_F^\alpha(x)(\lambda) = \int_0^\infty F(t,\alpha)\,e^{-\lambda G_\alpha(t)}\,x(t)\,dt$$

equals the classical Laplace transform of y: $\mathcal{L}_F^\alpha(x) = \mathcal{L}(y)$. Using $\mathcal{L}_F^\alpha(\mathcal{N}_F^\alpha x) = \lambda\mathcal{L}_F^\alpha(x) - x(0)$, applying the transform to (3.1) and solving $(\lambda I - A)\,\mathcal{L}(y) = x_0 + g(x) + \mathcal{L}(f\circ G_\alpha^{-1})$, we invert with the classical variation-of-constants formula to get, for $\theta\in[0,G_\alpha(\tau)]$,

$$y(\theta) = S(\theta)[x_0+g(x)] + \int_0^\theta S(\theta-\sigma)\,(f\circ G_\alpha^{-1})(\sigma)\,d\sigma.$$

Returning to t via $\theta = G_\alpha(t)$ and $\sigma = G_\alpha(s)$ (so $d\sigma = F(s,\alpha)\,ds$) yields the stated formula. □

Definition 3.2. A function $x \in \mathscr{C} = C([0,\tau],X)$ is called a mild solution of (3.1) if

$$x(t) = S\big(G_\alpha(t)\big)\,[x_0+g(x)] + \int_0^t F(s,\alpha)\,S\big(G_\alpha(t)-G_\alpha(s)\big)\,f(s,x(s))\,ds, \quad t\in[0,\tau].$$

Equivalently, the same identity can be written with $T(\cdot)$ as above.

To ensure the existence of mild solutions, we make the following assumptions:

- (H1) The nonlinear function $f : [0,\tau]\times X \to X$ is continuous in its second argument and uniformly bounded with respect to t. That is, there exists a bounded measurable function $\mu\in L^\infty([0,\tau],\mathbb{R}^+)$ such that

$$\|f(t,x)\| \le \mu(t), \quad \text{for all } (t,x)\in[0,\tau]\times X.$$

- (H2) For each $x\in X$, the mapping $t\mapsto f(t,x) : [0,\tau]\to X$ is continuous.
- (H3) The function $g : \mathscr{C}\to X$ is continuous and completely continuous (compact), meaning that it sends bounded subsets of $\mathscr{C}$ into relatively compact subsets of X.
- (H4) There exist $a,b>0$ with $\|g(x)\| \le a|x|_c + b$ for all $x\in\mathscr{C}$, where $|x|_c := \sup_{t\in[0,\tau]}\|x(t)\|$.
- (H5) $(S(\rho))_{\rho>0}$ is compact and $\sup_{\rho\in[0,G_\alpha(\tau)]}\|S(\rho)\| < \infty$ (equivalently, $\sup_{t\in[0,\tau]}\|T(t)\| < \infty$).

Theorem 3.3. Assume (H1)–(H5) hold. If

$$a \sup_{t\in[0,\tau]} \|T(t)\| \;=\; a \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \;<\; 1,$$

then the Cauchy problem (3.1) admits at least one mild solution on $[0,\tau]$.

Proof of Theorem 3.3 (in the $\mathcal{N}_F^\alpha$ setting). For $t \in [0,\tau]$, define

$$\Gamma(x)(t) := S\big(G_\alpha(t)\big)\,[x_0 + g(x)] \;+\; \int_0^t F(s,\alpha)\, S\big(G_\alpha(t) - G_\alpha(s)\big)\, f(s,x(s))\,ds,$$

and set

$$B_r := \{x \in \mathscr{C} : |x|_c \le r\}, \qquad \Omega(\Gamma) := \{x \in \mathscr{C} : \exists\,\lambda \in (0,1) \text{ with } x = \lambda\,\Gamma(x)\}.$$

Step 1: Γ is continuous. If $x_n \to x$ in $\mathscr{C}$, then for $t \in [0,\tau]$,

$$\Gamma(x_n)(t) - \Gamma(x)(t) = S(G_\alpha(t))\,[g(x_n) - g(x)] + \int_0^t F(s,\alpha)\, S\big(G_\alpha(t) - G_\alpha(s)\big)\,[f(s,x_n(s)) - f(s,x(s))]\,ds.$$

Hence

$$\big|\Gamma(x_n) - \Gamma(x)\big|_c \le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \Big(\|g(x_n) - g(x)\| + \int_0^\tau F(s,\alpha)\, \|f(s,x_n(s)) - f(s,x(s))\|\,ds \Big).$$

By (H1), $\|f(s,x_n(s)) - f(s,x(s))\| \le 2\mu(s)$ and $f(s,x_n(s)) \to f(s,x(s))$ a.e.; since $F(\cdot,\alpha)\mu(\cdot) \in L^1(0,\tau)$, the dominated convergence theorem yields $\int_0^\tau F(s,\alpha)\|f(s,x_n(s)) - f(s,x(s))\|ds \to 0$. The continuity of g (H3) gives $\|g(x_n) - g(x)\| \to 0$. Thus $|\Gamma(x_n) - \Gamma(x)|_c \to 0$.

Step 2: Γ is compact. *(i) Boundedness).* For $x \in B_r$,

$$\begin{aligned} |\Gamma(x)|_c &\le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \Big(\|x_0\| + \|g(x)\| + \int_0^\tau F(s,\alpha)\|f(s,x(s))\|\,ds \Big) \\ &\le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \Big(\|x_0\| + ar + b + \int_0^\tau F(s,\alpha)\mu(s)\,ds \Big) =: \delta. \end{aligned}$$

Thus $\Gamma(B_r) \subset B_\delta$.

(ii) Equicontinuity). For $0 \le t_1 < t_2 \le \tau$ and $x \in B_r$,

$$\begin{aligned} \Gamma(x)(t_2) - \Gamma(x)(t_1) = \big[S(G_\alpha(t_2)) - S(G_\alpha(t_1))\big] \Big(& S(G_\alpha(t_1))\, g(x) \\ & + \int_0^{t_1} F(s,\alpha)\, S(G_\alpha(t_1) - G_\alpha(s))\, f(s,x(s))\,ds \Big) \\ & + \int_{t_1}^{t_2} F(s,\alpha)\, S(G_\alpha(t_2) - G_\alpha(s))\, f(s,x(s))\,ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\Gamma(x)(t_2) - \Gamma(x)(t_1)\| \le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \Big(ar + b + \int_0^{t_1} F(s,\alpha)\mu(s)\,ds \Big) \|S(G_\alpha(t_2)) - S(G_\alpha(t_1))\| \\ + \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \int_{t_1}^{t_2} F(s,\alpha)\mu(s)\,ds. \end{aligned}$$

Due to the strong continuity of S and the integrability of $F\mu$ over $[0,\tau]$, the right-hand side tends to zero as $t_2 \to t_1^+$, uniformly with respect to $x \in B_r$. It follows that $\Gamma(B_r)$ forms an equicontinuous family of functions.

(iii) Pointwise relative compactness). At $t = 0$, $\Gamma(B_r)(0) = \{x_0 + g(x) : x \in B_r\}$ is relatively compact because g is compact (H3). For $t \in (0,\tau]$, fix $\varepsilon > 0$ and set

$$\sigma_\varepsilon(t) := t - G_\alpha^{-1}(G_\alpha(t) - \varepsilon) \in (0,t).$$

Split

$$\begin{aligned}\Gamma(x)(t) &= S(G_\alpha(t))\,[x_0 + g(x)] + \int_0^{t-\sigma_\varepsilon(t)} F(s,\alpha)\, S\big(G_\alpha(t) - G_\alpha(s)\big)\, f(s,x(s))\,ds \\ &\quad + \int_{t-\sigma_\varepsilon(t)}^{t} F(s,\alpha)\, S\big(G_\alpha(t) - G_\alpha(s)\big)\, f(s,x(s))\,ds \\ &=: \Gamma_\varepsilon(x)(t) + R_\varepsilon(x)(t).\end{aligned}$$

Note that

$$\Gamma_\varepsilon(x)(t) = S(\varepsilon)\left\{ S(G_\alpha(t) - \varepsilon)[x_0 + g(x)] + \int_0^{t-\sigma_\varepsilon(t)} F(s,\alpha)\, S\big(G_\alpha(t) - G_\alpha(s) - \varepsilon\big)\, f(s,x(s))\,ds \right\}.$$

Since $S(\varepsilon)$ is compact for $\varepsilon > 0$ and the bracketed set is bounded in X, $\{\Gamma_\varepsilon(x)(t) : x \in B_r\}$ is relatively compact. Moreover,

$$\|R_\varepsilon(x)(t)\| \le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\|\, \|\mu\|_{L^\infty} \int_{t-\sigma_\varepsilon(t)}^{t} F(s,\alpha)\,ds = \sup_\rho \|S(\rho)\|\, \|\mu\|_{L^\infty}\, \varepsilon \xrightarrow[\varepsilon\downarrow 0]{} 0,$$

uniformly in $x \in B_r$. By the Arzelà–Ascoli theorem, Γ is compact.

Step 3: A priori bound on $\Omega(\Gamma)$. If $x = \lambda\,\Gamma(x)$ for some $\lambda \in (0,1)$, then

$$\begin{aligned}|x|_c &= \lambda\,|\Gamma(x)|_c \\ &\le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \Big(\|x_0\| + \|g(x)\| + \int_0^\tau F(s,\alpha)\|f(s,x(s))\|\,ds \Big) \\ &\le \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| \Big(\|x_0\| + a|x|_c + b + \int_0^\tau F(s,\alpha)\mu(s)\,ds \Big).\end{aligned}$$

Hence

$$|x|_c \le \frac{\sup\limits_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\|}{1 - a \sup\limits_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\|} \Big(\|x_0\| + b + \int_0^\tau F(s,\alpha)\mu(s)\,ds \Big) < \infty,$$

because $a \sup_{\rho\in[0,G_\alpha(\tau)]} \|S(\rho)\| < 1$ by assumption. Thus $\Omega(\Gamma)$ is bounded.

Remark 3.3. Theorem 3.3 provides an existence result for mild solutions. Uniqueness is not investigated in this work and would require additional assumptions, such as a local Lipschitz condition on the nonlinear term f together with a suitable smallness condition.

Conclusion. Since Γ is continuous and compact, and $\Omega(\Gamma)$ is bounded, Schaefer's fixed point theorem guarantees the existence of a fixed point of Γ in $\mathscr{C}$. This fixed point provides a mild solution to the $\mathcal{N}_F^\alpha$ Cauchy problem.

□

4. Application

Consider the nonlocal fractional PDE

$$\begin{cases} \mathcal{N}_F^{1/2} u(t,\xi) = -\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \tanh\big(u(t,\xi)\big), & (t,\xi) \in [0,1]\times[0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,1], \\ u(0,\xi) = \displaystyle\int_0^1 w(t)\left(\int_0^\pi k(\xi,\sigma)\, \tanh(u(t,\sigma))\, d\sigma\right) dt \;+\; \eta(\xi), & \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where the data satisfy:

(C_1'') $w \in L^1(0,1)$.
(C_2'') $k \in L^2\big((0,\pi)\times(0,\pi)\big)$ and $K : X \to X$ defined by $(K\varphi)(\xi) = \int_0^\pi k(\xi,\sigma)\varphi(\sigma)\, d\sigma$ is the associated Hilbert–Schmidt operator with norm $\|K\|_{HS}$.
(C_3'') $\eta \in L^2(0,\pi)$.
(C_4'') The nonlinearity $\tanh : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with constant $L_{\tanh} \le 1$ and $|\tanh(s)| \le |s|$ for all $s \in \mathbb{R}$.

Let $X = L^2([0,\pi],\mathbb{R})$ and define the operator

$$A = -\frac{\partial^2}{\partial \xi^2}, \qquad D(A) = \big\{\varphi \in X :\ \varphi, \dot\varphi \text{ absolutely continuous},\ \ddot\varphi \in X,\ \varphi(0) = \varphi(\pi) = 0\big\}.$$

It is well known that A generates a compact contraction semigroup $(S(\rho))_{\rho\ge 0}$ on X with $\sup_{\rho\ge 0}\|S(\rho)\| \le 1$. Define the associated $\mathcal{N}_F^\alpha$-semigroup by $T(t) := S\big(G_\alpha(t)\big)$, so $(T(t))_{t>0}$ is compact and $\sup_{t\in[0,1]}\|T(t)\| \le 1$.

Next, set

$$x(t)(\xi) = u(t,\xi), \qquad f(t,x(t)) = \tanh\big(x(t)\big), \qquad g(x) = \int_0^1 w(t)\, K\big(\tanh(x(t))\big)\, dt + \eta.$$

Then (4.1) becomes the abstract problem

$$\begin{cases} \mathcal{N}_F^{1/2} x(t) = Ax(t) + f(t,x(t)), & t \in [0,1], \\ x(0) = g(x). \end{cases} \tag{4.2}$$

In this application, (H1) and (H2) are satisfied because tanh is bounded and globally Lipschitz; in particular $\|f(t,x)\|_X \le \|\tanh(x)\|_X \le \|x\|_X$ and $f(\cdot,x)$ is continuous for each $x \in X$. Here and in what follows, we denote by $\mathscr{C} = C([0,1],X)$.

Assumption (H3) holds since K is Hilbert–Schmidt (hence compact) and the composition $x \mapsto K(\tanh(x(\cdot)))$ is compact on $\mathscr{C} = C([0,1],X)$, while (C_3'') adds an affine compact perturbation by η. Moreover, using $|\tanh(s)| \le |s|$ and

Hölder/Fubini,

$$\|g(x)\| \le \int_0^1 |w(t)|\, \|K\|_{HS}\, \|\tanh(x(t))\|\, dt + \|\eta\| \le \|K\|_{HS}\, \|w\|_{L^1}\, |x|_c \;+\; \|\eta\|.$$

Thus (H4) holds with

$$a = \|K\|_{HS}\, \|w\|_{L^1}, \qquad b = \|\eta\|.$$

Finally, (H5) is already ensured by the heat semigroup properties and the time change via G_α.

Therefore,

$$a \sup_{t\in[0,1]} \|T(t)\| = \|K\|_{HS}\, \|w\|_{L^1} \;<\; 1$$

By Theorem 3.3, the fractional problem (4.2) has at least one mild solution on $[0, 1]$.

This example is intended to illustrate the applicability of the abstract existence result rather than to provide a detailed qualitative analysis of the fractional dynamics.

5. Conclusion and Future Work

In this work, we developed an abstract framework for evolution equations driven by the generalized conformable derivative $\mathcal{N}_F^\alpha$ with nonlocal initial conditions. By employing the time reparametrization $\rho = G_\alpha(t)$, we related $\mathcal{N}_F^\alpha$–C_0-semigroups to classical C_0-semigroups, derived growth estimates $\|T(t)\| \le Me^{\omega G_\alpha(t)}$, and obtained a variation-of-constants formula in the G_α-time. Under standard compactness and continuity assumptions, the existence of mild solutions was established via Schaefer's fixed point theorem, and uniqueness was ensured under a local Lipschitz condition. The framework was illustrated by a semilinear heat equation with a nonlocal initial condition driven by $\mathcal{N}_F^{1/2}$. For future work, one may investigate weaker compactness conditions, regularity and maximal L^p-estimates, extensions to state-dependent delays or stochastic perturbations, variable-order derivatives, numerical schemes adapted to G_α-time, and long-time behavior including dissipativity and attractors. These directions highlight the potential of the $\mathcal{N}_F^\alpha$ approach as a unifying framework for generalized conformable models with memory and nonlocal effects.

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