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EFFICIENT STOCHASTIC COLLOCATION TECHNIQUES FOR TRANSIENT HEAT EQUATIONS WITH UNCERTAIN INPUTS

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Abstract. This study develops an efficient numerical framework for transient heat equations with uncertain diffusion and Robin boundary parameters. The approach combines a stochastic collocation method to handle parametric uncertainty with a finite difference scheme for time integration. Convergence and stability properties of the method are established under suitable assumptions on the random inputs. Numerical results confirm the accuracy and computational efficiency of the proposed approach for uncertainty quantification in transient heat transfer problems.

Keywords. Stochastic collocation method, finite difference method, transient heat equation, Robin boundary condition, diffusion coefficient, uncertain inputs.
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1. Introduction

In recent years, the numerical analysis of partial differential equations with stochastic or random coefficients has attracted increasing interest, motivated by the growing need to model uncertainties present in numerous scientific and industrial fields. These uncertainties often arise from incomplete knowledge of the properties of materials, measurement errors or insufficiencies, or random variations of data such as model coefficients, source terms, boundary conditions, or the geometry of the domain.

To address these challenges, various methods have been developed to improve the accuracy of numerical predictions and obtain reliable forecasts that account for inherent uncertainties in the models. Among the most studied approaches are the multilevel Monte Carlo method [1, 2, 3, 5], the stochastic Galerkin method [6, 7, 8, 5], and the stochastic collocation method [9, 11, 20]. The latter has established itself as a particularly effective technique, notably due to its ability to exploit deterministic collocation points in the random variable space, allowing for a robust approximation of the stochastic dependence of the solution. It is often coupled with classical spatial or temporal discretization schemes, such as finite

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difference methods, to ensure a complete numerical resolution of the problem under consideration.

In this work, we focus on the numerical analysis of a transient heat model defined on a domain $D \subset \mathbb{R}^n$, characterized by random diffusion coefficients and random Robin boundary conditions (see [21] for Robin boundary conditions). The problem is formulated, for almost every $\xi \in \Gamma$, as follows:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x,\xi) - \nabla \cdot \big(a(\xi)\nabla y(t,x,\xi)\big) = f(t,x,\xi), & \text{in } (0,T] \times D, \\ a(\xi)\nabla y(t,x,\xi) \cdot \mathbf{n} = 0, & \text{on } (0,T] \times \partial D_0, \\ a(\xi)\nabla y(t,x,\xi) \cdot \mathbf{n} = \alpha(x,\xi)\big(u(t,x,\xi) - y(t,x,\xi)\big), & \text{on } (0,T] \times \partial D_1, \\ y(0,x,\xi) = y_0(x,\xi), & \text{in } D, \end{cases} \tag{1.1}$$

To account for uncertainties in the problem data, we assume that the parameters a (the diffusion coefficient), α (the Robin boundary parameter), the source term f, and the control function u are modeled as random fields.

Numerous uncertain factors influence heat and mass transfer processes, including random initial temperatures, ambient temperatures, material properties, thermal conductivities (diffusion coefficients), convective heat transfer coefficients (also known as Robin coefficients), as well as geometric variations. The problem stated in equation (1.1) involves random diffusion and Robin coefficients potentially varying with the time variable $t \in [0,T]$. However, for simplicity, they are assumed to be stationary, as indicated in [15, 16]. The model (1.1) is also relevant for boundary control problems, as discussed in [17, 18].

We propose to use a stochastic collocation method combined with a finite difference discretization to numerically solve this problem. We analyze the stability and error of the method under reasonable assumptions on the variability of the random coefficients around their means. Finally, numerical results illustrating the performance and robustness of the proposed method are presented to validate the theoretical findings.

The remainder of the paper is organized as follows. In Section 2, we introduce the mathematical formulation of the problem and the main notations used throughout the paper. Section 3 presents several regularity results for the solution. In Section 4, we establish a complete convergence result for the collocation method. Section 5 illustrates the theoretical findings through several numerical simulations. Finally, Section 6 concludes the paper.

2. The problem setting and Notation

Let $D \subset \mathbb{R}^n$ be a bounded convex polygonal domain, with spatial variable $x \in D$ and time variable $t \in [0,T]$. Consider a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with the sigma-algebra $\mathcal{A}$, where Ω is the sample space and $\mathbb{P}$ is a probability measure.

Let $\rho : \Gamma \to \mathbb{R}_+$ be a bounded joint probability density function associated with the random variable $\xi = [\xi_1(\omega), \dots, \xi_d(\omega)] \in \mathbb{R}^d$, with $\omega \in \Omega$ and $\Gamma := \prod_{n=1}^{d} \Gamma_n$, where $\Gamma_n = \xi_n(\Omega) \subset \mathbb{R}$ is the image of ξ_n.

We focus on the numerical analysis of a transient heat problem with random Robin boundary conditions. The problem is formulated as follows: find the random solution $y(x, t, \xi)$, for $(x, t, \xi) \in D \times (0, T] \times \Gamma$, such that, $\mathbb{P}$-almost surely, the following system is satisfied:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nabla \cdot (a(\xi)\nabla y) = f(t, x, \xi), & \text{in } (0, T] \times D, \\ a(\xi)\nabla y \cdot \mathbf{n} = 0, & \text{on } (0, T] \times \partial D_0, \\ a(\xi)\nabla y \cdot \mathbf{n} = \alpha(x, \xi) \left(u(t, x, \xi) - y(t, x, \xi)\right), & \text{on } (0, T] \times \partial D_1, \\ y(0, x, \xi) = y_0(x, \xi), & \text{in } D, \end{cases} \tag{2.1}$$

Here, $a(\xi)$ denotes a random diffusion coefficient, $\alpha(x, \xi)$ represents a random Robin-type transfer parameter, and $f(t, x, \xi)$ is a random source term. The vector $\mathbf{n}$ denotes the outward unit normal on the boundary $\partial D = \partial D_0 \cup \partial D_1$, while $u(t, x, \xi)$ is a random control function imposed on ∂D_1, modeling, for instance, an external temperature or a prescribed random flux.

Remark 2.1. In this work, the notations for the Laplacian Δ and the gradient ∇ refer solely to derivatives with respect to the spatial variable x and $|D|$ is the size of D. The functions a, α, f, and u are assumed to possess sufficient smoothness.

Let $\{\xi_k\}_{k=1}^{(N+1)^d} \subset \Gamma$ be a set of Gauss–Lobatto collocation points, where $N + 1$ is the number of points in each random dimension.

At each collocation point ξ_k, $k = 1, \dots, (N + 1)^d$, we solve the following deterministic problem parameterized by ξ_k:

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t}(t, x, \xi_k) - \nabla \cdot \big(a(\xi_k)\nabla \bar{y}(t, x, \xi_k)\big) = f(t, x, \xi_k), & \text{in } (0, T] \times D, \\ a(\xi_k)\nabla \bar{y}(t, x, \xi_k) \cdot \mathbf{n} = 0, & \text{on } (0, T] \times \partial D_0, \\ a(\xi_k)\nabla \bar{y}(t, x, \xi_k) \cdot \mathbf{n} = \alpha(x, \xi_k)\big(u(t, x, \xi_k) - \bar{y}(t, x, \xi_k)\big), & \text{on } (0, T] \times \partial D_1, \\ \bar{y}(0, x, \xi_k) = \bar{y}_0(x, \xi_k), & \text{in } D. \end{cases} \tag{2.2}$$

The full approximate solution $y^N(t, x, \xi)$ is then obtained by tensorized Lagrange interpolation over the random space. :

$$y^N(t, x, \xi) = \sum_{k=1}^{(N+1)^d} \bar{y}(t, x, \xi_k)\, L_k(\xi), \tag{2.3}$$

where $L_k(\xi)$ denotes the Lagrange interpolation polynomials associated with the points ξ_k.

3. Regularity analysis for the solution of our model problem

In this section, we analyze the regularity properties of the solution to problem (2.1). These properties play a key role in establishing the convergence of the numerical method introduced earlier and will be exploited in the subsequent section. We begin by recalling a Gronwall-type inequality, which will be repeatedly used in the forthcoming analysis.

Lemma 3.1 (Gronwall inequality)**.** *If $h(t)$ satisfies*

$$\frac{dh(t)}{dt} \leq ah(t) + b$$

for some constant $a \neq 0$ and b, then we have

$$h(t) \leq e^{at}\left(h(0) + \frac{b}{a}\right), \quad \forall t \geq 0.$$

We have the following Lemmas concerning the regularity of the system.

Lemma 3.2. *for $t \in [0;T]$ we have*

$$\int_\Gamma \int_D \left(|y|^2\right) \rho(\xi)(t)\, dx\, d\xi \leq e^{C_1 T} \int_\Gamma \int_D \left(|y_0|^2\right) \rho(\xi)\, dx\, d\xi,$$

where

$$C_1 := \max_{\overline{D}\times\overline{\Gamma}} \{|\alpha u|; 2|f|\}\, |D|^{1/2}$$

Proof. We multiply the equations in (2.2) by $2\rho(\xi)y$. Then we integrate over $D \times \Gamma$. The following equation is a direct result of applying Green's formula and using the boundary conditions:

$$\begin{aligned}\frac{d}{dt}\int_\Gamma \int_D |y|^2 \rho(\xi)\, dx\, d\xi \quad &+ \int_\Gamma \int_D 2a|\nabla y|^2 \rho(\xi)\, dx\, d\xi + \int_\Gamma \int_{\partial D_1} |y|^2 \rho(\xi)\, dx\, d\xi \\ &= \int_\Gamma \int_{\partial D_1} \alpha u y\, \rho(\xi)\, dx\, d\xi + \int_\Gamma \int_D 2fy\, \rho(\xi)\, dx\, d\xi\end{aligned} \tag{3.1}$$

then

$$\frac{d}{dt}\int_\Gamma \int_D |y|^2 \rho(\xi)\, dx\, d\xi \leq \int_\Gamma \int_{\partial D_1} \alpha u y\, \rho(\xi)\, dx\, d\xi + \int_\Gamma \int_D 2fy\, \rho(\xi)\, dx\, d\xi \tag{3.2}$$

then using inequality $\|f\|_{L^1(D)} \leq |D|^{1/2} \|f\|_{L^2(D)}$ we get

$$\frac{d}{dt}\int_\Gamma \int_D |y|^2 \rho(\xi)\, dx\, d\xi \leq \max_{\overline{D}\times\overline{\Gamma}} \{|\alpha u|; 2|f|\}\, |D|^{1/2} \int_\Gamma \int_D |y|^2 \rho(\xi)(t)\, dx\, d\xi \tag{3.3}$$

Applying the Gronwall inequality yields

$$\int_\Gamma \int_D |y|^2 \rho(\xi)(t)\, dx\, d\xi \leq e^{C_1 T} \int_\Gamma \int_D |y_0|^2 \rho(\xi)\, dx\, d\xi,$$

which completes the proof. □

Lemma 3.3. *Let $t \in [0,T]$, we have*

$$\int_\Gamma \int_D \left(\left|\frac{\partial y}{\partial t}\right|^2\right) \rho(\xi)(t)\, dx\, d\xi \leq C_2 e^{C_1 T} \int_\Gamma \int_D \left(\left|\frac{\partial y}{\partial t}(0)\right|^2\right) \rho(\xi)\, dx\, d\xi, \tag{3.4}$$

where

$$C_2 := \max_{\overline{D}\times\overline{\Gamma}} \left\{ |\alpha \frac{\partial u}{\partial t}|; 2|\alpha \frac{\partial f}{\partial t}|; |\alpha| \right\},$$

Proof. Taking the time derivative of Eq. (2.2), we obtain

$$\frac{\partial^2 y}{\partial t^2} - \nabla \cdot \left(a(\xi) \nabla \frac{\partial y}{\partial t} \right) = \frac{\partial f}{\partial t}(t, x, \xi), \quad \text{in } [0, T] \times D \times \Gamma. \tag{3.5}$$

Following the same steps as in the proof of Lemma (3.2), by multiplying this time the equations of (3.5) by $2\rho(\xi)\frac{\partial y}{\partial t}$, one obtains

$$\int_\Gamma \int_D \left| \frac{\partial y}{\partial t}(t) \right|^2 \rho(\xi)\, dx\, d\xi - \int_\Gamma \int_D 2\nabla \left(a(\xi) \nabla \frac{\partial y}{\partial t} \right) \frac{\partial y}{\partial t} \rho(\xi)\, dx\, d\xi = \int_\Gamma \int_D \frac{\partial f}{\partial t} \frac{\partial y}{\partial t} \rho(\xi)\, dx\, d\xi \tag{3.6}$$

then using the green formula

$$\begin{aligned} &\int_\Gamma \int_D \left| \frac{\partial y}{\partial t}(t) \right|^2 \rho(\xi)\, dx\, d\xi + \int_\Gamma \int_D 2a(\xi) \left| \nabla \frac{\partial y}{\partial t} \right|^2 \rho(\xi)\, dx\, d\xi \\ &\quad - \int_\Gamma \int_{\partial D_1} 2\alpha(x, \xi) \left(\frac{\partial u}{\partial t} - \frac{\partial y}{\partial t} \right) \frac{\partial y}{\partial t} \rho(\xi)\, dx\, d\xi \\ &= \int_\Gamma \int_D \frac{\partial f}{\partial t} \frac{\partial y}{\partial t} \rho(\xi)\, dx\, d\xi \end{aligned} \tag{3.7}$$

then the limit condition together with Grönwall's inequality yields

$$\int_\Gamma \int_D \left(\left| \frac{\partial y}{\partial t}(t) \right|^2 \right) \rho(\xi)\, dx\, d\xi \leq e^{C_2 T} \int_\Gamma \int_D \left(\left| \frac{\partial y}{\partial t}(0) \right|^2 \right) \rho(\xi)\, dx\, d\xi. \tag{3.8}$$

where

$$C_2 := |D| \times \max_{\overline{D}\times\overline{\Gamma}} \left\{ |\alpha \frac{\partial u}{\partial t}|; 2|\alpha \frac{\partial f}{\partial t}|; |\alpha| \right\},$$

□

Lemma 3.4. *Let $t \in [0, T]$, we have*

$$\int_\Gamma \int_D |\nabla y|^2 \rho(\xi)\, dx\, d\xi \leq C_1' e^{C_1 T} \int_\Gamma \int_D \left(|y_0|^2 \right) \rho(\xi)\, dx\, d\xi, \tag{3.9}$$

where

$$C_1' = \frac{C_1}{a_{min}}$$

Proof. We multiply the equations in (2.2) by $2\rho(\xi)y$. Then we integrate over $D \times \Gamma$. The following equation is a direct result of applying Green's formula and using the boundary conditions:

$$\begin{aligned} \frac{d}{dt} \int_\Gamma \int_D |y|^2 \rho(\xi)\, dx\, d\xi \quad &+ \int_\Gamma \int_D 2a |\nabla y|^2 \rho(\xi)\, dx\, d\xi + \int_\Gamma \int_{\partial D_1} |y|^2 \rho(\xi)\, dx\, d\xi \\ &= \int_\Gamma \int_{\partial D_1} \alpha(x, \xi) u y\, \rho(\xi)\, dx\, d\xi + \int_\Gamma \int_D 2 f y\, \rho(\xi)\, dx\, d\xi \end{aligned} \tag{3.10}$$

then we have

$$\frac{d}{dt} \int_\Gamma \int_D |y|^2 \rho(\xi)\, dx\, d\xi \leq C_1 \int_\Gamma \int_D |y|^2 \rho(\xi)(t)\, dx\, d\xi \tag{3.11}$$

and

$$\int_\Gamma\int_D 2a|\nabla y|^2\,\rho(\xi)\,dx\,d\xi \le -\frac{d}{dt}\int_\Gamma\int_D |y|^2\,\rho(\xi)\,dx\,d\xi + \int_\Gamma\int_{\partial D_1}\alpha(x,\xi)uy\,\rho(\xi)\,dx\,d\xi + \int_\Gamma\int_D 2fy\,\rho(\xi)\,dx\,d\xi \tag{3.12}$$

Then, taking the absolute value and applying Lemma (3.2), we obtain:

$$\int_\Gamma\int_D |\nabla y|^2\,\rho(\xi)\,dx\,d\xi \le C_1' e^{C_1 T}\int_\Gamma\int_D \left(|y_0|^2\right)\rho(\xi)\,dx\,d\xi, \tag{3.13}$$

where

$$C_1' = \frac{C_1}{a_{min}}$$

□

Theorem 3.5. *Let $t\in[0,T]$, we have*

$$\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2 \rho(\xi)\,dx\,d\xi \le e^{C_4 T}\left(\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2 (0)\rho(\xi)\,dx\,d\xi + (\frac{C_1C_1'}{C_4}+\frac{C_5}{C_4})e^{C_1 T}\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi\right)$$

where the constants are explicitly given by:

$$C_4 = \max_{\overline{D}\times\overline{\Gamma}}\left\{|\frac{\partial f}{\partial \xi_k}||D|,\ |\alpha\frac{\partial u}{\partial \xi_k}|,\ |u\frac{\partial \alpha}{\partial \xi_k}|,\ |\frac{\partial \alpha}{\partial \xi_k}|\right\},$$
$$C_5 = \max_{\overline{D}\times\overline{\Gamma}}\left\{\frac{\left|\frac{\partial a}{\partial \xi_k}\right|}{2|a|}\right\}.$$

Proof. We differentiate the equation with respect to ξ_k:

$$\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial \xi_k}\right) = \nabla\cdot\left(\frac{\partial a}{\partial \xi_k}\nabla y + a\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right) + \frac{\partial f}{\partial \xi_k}. \tag{3.14}$$

The boundary conditions become:

$$\begin{cases} \left(\frac{\partial a}{\partial \xi_k}\nabla y + a\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right)\cdot\mathbf{n} = 0, & \text{on } \partial D_0,\\ \left(\frac{\partial a}{\partial \xi_k}\nabla y + a\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right)\cdot\mathbf{n} = \frac{\partial \alpha}{\partial \xi_k}(u-y) + \alpha\left(\frac{\partial u}{\partial \xi_k} - \frac{\partial y}{\partial \xi_k}\right), & \text{on } \partial D_1. \end{cases} \tag{3.15}$$

We multiply the differentiated equation by $2\rho(\xi)\frac{\partial y}{\partial \xi}$ and integrate over $D\times\Gamma$:

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi &= -2\int_\Gamma\int_D a\left|\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right|^2\rho(\xi)\,dx\,d\xi - 2\int_\Gamma\int_D \frac{\partial a}{\partial \xi_k}\nabla y\cdot\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\rho(\xi)\,dx\,d\xi\\ &+2\int_\Gamma\int_D \frac{\partial f}{\partial \xi_k}\frac{\partial y}{\partial \xi_k}\rho(\xi)\,dx\,d\xi\\ &+2\int_\Gamma\int_{\partial D_1}\left[\frac{\partial \alpha}{\partial \xi_k}(u-y) + \alpha\left(\frac{\partial u}{\partial \xi_k} - \frac{\partial y}{\partial \xi_k}\right)\right]\cdot\frac{\partial y}{\partial \xi_k}\rho(\xi)\,dx\,d\xi.\end{aligned}$$

Let us decompose the boundary terms:

$$2\int_\Gamma\int_{\partial D_1}\frac{\partial \alpha}{\partial \xi_k}(u-y)\cdot\frac{\partial y}{\partial \xi_k}\rho(\xi)\,dx\,d\xi + 2\int_\Gamma\int_{\partial D_1}\alpha\frac{\partial u}{\partial \xi_k}\cdot\frac{\partial y}{\partial \xi_k}\rho(\xi)\,dx\,d\xi - 2\int_\Gamma\int_{\partial D_1}\alpha\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi$$

We apply the Cauchy–Schwarz inequality to estimate all the terms:

- $$\left|\int_\Gamma\int_D \frac{\partial a}{\partial \xi_k}\nabla y\cdot\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\rho(\xi)dxd\xi\right| \le \frac{1}{2}\int_\Gamma\int_D a\left|\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right|^2\rho(\xi)dxd\xi + C_5\int_\Gamma\int_D |\nabla y|^2\rho(\xi)dxd\xi$$

  where
  $$C_5 = \max_{\overline{D}\times\overline{\Gamma}}\left\{\frac{\left|\frac{\partial a}{\partial \xi_k}\right|}{2|a|}\right\}.$$

- We also have
  $$\left|\frac{\partial f}{\partial \xi_k}\cdot\frac{\partial y}{\partial \xi_k}\right| \le \left\|\frac{\partial f}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\cdot\left|\frac{\partial y}{\partial \xi_k}\right|,$$

  and thus:
  $$\left|\int_\Gamma\int_D \frac{\partial f}{\partial \xi_k}\cdot\frac{\partial y}{\partial \xi_k}\,\rho(\xi)\,dx\,d\xi\right| \le \left\|\frac{\partial f}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|\rho(\xi)\,dx\,d\xi.$$

  Then, we apply the Cauchy–Schwarz inequality:
  $$\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|\rho(\xi)\,dx\,d\xi \le |D|\left(\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\right)^{1/2}.$$

  Therefore, by combining:
  $$\left|\int_\Gamma\int_D \frac{\partial f}{\partial \xi_k}\cdot\frac{\partial y}{\partial \xi_k}\,\rho(\xi)\,dx\,d\xi\right| \le |D|\left\|\frac{\partial f}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\left(\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\right).$$

- $$\begin{aligned}\left|\int_\Gamma\int_{\partial D_1}\alpha\frac{\partial u}{\partial \xi_k}\cdot\frac{\partial y}{\partial \xi_k}\rho(\xi)\,dx\,d\xi\right| &\le \left\|\alpha\frac{\partial u}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_{\partial D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\\ &\le \left\|\alpha\frac{\partial u}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_{D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\end{aligned}$$

- $$\begin{aligned}\left|\int_\Gamma\int_{\partial D_1}\frac{\partial \alpha}{\partial \xi_k}(u-y)\cdot\frac{\partial y}{\partial \xi_k}\rho(\xi)\,dx\,d\xi\right| &\le \left\|u\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_{D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\\ &+\left\|\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_{D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\\ &+\left\|\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_{D_1}|y|^2\rho(\xi)\,dx\,d\xi\end{aligned}$$

By collecting terms, we obtain:

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi &\le C_1\int_\Gamma\int_D|\nabla y|^2\rho(\xi)\,dx\,d\xi + \left\|\frac{\partial f}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)}|D|\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi}\right|^2\rho(\xi)\,dx\,d\xi\\ &+\left\|\alpha\frac{\partial u}{\partial \xi_k}\right\|_{L^\infty}\int_\Gamma\int_{\partial D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi + \left\|u\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty}\int_\Gamma\int_{\partial D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi\\ &+\left\|\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty}\int_\Gamma\int_{\partial D_1}\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi + \left\|\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty}\int_\Gamma\int_{\partial D_1}|y|^2\rho(\xi)\,dx\,d\xi.\end{aligned}$$

then we have

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi &\le C_1\int_\Gamma\int_D|\nabla y|^2\rho(\xi)\,dx\,d\xi\\ &+C_4\int_\Gamma\int_D\left|\frac{\partial y}{\partial \xi_k}\right|^2\rho(\xi)\,dx\,d\xi + C_5\int_\Gamma\int_{\partial D_1}|y|^2\rho(\xi)\,dx\,d\xi,\end{aligned}$$

where the constants are explicitly given by:

$$C_4 = \max_{\overline{D}\times\overline{\Gamma}} \left\{ |D| \left|\frac{\partial f}{\partial \xi_k}\right|, \left|\alpha\frac{\partial u}{\partial \xi_k}\right|, \left|u\frac{\partial \alpha}{\partial \xi_k}\right|, \left|\frac{\partial \alpha}{\partial \xi_k}\right| \right\},$$
$$C_5 = \max_{\overline{D}\times\overline{\Gamma}} \left\{ \frac{\left|\frac{\partial a}{\partial \xi_k}\right|}{2|a|} \right\}.$$

By Lemma (3.2) and Lemma (3.4), we obtain:

$$\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2 \rho(\xi)\,dx\,d\xi \le e^{C_4 T}\left(\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2(0)\rho(\xi)\,dx\,d\xi + (\frac{C_1C_1'}{C_4}+\frac{C_5}{C_4})e^{C_1T}\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi\right)$$

Then, Grönwall's inequality concludes the proof.

□

Lemma 3.6. *Let $t \in [0,T]$, we have*

$$\int_\Gamma\int_D \left|\nabla\frac{\partial y}{\partial t}\right|^2(t)\rho(\xi)\,dx\,d\xi \le e^{C_6T}\left[\int_\Gamma\int_D \left|\nabla\frac{\partial y_0}{\partial t}\right|^2\rho(\xi)\,dx\,d\xi + \frac{\|\nabla\alpha\|_{L^\infty}}{C_6}C_2e^{C_1T}\int_\Gamma\int_D\left|\frac{\partial y}{\partial t}(0)\right|^2\rho(\xi)\,dx\,d\xi,\right],$$

where

$$C_6 = \max_{\overline{D}\times\overline{\Gamma}}\left\{|D|\,|\alpha|\left|\nabla\frac{\partial u}{\partial t}\right|, |D|\,|\alpha|, |D|\,|\nabla\alpha|\left|\frac{\partial u}{\partial t}\right|, |\nabla\alpha|, 2\,|D|\left|\nabla\frac{\partial f}{\partial t}\right|\right\}.$$

Proof. The proof follows the same strategy as in the previous lemma, applied to the *gradient of the time derivative* of the solution, namely $\nabla\frac{\partial y}{\partial t}$. We start from the stochastic parabolic equation :

$$\frac{\partial y}{\partial t} = \nabla\cdot(a(\xi)\nabla y) + f(t,x,\xi).$$

By differentiating with respect to time:

$$\frac{\partial^2 y}{\partial t^2} = \nabla\cdot\left(a(\xi)\nabla\frac{\partial y}{\partial t}\right) + \frac{\partial f}{\partial t}(t,x,\xi).$$

We apply the spatial gradient to both sides:

$$\nabla\left(\frac{\partial^2 y}{\partial t^2}\right) = \nabla\left[\nabla\cdot\left(a(\xi)\nabla\frac{\partial y}{\partial t}\right)\right] + \nabla\left(\frac{\partial f}{\partial t}\right).$$

We multiply by $2\rho(\xi)\nabla\frac{\partial y}{\partial t}$ and integrate over $D\times\Gamma$:

$$\begin{aligned}
\frac{d}{dt}\int_\Gamma\int_D\left|\nabla\frac{\partial y}{\partial t}\right|^2\rho(\xi)\,dx\,d\xi &= 2\int_\Gamma\int_D \nabla\left[\nabla\cdot\left(a(\xi)\nabla\frac{\partial y}{\partial t}\right)\right]\cdot\nabla\frac{\partial y}{\partial t}\,\rho(\xi)\,dx\,d\xi\\
&\quad + 2\int_\Gamma\int_D \nabla\frac{\partial f}{\partial t}\cdot\nabla\frac{\partial y}{\partial t}\,\rho(\xi)\,dx\,d\xi\\
&= -\int_D a(\xi)\left|\nabla^2\frac{\partial y}{\partial t}\right|^2 dx + \int_{\partial D}\nabla\left(a(\xi)\nabla\frac{\partial y}{\partial t}\right)\cdot\mathbf{n}\,\nabla\frac{\partial y}{\partial t}\,dx,\\
&\quad + 2\int_\Gamma\int_D \nabla\frac{\partial f}{\partial t}\cdot\nabla\frac{\partial y}{\partial t}\,\rho(\xi)\,dx\,d\xi
\end{aligned}$$

The differentiated Robin boundary condition imposes on ∂D_1:

$$a(\xi)\nabla\frac{\partial y}{\partial t}\cdot\mathbf{n} = \alpha(x,\xi)\left(\frac{\partial u}{\partial t} - \frac{\partial y}{\partial t}\right).$$

Taking the spatial gradient ∇ on both sides (which acts on x):

$$a(\xi)\nabla\left[\nabla\frac{\partial y}{\partial t}\cdot\mathbf{n}\right]=\nabla\left[\alpha(x,\xi)\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right)\right].$$

Expanding the right-hand side using the product rule:

$$a(\xi)\nabla\left[\nabla\frac{\partial y}{\partial t}\cdot\mathbf{n}\right]=(\nabla\alpha(x,\xi))\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right)+\alpha(x,\xi)\nabla\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right).$$

It follows that the boundary term becomes:

$$\begin{aligned}&\int_{\partial D_1}\nabla\left(a(\xi)\nabla\frac{\partial y}{\partial t}\right)\cdot\mathbf{n}\,\nabla\frac{\partial y}{\partial t}\,dx\\&=\int_{\partial D_1}a(\xi)\nabla\left[\nabla\frac{\partial y}{\partial t}\cdot\mathbf{n}\right]\cdot\nabla\frac{\partial y}{\partial t}\,dx\\&=\int_{\partial D_1}\left[(\nabla\alpha)\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right)+\alpha\nabla\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right)\right]\cdot\nabla\frac{\partial y}{\partial t}\,dx.\end{aligned}$$

Each boundary term will be estimated using the Cauchy–Schwarz inequalities:

$$\begin{aligned}&\left|\int_{\partial D_1}(\nabla\alpha)\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right)\cdot\nabla\frac{\partial y}{\partial t}\,dx\right|\\&\leq|D|\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\|\frac{\partial u}{\partial t}\|_{L^\infty}\int_{\partial D_1}\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx\\&+\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx+\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\left|\frac{\partial y}{\partial t}\right|^2\,dx\\&\leq|D|\|\nabla\alpha\|_{L^\infty}\|\frac{\partial u}{\partial t}\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx\\&+\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx+\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\left|\frac{\partial y}{\partial t}\right|^2\,dx\\&\leq\max\{|D|\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\|\frac{\partial u}{\partial t}\|_{L^\infty(D\times\Gamma)};\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\}\int_D\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx+\|\nabla\alpha\|_{L^\infty(D\times\Gamma)}\int_D\left|\frac{\partial y}{\partial t}\right|^2\,dx\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\partial D_1}\alpha\nabla\left(\frac{\partial u}{\partial t}-\frac{\partial y}{\partial t}\right)\cdot\nabla\frac{\partial y}{\partial t}\,dx\right|\\&\leq\|\alpha\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\left|\nabla\frac{\partial u}{\partial t}\right|\cdot\left|\nabla\frac{\partial y}{\partial t}\right|\,dx+\|\alpha\|_{L^\infty(D\times\Gamma)}\int_{\partial D_1}\cdot\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx\\&\leq\max\{|D|\|\alpha\|_{L^\infty(D\times\Gamma)}\|\nabla\frac{\partial u}{\partial t}\|_{L^\infty(D\times\Gamma)};\|\alpha\|_{L^\infty}\}\int_D\left|\nabla\frac{\partial y}{\partial t}\right|^2\,dx\end{aligned}$$

For the second term in the main equation, we use the Cauchy–Schwarz inequality:

$$2\int_\Gamma\int_D\nabla\frac{\partial f}{\partial t}\cdot\nabla\frac{\partial y}{\partial t}\,\rho(\xi)\,dx\,d\xi\leq2|D|\|\nabla\frac{\partial f}{\partial t}\|_{L^\infty(D\times\Gamma)}\int_\Gamma\int_D\left|\nabla\frac{\partial y}{\partial t}\right|^2\rho(\xi)\,dx\,d\xi$$

By collecting all the terms, we obtain the following differential inequality:

$$\frac{d}{dt}\int_\Gamma\int_D\left|\nabla\frac{\partial y}{\partial t}\right|^2\rho(\xi)\,dx\,d\xi.\leq C_6\int_\Gamma\int_D\left|\nabla\frac{\partial y}{\partial t}\right|^2\rho(\xi)\,dx\,d\xi+\|\nabla\alpha\|_{L^\infty}\int_\Gamma\int_D\left|\frac{\partial y}{\partial t}\right|^2\rho(\xi)\,dx\,d\xi.$$

where

$$C_6 = \max_{\overline{D}\times\overline{\Gamma}} \left\{ |D|\,|\alpha| \left|\nabla \frac{\partial u}{\partial t}\right|, |D|\,|\alpha|, |D|\,|\nabla \alpha| \left|\frac{\partial u}{\partial t}\right|, |\nabla \alpha|, 2\,|D| \left|\nabla \frac{\partial f}{\partial t}\right| \right\}.$$

By Grönwall's lemma and Lemma (3.3), we deduce that

$$\int_\Gamma \int_D \left|\nabla \frac{\partial y}{\partial t}\right|^2 (t)\rho\, dx\, d\xi \leq e^{C_6 T} \left[\int_\Gamma \int_D \left|\nabla \frac{\partial y}{\partial t}(0)\right|^2 \rho\, dx\, d\xi + \frac{\|\nabla \alpha\|_{L^\infty}}{C_6} C_2 e^{C_1 T} \int_\Gamma \int_D \left|\frac{\partial y}{\partial t}(0)\right|^2 \rho(\xi)\, dx\, d\xi, \right],$$

□

Theorem 3.7. *Let y be the solution of the parabolic problem with Robin boundary conditions depending on $\xi \in \Gamma \subset \mathbb{R}^d$, where the functions a, α, f, u are sufficiently regular such that*

$$C_a = \left(2\,\|a\|_{L^\infty(\Gamma)} - |D| \left\| \frac{\partial a}{\partial \xi_k} \right\|_{L^\infty(\Gamma)} \right) > 0.$$

Then, for all $t \in [0, T]$, the following estimate holds:

$$\int_\Gamma \int_D \left|\nabla \left(\frac{\partial y}{\partial \xi_k}\right)\right|^2 \rho\, dx\, d\xi \leq \frac{1}{C_a} \left[\left(C_7 2 C_1 e^{C_1 T} + C_8 e^{C_4 T} \left(\frac{2C_1^2}{C_4} e^{C_1 T} + \frac{C_5}{C_4} e^{C_1 T} \right) + C_9 e^{C_1 T} \right) \int_\Gamma \int_D |y_0|^2 \rho(\xi)\, dx\, d\xi \right.$$
$$\left. + C_8 e^{C_4 T} \int_\Gamma \int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2 (0) \rho(\xi)\, dx\, d\xi \right].$$

where the constants are explicitly given by:

$$C_7 = \max_{\overline{D}\times\overline{\Gamma}} \{C_1; |D|\},$$
$$C_8 = \max_{\overline{D}\times\overline{\Gamma}} \{C_4; 2\left|\frac{\partial f}{\partial \xi_k}\right|; 2|D| \left|\frac{\partial \alpha}{\partial \xi_k} u\right|; 2|D| \left|\frac{\partial u}{\partial \xi_k}\alpha\right|; 2|D| \left|\frac{\partial \alpha}{\partial \xi_k}\right|; 2|D|\,|\alpha|\},$$
$$C_9 = \max_{\overline{D}\times\overline{\Gamma}} \{C_5; 2|D| \left|\frac{\partial \alpha}{\partial \xi_k}\right|\}.$$

Proof. We differentiate the equation with respect to ξ_k :

$$\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial \xi_k}\right) = \nabla \cdot \left(\frac{\partial a}{\partial \xi_k} \nabla y + a \nabla \left(\frac{\partial y}{\partial \xi_k}\right) \right) + \frac{\partial f}{\partial \xi_k}.$$

The boundary conditions become:

$$\begin{cases} \left(\frac{\partial a}{\partial \xi_k} \nabla y + a \nabla \left(\frac{\partial y}{\partial \xi_k}\right) \right) \cdot \mathbf{n} = 0, & \text{on } \partial D_0, \\ \left(\frac{\partial a}{\partial \xi_k} \nabla y + a \nabla \left(\frac{\partial y}{\partial \xi_k}\right) \right) \cdot \mathbf{n} = \frac{\partial \alpha}{\partial \xi_k}(u - y) + \alpha \left(\frac{\partial u}{\partial \xi_k} - \frac{\partial y}{\partial \xi_k} \right), & \text{on } \partial D_1. \end{cases}$$

We multiply the differentiated equation by $2\rho(\xi)\frac{\partial y}{\partial \xi_k}$ and integrate over $D \times \Gamma$:

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\int_D\left|\frac{\partial y}{\partial\xi_k}\right|^2\rho\,dx\,d\xi = &-2\int_\Gamma\int_D a\left|\nabla\left(\frac{\partial y}{\partial\xi_k}\right)\right|^2\rho\,dx\,d\xi\\ &-2\int_\Gamma\int_D\frac{\partial a}{\partial\xi_k}\nabla y\cdot\nabla\left(\frac{\partial y}{\partial\xi_k}\right)\rho\,dx\,d\xi\\ &+2\int_\Gamma\int_D\frac{\partial f}{\partial\xi_k}\frac{\partial y}{\partial\xi_k}\rho\,dx\,d\xi\\ &+2\int_\Gamma\int_{\partial D_1}\left[\frac{\partial\alpha}{\partial\xi_k}(u-y)+\alpha\left(\frac{\partial u}{\partial\xi_k}-\frac{\partial y}{\partial\xi_k}\right)\right]\cdot\frac{\partial y}{\partial\xi_k}\rho\,d\sigma\,d\xi.\end{aligned}$$

then we have as in proof of Theorem (3.5)

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\int_D\left|\frac{\partial y}{\partial\xi_k}\right|^2\rho(\xi)\,dx\,d\xi \le\ & C_1\int_\Gamma\int_D|\nabla y|^2\rho(\xi)\,dx\,d\xi\\ &+C_4\int_\Gamma\int_D\left|\frac{\partial y}{\partial\xi_k}\right|^2\rho(\xi)\,dx\,d\xi+C_5\int_\Gamma\int_{\partial D_1}|y|^2\rho(\xi)\,d\sigma\,d\xi,\end{aligned}$$

and

$$\begin{aligned}2\int_\Gamma\int_D a\left|\nabla\left(\frac{\partial y}{\partial\xi_k}\right)\right|^2\rho\,dx\,d\xi = &-\frac{d}{dt}\int_\Gamma\int_D\left|\frac{\partial y}{\partial\xi_k}\right|^2\rho\,dx\,d\xi-2\int_\Gamma\int_D\frac{\partial a}{\partial\xi_k}\nabla y\cdot\nabla\left(\frac{\partial y}{\partial\xi_k}\right)\rho(\xi)\,dx\,d\xi\\ &+2\int_\Gamma\int_D\frac{\partial f}{\partial\xi_k}\frac{\partial y}{\partial\xi_k}\rho(\xi)\,dx\,d\xi\\ &+2\int_\Gamma\int_{\partial D_1}\left[\frac{\partial\alpha}{\partial\xi_k}(u-y)+\alpha\left(\frac{\partial u}{\partial\xi_k}-\frac{\partial y}{\partial\xi_k}\right)\right]\cdot\frac{\partial y}{\partial\xi_k}\rho(\xi)\,dx\,d\xi.\end{aligned}$$

then using Chauchy Shwartz inequality and Theorem (3.5) and Lemmas (3.2) and (3.3), we find that

$$
\begin{aligned}
C_a \int_\Gamma \int_D \left|\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right|^2 \rho(\xi)\,dx\,d\xi \leq{} & C_1 \int_\Gamma \int_D |\nabla y|^2 \rho(\xi)\,dx\,d\xi \\
& + C_4 \int_\Gamma \int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2 \rho(\xi)\,dx\,d\xi + C_5 \int_\Gamma \int_{\partial D_1} |y|^2 \rho(\xi)\,dx\,d\xi, \\
& + |D| \int_\Gamma \int_D |\nabla y|^2 \rho(\xi)\,dx\,d\xi \\
& + 2\left\|\frac{\partial f}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)} \int_\Gamma \int_D |\frac{\partial y}{\partial \xi_k}|^2 \rho(\xi)\,dx\,d\xi \\
& + 2|D|\left\|\frac{\partial \alpha}{\partial \xi_k} u\right\|_{L^\infty(D\times\Gamma)} \int_\Gamma \int_D |\frac{\partial y}{\partial \xi_k}|^2 \rho(\xi)\,dx\,d\xi \\
& + 2|D|\left\|\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)} \int_\Gamma \int_D |\frac{\partial y}{\partial \xi_k}|^2 \rho(\xi)\,dx\,d\xi \\
& + 2|D|\left\|\frac{\partial \alpha}{\partial \xi_k}\right\|_{L^\infty(D\times\Gamma)} \int_\Gamma \int_D |y|^2 \rho(\xi)\,dx\,d\xi \\
& + 2|D|\left\|\frac{\partial u}{\partial \xi_k}\alpha\right\|_{L^\infty(D\times\Gamma)} \int_\Gamma \int_D |\frac{\partial y}{\partial \xi_k}|^2 \rho(\xi)\,dx\,d\xi \\
& + 2|D|\,\|\alpha\|_{L^\infty(D\times\Gamma)} \int_\Gamma \int_D |\frac{\partial y}{\partial \xi_k}|^2 \rho(\xi)\,dx\,d\xi
\end{aligned}
$$

It follows that

$$
\begin{aligned}
C_a \int_\Gamma \int_D \left|\nabla\left(\frac{\partial y}{\partial \xi_k}\right)\right|^2 \rho(\xi)\,dx\,d\xi \leq{} & C_7 \int_\Gamma \int_D |\nabla y|^2 \rho(\xi)\,dx\,d\xi \\
& + C_8 \int_\Gamma \int_D \left|\frac{\partial y}{\partial \xi_k}\right|^2 \rho(\xi)\,dx\,d\xi + C_5 \int_\Gamma \int_{\partial D_1} |y|^2 \rho(\xi)\,dx\,d\xi, \\
& + |D| \int_\Gamma \int_D |\nabla y|^2 \rho(\xi)\,dx\,d\xi + C_9 \int_\Gamma \int_D |y|^2 \rho(\xi)\,dx\,d\xi
\end{aligned}
$$

where the constants are explicitly given by:

$$
\begin{aligned}
C_a &= \left(2\,\|a\|_{L^\infty(D\times\Gamma)} - |D|\left\|\frac{\partial a}{\partial \xi}\right\|_{L^\infty(\Gamma)}\right) \\
C_7 &= \max_{\overline{D}\times\overline{\Gamma}}\{C_1; |D|\}, \\
C_8 &= \max_{\overline{D}\times\overline{\Gamma}}\{C_4; 2\left|\frac{\partial f}{\partial \xi}\right|; 2|D|\left|\frac{\partial \alpha}{\partial \xi}u\right|; 2|D|\left|\frac{\partial u}{\partial \xi}\alpha\right|; 2|D|\left|\frac{\partial \alpha}{\partial \xi}\right|; 2|D|\,|\alpha|\}, \\
C_9 &= \max_{\overline{D}\times\overline{\Gamma}}\{C_5; 2|D|\left|\frac{\partial \alpha}{\partial \xi}\right|\}
\end{aligned}
$$

As a consequence, using Theorem (3.5) and Lemmas (3.2) and (3.3), we obtain that

$$
\begin{aligned}
C_a \int_\Gamma \int_D \left| \nabla \left(\frac{\partial y}{\partial \xi} \right) \right|^2 \rho \, dx \, d\xi \le C_7 2C_1 e^{C_1 T} & \int_\Gamma \int_D \left(|y^0|^2 \right) \rho(\xi) \, dx \, d\xi \\
& + C_8 e^{C_4 T} (\int_\Gamma \int_D \left| \frac{\partial y}{\partial \xi} \right|^2 (0) \rho(\xi) \, dx \, d\xi \\
& + \frac{C_1}{C_4} 2C_1 e^{C_1 T} \int_\Gamma \int_D |y^0|^2 \rho(\xi) \, dx \, d\xi \\
& + \frac{C_5}{C_4} e^{C_1 T} \int_\Gamma \int_D |y^0|^2 \rho(\xi) \, dx \, d\xi) \\
& + C_9 e^{C_1 T} \int_\Gamma \int_D \left(|y^0|^2 \right) \rho(\xi) \, dx \, d\xi,
\end{aligned}
$$

then

$$
\begin{aligned}
\int_\Gamma \int_D \left| \nabla \left(\frac{\partial y}{\partial \xi} \right) \right|^2 \rho \, dx \, d\xi \le \frac{1}{C_a} \Bigg[& \left(C_7 2C_1 e^{C_1 T} + C_8 e^{C_4 T} \left(\frac{2C_1^2}{C_4} e^{C_1 T} + \frac{C_5}{C_4} e^{C_1 T} \right) + C_9 e^{C_1 T} \right) \int_\Gamma \int_D |y^0|^2 \rho(\xi) \, dx \, d\xi \\
& + C_8 e^{C_4 T} \int_\Gamma \int_D \left| \frac{\partial y}{\partial \xi} \right|^2 (0) \rho(\xi) \, dx \, d\xi \Bigg].
\end{aligned}
$$

□

Theorem 3.8. *Let $y(t, x, \xi)$ be the solution of the parabolic problem with Robin boundary conditions depending on $\xi \in \Gamma \subset \mathbb{R}^d$, where the functions a, α, f, u are sufficiently regular. Then, for all $t \in [0, T]$, the following estimate holds:*

$$
\int_\Gamma \int_D y_{kk}^2(t) \, \rho(\xi) \, dx \, d\xi \le e^{C_c' T} \left(\int_\Gamma \int_D y_{kk}^2(0) \, \rho(\xi) \, dx \, d\xi + \frac{C_a'}{C_c'} \int_\Gamma \int_D |y_0|^2 \rho(\xi) \, dx \, d\xi + \frac{C_b'}{C_c'} \int_\Gamma \int_D |y_k|^2 (0) \rho(\xi) \, dx \, d\xi \right).
$$

Where

$$
\begin{aligned}
C_{10} &= |D| \cdot \max_{\overline{D} \times \overline{\Gamma}} \left\{ \frac{8}{|a|} \left| \frac{\partial a}{\partial \xi_k} \right|^2 \right\}, \\
C_{11} &= |D| \cdot \max_{\overline{D} \times \overline{\Gamma}} \left\{ \frac{4}{|a|} \left| \frac{\partial^2 a}{\partial \xi_k^2} \right|^2 \right\}, \\
C_{12} &= 2|D| \cdot \max_{\overline{D} \times \overline{\Gamma}} \left| \frac{\partial^2 f}{\partial \xi_k^2} \right|, \\
C_{13} &= 2|D| \cdot \max_{\overline{D} \times \overline{\Gamma}} \left\{ \left| \frac{\partial^2 \alpha}{\partial \xi_k^2} u \right|, \left| \frac{\partial^2 \alpha}{\partial \xi_k^2} \right|, \left| \frac{\partial \alpha}{\partial \xi_k} u_k \right|, \left| \alpha \frac{\partial^2 u}{\partial \xi_k^2} \right| \right\}, \\
C_{14} &= 2|D| \cdot \max_{\overline{D} \times \overline{\Gamma}} \left| \frac{\partial^2 \alpha}{\partial \xi_k^2} \right|, \\
C_{15} &= 2|D| \cdot \max_{\overline{D} \times \overline{\Gamma}} \left| \frac{\partial \alpha}{\partial \xi_k} \right|; \\
C_a' &= \frac{1}{C_a} \Bigg(2C_1 C_7 e^{C_1 T} + C_8 e^{C_4 T} \left(\frac{2C_1^2}{C_4} e^{C_1 T} + \frac{C_5}{C_4} e^{C_1 T} \right) + C_9 e^{C_1 T} + 2C_1 C_{11} e^{C_1 T} + C_{14} e^{C_1 T} \\
& \qquad + \frac{C_1 C_4}{2C_1} C_{15} e^{(C_1 + C_4) T} + \frac{C_5}{C_4} C_{15} e^{(C_1 + C_4) T} \Bigg), \\
C_b' &= \frac{C_8}{C_a} e^{C_4 T} + C_{15} e^{C_4 T}, \\
C_c' &= C_{12} + C_{13}.
\end{aligned}
$$

Proof. We differentiate the equation twice with respect to ξ_k. Denoting $y_k = \frac{\partial y}{\partial \xi_k}$ and $y_{kk} = \frac{\partial^2 y}{\partial \xi_k^2}$, we obtain:

$$\partial_t y_{kk} = \nabla \cdot \left(\frac{\partial^2 a}{\partial \xi_k^2} \nabla y + 2 \frac{\partial a}{\partial \xi_k} \nabla y_k + a \nabla y_{kk} \right) + \frac{\partial^2 f}{\partial \xi_k^2}.$$

The Robin boundary condition, differentiated twice on ∂D_1, gives:

$$\left(a \nabla y_{kk} + 2 \frac{\partial a}{\partial \xi_k} \nabla y_k + \frac{\partial^2 a}{\partial \xi_k^2} \nabla y \right) \cdot \mathbf{n} + \alpha y_{kk} = \frac{\partial^2 \alpha}{\partial \xi_k^2} (u - y) + 2 \frac{\partial \alpha}{\partial \xi_k} (u_k - y_k) + \alpha \frac{\partial^2 u}{\partial \xi_k^2},$$

where $u_k = \frac{\partial u}{\partial \xi_k}$.

We multiply the equation by $2 y_{kk} \rho(\xi)$, and integrate over $D \times \Gamma$.

$$\begin{aligned}
\frac{d}{dt} \int_\Gamma \int_D y_{kk}^2 \rho(\xi)\, dx\, d\xi &+ 2 \int_\Gamma \int_D a |\nabla y_{kk}|^2 \rho(\xi)\, dx\, d\xi + 2 \int_\Gamma \int_{\partial D_1} \alpha y_{kk}^2 \rho(\xi)\, dx\, d\xi \\
&= -4 \int_\Gamma \int_D \frac{\partial a}{\partial \xi_k} \nabla y_k \cdot \nabla y_{kk} \rho(\xi)\, dx\, d\xi \\
&- 2 \int_\Gamma \int_D \frac{\partial^2 a}{\partial \xi_k^2} \nabla y \cdot \nabla y_{kk} \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_D \frac{\partial^2 f}{\partial \xi_k^2} y_{kk} \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_{\partial D_1} \left(\frac{\partial^2 \alpha}{\partial \xi_k^2} (u - y) + 2 \frac{\partial \alpha}{\partial \xi_k} (u_k - y_k) + \alpha \frac{\partial^2 u}{\partial \xi_k^2} \right) y_{kk} \rho(\xi)\, dx\, d\xi.
\end{aligned}$$

We take the modulus of both sides

$$\begin{aligned}
\frac{d}{dt} \int_\Gamma \int_D y_{kk}^2 \rho(\xi)\, dx\, d\xi &+ 2 \int_\Gamma \int_D a |\nabla y_{kk}|^2 \rho + 2 \int_\Gamma \int_{\partial D_1} \alpha y_{kk}^2 \rho(\xi)\, dx\, d\xi \\
&\leq 4 \int_\Gamma \int_D |\frac{2\sqrt{a}}{2\sqrt{a}} \frac{\partial a}{\partial \xi_k} \nabla y_k \cdot \nabla y_{kk}| \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_D |\frac{\sqrt{2a}}{\sqrt{2a}} \frac{\partial^2 a}{\partial \xi_k^2} \nabla y \cdot \nabla y_{kk}| \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_D |\frac{\partial^2 f}{\partial \xi_k^2} y_{kk}| \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_{\partial D_1} |\left(\frac{\partial^2 \alpha}{\partial \xi_k^2} (u - y) + 2 \frac{\partial \alpha}{\partial \xi_k} (u_k - y_k) + \alpha \frac{\partial^2 u}{\partial \xi_k^2} \right) y_{kk}| \rho(\xi)\, dx\, d\xi.
\end{aligned}$$

We apply the Cauchy–Schwarz inequality to all the terms on the right-hand side, and we obtain:

$$\begin{aligned}
\frac{d}{dt} \int_\Gamma \int_D y_{kk}^2 \rho(\xi)\, dx\, d\xi &+ 2 \int_\Gamma \int_D a |\nabla y_{kk}|^2 \rho(\xi)\, dx\, d\xi + 2 \int_\Gamma \int_{\partial D_1} \alpha y_{kk}^2 \rho(\xi)\, dx\, d\xi \\
&\leq \int_\Gamma \int_D (|\frac{8}{a} \frac{\partial a}{\partial \xi_k} \nabla y_k|^2 + a |\nabla y_{kk}|^2) \rho(\xi)\, dx\, d\xi \\
&+ \int_\Gamma \int_D (|\frac{4}{a} \frac{\partial^2 a}{\partial \xi_k^2} \nabla y|^2 + a |\nabla y_{kk}|^2) \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_D |\frac{\partial^2 f}{\partial \xi_k^2} y_{kk}| \rho(\xi)\, dx\, d\xi \\
&+ 2 \int_\Gamma \int_{\partial D_1} |\left(\frac{\partial^2 \alpha}{\partial \xi_k^2} (u - y) + 2 \frac{\partial \alpha}{\partial \xi_k} (u_k - y_k) + \alpha \frac{\partial^2 u}{\partial \xi_k^2} \right) y_{kk}| \rho(\xi)\, dx\, d\xi.
\end{aligned}$$

we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_\Gamma\int_D y_{kk}^2\rho(\xi)\,dx\,d\xi \le{}& |D|\max_{\overline{D}\times\overline{\Gamma}}\{\frac{8}{|a|}|\frac{\partial a}{\partial \xi_k}|^2\}\int_\Gamma\int_D(|\nabla y_k|^2)\rho(\xi)\,dx\,d\xi\\
&+|D|\max_{\overline{D}\times\overline{\Gamma}}\{\frac{4}{|a|}|\frac{\partial^2 a}{\partial \xi_k^2}|^2\}\int_\Gamma\int_D(|\nabla y|^2)\rho(\xi)\,dx\,d\xi\\
&+2|D|\max_{\overline{D}\times\overline{\Gamma}}\{|\frac{\partial^2 f}{\partial \xi_k^2}|\}\int_\Gamma\int_D|y_{kk}|^2\rho(\xi)\,dx\,d\xi\\
&+2|D|\max_{\overline{D}\times\overline{\Gamma}}\{|\frac{\partial^2 \alpha}{\partial \xi_k^2}u|;|\frac{\partial^2 \alpha}{\partial \xi_k^2}|;|\frac{\partial \alpha}{\partial \xi_k}u_k|;|\frac{\partial^2 u}{\partial \xi_k^2}\alpha|\}\int_\Gamma\int_D|y_{kk}|^2\rho(\xi)\,dx\,d\xi.\\
&+2|D|\max_{\overline{D}\times\overline{\Gamma}}\{|\frac{\partial^2 \alpha}{\partial \xi_k^2}|\}\int_\Gamma\int_D|y|^2\rho(\xi)\,dx\,d\xi.\\
&+2|D|\max_{\overline{D}\times\overline{\Gamma}}\{|\frac{\partial \alpha}{\partial \xi_k}|\}\int_\Gamma\int_D|y_k|^2\rho(\xi)\,dx\,d\xi.
\end{aligned}
$$

It follows that;

$$
\begin{aligned}
\frac{d}{dt}\int_\Gamma\int_D y_{kk}^2\,\rho(\xi)\,dx\,d\xi \le{}& C_{10}\int_\Gamma\int_D|\nabla y_k|^2\,\rho(\xi)\,dx\,d\xi + C_{11}\int_\Gamma\int_D|\nabla y|^2\,\rho(\xi)\,dx\,d\xi\\
&+C_{12}\int_\Gamma\int_D y_{kk}^2\,\rho(\xi)\,dx\,d\xi + C_{13}\int_\Gamma\int_D y_{kk}^2\,\rho(\xi)\,dx\,d\xi\\
&+C_{14}e^{C_1T}\int_\Gamma\int_D|y_0|^2\rho(\xi)\,dx\,d\xi + C_{15}\int_\Gamma\int_D y_k^2\,\rho(\xi)\,dx\,d\xi
\end{aligned}
$$

where

$$
\begin{aligned}
C_{10} &= |D|\cdot\max_{\overline{D}\times\overline{\Gamma}}\left\{\frac{8}{|a|}\left|\frac{\partial a}{\partial \xi_k}\right|^2\right\},\\
C_{11} &= |D|\cdot\max_{\overline{D}\times\overline{\Gamma}}\left\{\frac{4}{|a|}\left|\frac{\partial^2 a}{\partial \xi_k^2}\right|^2\right\},\\
C_{12} &= 2|D|\cdot\max_{\overline{D}\times\overline{\Gamma}}\left|\frac{\partial^2 f}{\partial \xi_k^2}\right|,\\
C_{13} &= 2|D|\cdot\max_{\overline{D}\times\overline{\Gamma}}\left\{\left|\frac{\partial^2 \alpha}{\partial \xi_k^2}u\right|,\left|\frac{\partial^2 \alpha}{\partial \xi_k^2}\right|,\left|\frac{\partial \alpha}{\partial \xi_k}u_k\right|,\left|\alpha\frac{\partial^2 u}{\partial \xi_k^2}\right|\right\},\\
C_{14} &= 2|D|\cdot\max_{\overline{D}\times\overline{\Gamma}}\left|\frac{\partial^2 \alpha}{\partial \xi_k^2}\right|,\\
C_{15} &= 2|D|\cdot\max_{\overline{D}\times\overline{\Gamma}}\left|\frac{\partial \alpha}{\partial \xi_k}\right|;
\end{aligned}
$$

Using the preceding lemmas and theorems, we obtain:

$$\begin{aligned}
\frac{d}{dt}\int_\Gamma\int_D y_{kk}^2\,\rho\,dx\,d\xi \le \frac{C_{10}}{C_a}\Bigg[&\left(C_7 2C_1 e^{C_1T} + C_8 e^{C_4T}\left(\frac{2C_1^2}{C_4}e^{C_1T} + \frac{C_5}{C_4}e^{C_1T}\right) + C_9 e^{C_1T}\right)\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi \\
&+ C_8 e^{C_4T}\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi}\right|^2(0)\rho(\xi)\,dx\,d\xi\Bigg]. \\
&+ C_{11}2C_1e^{C_1T}\int_\Gamma\int_D \left(|y_0|^2\right)\rho(\xi)\,dx\,d\xi \\
&+ C_{12}\int_\Gamma\int_D y_{kk}^2\,\rho\,dx\,d\xi + C_{13}\int_\Gamma\int_D y_{kk}^2\,\rho\,dx\,d\xi \\
&+ C_{14}e^{C_1T}\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi + C_{15}e^{C_4T}(\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi}\right|^2(0)\rho(\xi)\,dx\,d\xi \\
&+ \frac{C_1}{C_4}2C_1e^{C_1T}\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi \\
&+ \frac{C_5}{C_4}e^{C_1T}\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi),
\end{aligned}$$

Consequently, we have

$$\frac{d}{dt}\int_\Gamma\int_D y_{kk}^2\,\rho\,dx\,d\xi \le C_a'\int_\Gamma\int_D |y_0|^2\rho(\xi)\,dx\,d\xi + C_b'\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi}\right|^2(0)\rho(\xi)\,dx\,d\xi + C_c'\int_\Gamma\int_D y_{kk}^2\,\rho\,dx\,d\xi,$$

where

$$\begin{aligned}
C_a' &= \frac{1}{C_a}\Bigg(2C_1C_7e^{C_1T} + C_8e^{C_4T}\left(\frac{2C_1^2}{C_4}e^{C_1T} + \frac{C_5}{C_4}e^{C_1T}\right) + C_9e^{C_1T} + 2C_1C_{11}e^{C_1T} + C_{14}e^{C_1T} \\
&\qquad + \frac{C_1C_4}{2C_1}C_{15}e^{(C_1+C_4)T} + \frac{C_5}{C_4}C_{15}e^{(C_1+C_4)T}\Bigg), \\
C_b' &= \frac{C_8}{C_a}e^{C_4T} + C_{15}e^{C_4T}, \\
C_c' &= C_{12} + C_{13}.
\end{aligned}$$

Finally, using Gronwall's lemma, we obtain:

$$\int_\Gamma\int_D y_{kk}^2(t)\,\rho\,dx\,d\xi \le e^{C_c'T}\left(\int_\Gamma\int_D y_{kk}^2(0)\,\rho\,dx\,d\xi + \frac{C_a'}{C_c'}\int_\Gamma\int_D |y_0|^2\rho\,dx\,d\xi + \frac{C_b'}{C_c'}\int_\Gamma\int_D \left|\frac{\partial y}{\partial \xi}\right|^2(0)\rho\,dx\,d\xi\right).$$

□

4. Convergence Estimate for the Stochastic Collocation Method

In this section, we derive a convergence estimate for the stochastic collocation approach by exploiting the previously established regularity properties together with suitable interpolation error bounds.

Lemma 4.1 (Interpolation Error Estimates)**.** *Let $I_N^\xi u$ denote the polynomial of degree N that interpolates u at the $(N+1)$ Gauss, Gauss-Radau, or Gauss-Lobatto points $\{\xi_k\}_{k=0}^N$, i.e.,*

$$I_N^\xi u(\xi) = \sum_{k=0}^N u(\xi_k)L_k(\xi),$$

where $L_k(\xi)$ are the associated Lagrange basis polynomials. Then, we have the following interpolation error bounds:

- *In the L^2-norm:*

$$\|u - I_N^\xi u\|_{L^2(-1,1)} \le CN^{-m}|u|_{H^m(-1,1)}, \quad \forall u \in H^m(-1,1),\ m \ge 1. \tag{4.1}$$

- *In the H^ℓ-norm:*

$$\|u - I_N^\xi u\|_{H^\ell(-1,1)} \le CN^{2\ell-\frac{1}{2}-m}|u|_{H^m(-1,1)}, \quad \forall u \in H^m(-1,1),\ m \ge \ell \ge 1. \tag{4.2}$$

- *For the Gauss-Lobatto interpolation, we have the optimal error estimate:*

$$\left\|(u - I_N^\xi u)'\right\|_{L^2(-1,1)} \le CN^{1-m}|u|_{H^m(-1,1)}, \quad \forall u \in H^m(-1,1),\ m \ge 1. \tag{4.3}$$

Proof. See [14], pp. 289–290. □

To present the error estimate, we first recall the definition of the mean (or expectation) of a function u:

$$\mathbb{E}[u] = \int_\Gamma \int_D u(x,t,\xi)\rho(\xi)\,dx\,d\xi. \tag{4.4}$$

Its mean square (second moment) is defined as:

$$\mathbb{V}[u] = \left(\int_\Gamma \int_D |u(x,t,\xi)|^2\rho(\xi)\,dx\,d\xi\right)^{1/2}. \tag{4.5}$$

Theorem 4.2. *[Error Estimate for the Heat Equation with Random Robin Boundary Conditions]*

Let y be the solution of system (2.2), and let y^N denote the approximate solution obtained via the stochastic collocation method. If the assumptions of Theorems (3.5), (3.2), and (3.3) are satisfied, then the following estimates for the mean and the mean square errors hold: for any $t \in (0,T]$, there exists a constant $C_T > 0$, independent of N, such that

$$\mathbb{V}\big[y - y^N\big] \le C_T N^{-2}, \tag{4.6}$$
$$\mathbb{E}\big[y - y^N\big] \le C_T N^{-2}, \tag{4.7}$$
$$\mathbb{V}\big[\nabla(y - y^N)\big] \le C_T N^{-1}, \tag{4.8}$$
$$\mathbb{E}\big[\nabla(y - y^N)\big] \le C_T N^{-1}, \tag{4.9}$$

For the Gauss–Lobatto interpolation, the following error estimate holds for the derivative of the solution with respect to the random variables: for all $0 < t \le T$ and $k = 1,\dots,d$,

$$\mathbb{V}\left[\partial_{\xi_k}(y - y^N)\right] \le C_T N^{-1}, \tag{4.10}$$
$$\mathbb{E}\left[\partial_{\xi_k}(y - y^N)\right] \le C_T N^{-1}. \tag{4.11}$$

.

Proof. The proof relies on the spatial and stochastic regularity of the solution y and its derivatives, as well as on the spectral approximation properties of the stochastic collocation method.

Let $m = 2$. For any fixed x, using inequality (4.1) from Lemma (4.1) for $u = y$, respectively, we obtain

$$\int_\Gamma \left(|y - y^N|^2\right) \rho(\xi)\, d\xi \leq CN^{-4} \int_\Gamma \left(\left|\partial_\xi^2 y\right|^2\right) \rho(\xi)\, d\xi \tag{4.12}$$

Integrating with respect to x over D and using Theorem (3.8), we obtain Eq.(4.6). Similarly, using Eq.(4.2) of Lemma (4.1) and the higher regularity proved in Theorem(3.8), we obtain Eq.(4.10).

Let now $m = 1$. Again, by the inequality(4.1) of Lemma(4.1) for $u = \nabla y$, respectively, we get

$$\int_\Gamma \left(\left|\nabla(y - y^N)\right|^2\right) \rho(\xi)\, d\xi \leq CN^{-2} \int_\Gamma \left(|\partial_\xi \nabla y|^2\right) \rho(\xi)\, d\xi. \tag{26}$$

We integrate with respect to x over D and we use Theorem (3.7). We immediately get Eq. (4.8).

Finally, Eqs.(4.7), (4.9) and (4.11) follow from the standard inequality

$$\|u\|_{L^1} \leq C'\|u\|_{L^2}$$

and the estimates(4.6), (4.8), and (4.10).

□

5. Numerical analysis

5.1. **Explicit Finite Difference Scheme.** Consider a uniform discretization of $D = [0,1]^2$ with mesh size h, and a time discretization with time step Δt. We denote $y_{i,j}^n \approx y(n\Delta t, x_i, y_j, \xi)$, where $x_i = ih, y_j = jh$.

The equation is approximated by:

$$\frac{y_{i,j}^{n+1} - y_{i,j}^n}{\Delta t} = a(\xi) \frac{y_{i+1,j}^n + y_{i-1,j}^n + y_{i,j+1}^n + y_{i,j-1}^n - 4y_{i,j}^n}{h^2} + f_{i,j}^n$$

Boundary Conditions. Homogeneous Neumann condition on ∂D_0:

$$\left.\frac{\partial y}{\partial n}\right|_{\partial D_0} = 0 \quad \Rightarrow \quad y_{-1,j}^n = y_{1,j}^n \text{ (left boundary)}, \quad y_{N+1,j}^n = y_{N-1,j}^n \text{ (right boundary)}.$$

Robin condition on ∂D_1:

$$-a(\xi)\frac{y_{N+1,j}^n - y_{N-1,j}^n}{2h} = \alpha(x_N, \xi)\left(u_{N,j}^n - y_{N,j}^n\right) \quad \Rightarrow \quad y_{N+1,j}^n = y_{N-1,j}^n - \frac{2h}{a(\xi)}\alpha(x_N,\xi)\left(u_{N,j}^n - y_{N,j}^n\right)$$

For discrete functions defined on the grid

$$\mathcal{M} := \{M_{i,j} \mid i = 0, 1, \ldots, N_x + 1,\ j = 0, 1, \ldots, N_y + 1\},$$

we introduce the following norm:

$$\|\mathcal{M}\|_{\ell^2(D)} = \left(\sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} (M_{i,j})^2\, \Delta x\, \Delta y\right)^{1/2}. \tag{32}$$

In the remainder of this work, particularly in the theoretical analysis, we assume that the solution of the system possesses the following regularity property: for every fixed random vector ξ, we have

$$y \in \mathcal{C}^1\left([0,T], \mathcal{C}^3(\overline{D})\right). \tag{33}$$

Theorem 5.1. *[Stability and convergence of the explicit scheme] Let $D = [0,1]^2$ be uniformly discretized with spatial step size h and time step $\Delta t > 0$. Consider the numerical solution $y_{i,j}^{n,\xi}$ given by the explicit scheme*

$$\frac{y_{i,j}^{n+1,\xi} - y_{i,j}^{n,\xi}}{\Delta t} = a(\boldsymbol{\xi})\frac{y_{i+1,j}^{n,\xi} + y_{i-1,j}^{n,\xi} + y_{i,j+1}^{n,\xi} + y_{i,j-1}^{n,\xi} - 4y_{i,j}^{n,\xi}}{h^2} + f_{i,j}^{n,\xi}, \tag{5.1}$$

with homogeneous Neumann boundary conditions on ∂D_0 and Robin boundary conditions on ∂D_1, defined as in the problem statement.

For any fixed random vector $\boldsymbol{\xi}$, if the time step Δt satisfies the following CFL-type stability condition (see [22]):

$$\Delta t \leq \frac{h^2}{4\, a_{\max}}, \tag{5.2}$$

where

$$a_{\max} := \sup_{\xi \in \Gamma} a(\boldsymbol{\xi}), \tag{5.3}$$

then there exists a positive constant C_T', independent of Δt and h, such that for all n with $0 \leq n \leq N_T := \lfloor T/\Delta t \rfloor$,

$$\|y(t_n,\cdot,\cdot,\boldsymbol{\xi}) - y^{n,\xi}\|_{\ell^2(D)} \leq C_T'\left(\Delta t + h^2\right). \tag{5.4}$$

Proof. We consider the stochastic heat equation discretized using an explicit time scheme and centered finite differences in space. Let $y(t,x,y,\boldsymbol{\xi})$ denote the exact solution, and $y_{i,j}^{n,\xi}$ the numerical approximation at the point (t_n, x_i, y_j).

We define the pointwise error at the n^{th} time step as:

$$e_{i,j}^{n,\xi} := y(t_n, x_i, y_j, \boldsymbol{\xi}) - y_{i,j}^{n,\xi}. \tag{5.5}$$

Injecting the exact solution y into the explicit scheme yields the error equation:

$$e_{i,j}^{n+1,\xi} = e_{i,j}^{n,\xi} + \Delta t\, \delta_h e_{i,j}^{n,\xi} + \Delta t\, \tau_{i,j}^{n,\xi}, \tag{5.6}$$

where δ_h denotes the discrete Laplace operator, and $\tau_{i,j}^{n}$ is the truncation error term representing the error due to the approximation of derivatives.

Assuming that y is sufficiently smooth , the truncation error satisfies:

$$|\tau_{i,j}^{n,\xi}| \leq C(\Delta t + h^2), \tag{5.7}$$

where C is a constant independent of n, i, j, Δt, and h.

Multiplying the error equation by $e_{i,j}^{n+1,\xi}$ and summing over all grid points, we obtain:

$$\sum_{i,j} e_{i,j}^{n+1,\xi} e_{i,j}^{n+1,\xi} h^2 = \sum_{i,j} e_{i,j}^{n,\xi} e_{i,j}^{n+1,\xi} h^2 + \Delta t \sum_{i,j} \delta_h e_{i,j}^{n} e_{i,j}^{n+1} h^2 + \Delta t \sum_{i,j} \tau_{i,j}^{n,\xi} e_{i,j}^{n+1,\xi} h^2. \tag{5.8}$$

Each term is handled using discrete integration by parts identities and the boundary conditions (Neumann and Robin), which yields an inequality of the form:

$$\|e^{n+1,\xi}\|^2_{\ell^2(D)} \leq (1 + C\Delta t)\,\|e^{n,\xi}\|^2_{\ell^2(D)} + C\Delta t(\Delta t + h^2)^2, \tag{5.9}$$

where C is a constant depending on the coefficients of the problem but independent of Δt, h, and n.

The stability of the explicit scheme is ensured under the CFL condition:

$$\Delta t \leq \frac{h^2}{4a_{\max}}, \tag{5.10}$$

where $a_{\max}$ is the upper bound of the random diffusion coefficients. Under this condition, the inequality becomes:

$$\|e^{n+1,\xi}\|^2_{\ell^2(D)} \leq (1 + \tilde{C}\Delta t)\|e^{n,\xi}\|^2_{\ell^2(D)} + \tilde{C}\Delta t(\Delta t + h^2)^2. \tag{5.11}$$

Iterating this inequality up to the final time $T = N\Delta t$, and applying the discrete Grönwall lemma, we obtain:

$$\|e^{N,\xi}\|^2_{\ell^2(D)} \leq C_T\left(\Delta t^2 + h^4\right), \tag{5.12}$$

where C_T is a constant depending on T but independent of Δt and h.

We conclude that the method is convergent of order 1 in time and order 2 in space:

$$\|e^{N,\xi}\|_{\ell^2(D)} \leq C'_T(\Delta t + h^2). \tag{5.13}$$

□

5.2. **Mean Square Error for the Finite Difference Scheme with Stochastic Collocation.** We now present the main result of this section, where we analyze the global error in solving the transient heat equation with random diffusion coefficient and random Robin boundary conditions and random initial data using the finite difference scheme coupled with a stochastic collocation method.

Let $y(t,x,y,\boldsymbol{\xi})$ denote the exact solution of the problem, $y^n_{i,j}(\boldsymbol{\xi})$ the semi-discrete (in time) solution obtained using the finite difference scheme for a fixed random vector $\boldsymbol{\xi}$, and $y^n_{i,j,h,\Delta t}(\boldsymbol{\xi})$ the fully discrete solution obtained by the stochastic collocation method.

Then, the discrete mean square error is defined by

$$\left(\int_\Gamma \|y(t_n,\cdot,\cdot,\boldsymbol{\xi}) - y^n_{h,\Delta t}(\cdot,\cdot,\boldsymbol{\xi})\|^2_{\ell^2(D)}\,\rho(\boldsymbol{\xi})\,d\boldsymbol{\xi}\right)^{1/2}, \tag{5.14}$$

Using the inequality

$$\begin{aligned}\left(\int_\Gamma \|y - y^n_{h,\Delta t}\|^2_{\ell^2(D)}\,\rho\,d\boldsymbol{\xi}\right)^{1/2} &\leq \left(\int_\Gamma \left(\|y - y^n\|_{\ell^2(D)} + \|y^n - y^n_{h,\Delta t}\|_{\ell^2(D)}\right)^2 \rho\,d\boldsymbol{\xi}\right)^{1/2}\\ &\leq \left(2\int_\Gamma \left(\|y - y^n\|^2_{\ell^2(D)} + \|y^n - y^n_{h,\Delta t}\|^2_{\ell^2(D)}\right)\rho\,d\boldsymbol{\xi}\right)^{1/2},\end{aligned} \tag{5.15}$$

and applying the spatial and temporal convergence theorems (4.2) and (5.1) , we obtain:

$$\left(\int_\Gamma \|y - y^n_{h,\Delta t}\|^2_{\ell^2(D)}\, \rho\, d\boldsymbol{\xi}\right)^{1/2} \leq C_T\left(N^{-2} + h^2\right), \tag{5.16}$$

where N denotes the number of collocation points, and $h, \Delta t$ are the spatial and temporal discretization steps.

5.3. **Numerical experiments.** We consider the model (1.1) with the spatial domain $D = [-\frac{1}{4}, \frac{1}{4}] \times [0,1]$ and the final time $T = 1$. Let $\partial D_1 = \Gamma_l \cup \Gamma_r$ and $\partial D_0 = \Gamma_u \cup \Gamma_d$, where Γ_l and Γ_r denote the left and right boundaries of D, respectively, while Γ_u and Γ_d represent the upper and lower boundaries.

Inspired by the numerical experiments in [19], we construct an exact solution to the model given by: $\forall x = (x_1, x_2) \in D,\ t \in [0,T]$,

$$y(x,t,\xi) = \left(1 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2\right)(T-t)\cos(\pi x_1)\cos(\pi x_2). \tag{5.17}$$

We take:

$$a(\xi) = \frac{1 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2}{2}, \quad \alpha(x,\xi) = \frac{1 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2}{2}\cos(\pi x_1 x_2), \tag{5.18}$$

where $\xi_k \sim \mathcal{U}(0,1)$ are independent and identically distributed random variables.

The initial condition y_0, the boundary control u, and the source term f are chosen such that the prescribed exact solution is satisfied.

TABLE 1. Mean-square error $\mathbb{E}\left[\|y - y_h^N\|^2_{l^2(D)}\right]$ and its variance as functions of h for different values of N

h	$\mathbb{E}[\|y - y_h^7\|^2]$	$\mathbb{V}[\|y - y_h^7\|^2]$	$\mathbb{E}[\|y - y_h^{10}\|^2]$	$\mathbb{V}[\|y - y_h^{10}\|^2]$	$\mathbb{E}[\|y - y_h^{80}\|^2]$	$\mathbb{V}[\|y - y_h^{80}\|^2]$
0.1	0.414	0.6881	0.214	0.481	0.104	0.181
0.05	0.0712	0.0963	0.0212	0.026	0.0103	0.0106
0.025	0.0097	0.0125	0.0027	0.0025	0.0007	0.0009
0.0125	0.0052	0.0036	0.0007	0.0008	0.0002	0.0005

To test the convergence rate, we vary the partition size in the x_1 and x_2 directions $h_{x_1} = h_{x_2} = h$ from 0.1 to 0.0125. We set time partition $\Delta t = 0.1 \times h$ to guarantee the stablity of our schema.

The analysis of the results presented in Table (1) highlights a dual convergence of the proposed scheme, both spatial and stochastic. Indeed, for a fixed number of stochastic collocation points N, reducing the spatial discretization step h leads to a significant decrease in both the mean square error and its variance. This confirms the spatial convergence of the scheme: a finer mesh improves accuracy and reduces the variability of the results. Conversely, for a fixed spatial step h, increasing the number of collocation points N also improves the approximation, as evidenced by the joint decrease in the mean error and its variance. This behavior confirms the stochastic convergence of the scheme, in accordance with the theoretical results established earlier: a denser sampling in the random space

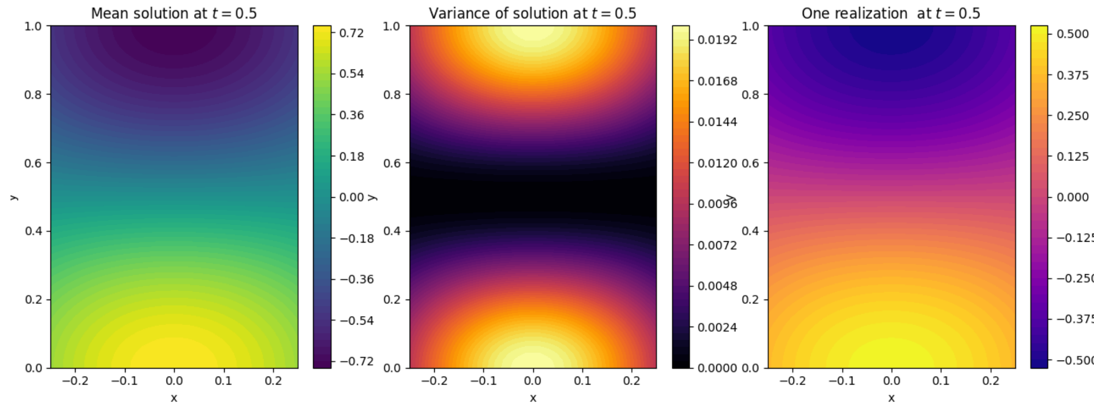

FIGURE 1. Comparison of a random sample solution and its mean and variance obtained with $\xi = (0.3107, 0.8058, 0.1270, 0.9132, 0.0124)$.

allows for a better capture of uncertainty effects and enhances the statistical robustness of the approximate solution.

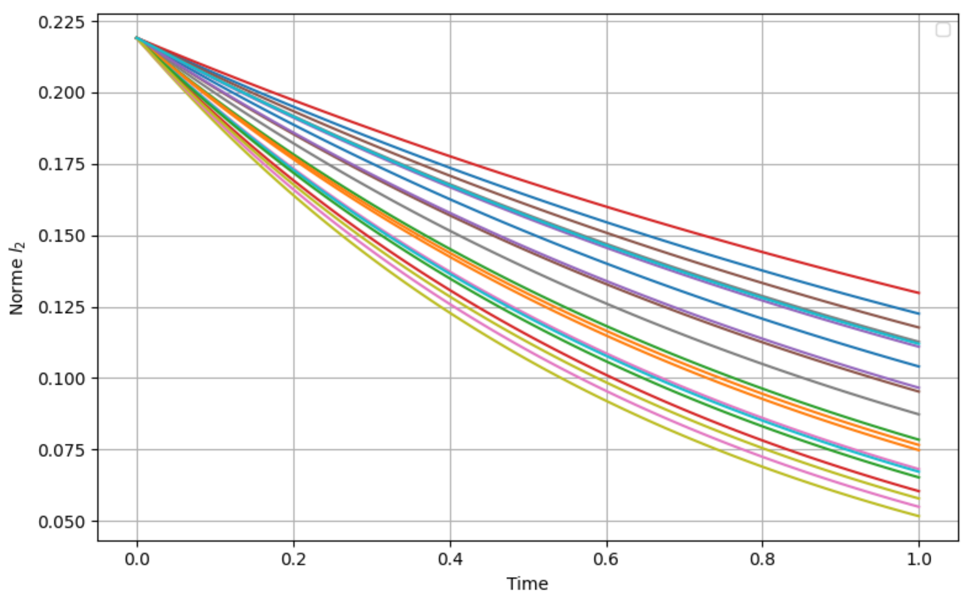

FIGURE 2. l_2 norm of the solution for the first 20 realizations.

L^2 norm of the solution for the first 10 realizations In Figure 1, we present one realization of the solution at $t = 0.5$, along with its mean and variance, obtained by solving the same problem over the domain $[-0.25, 0.25] \times [0, 1]$. The initial conditions and boundary values are set using the exact solution described above. In figure 2 the first 20 realization is shown. Then If we choose the 95% confidence level, we obtain the following result in Figure 3 in which the confidence

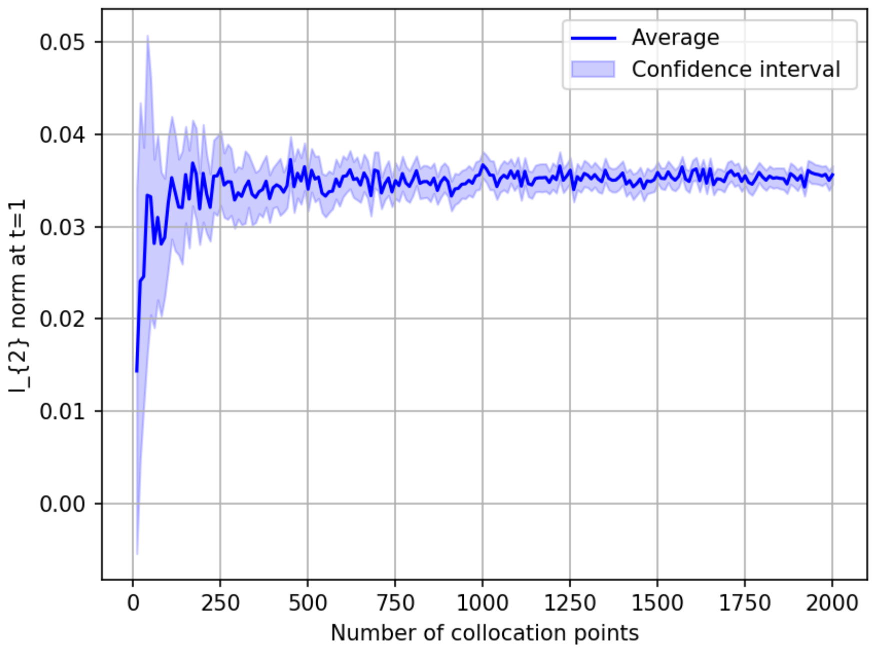

FIGURE 3. Confidence interval vs the number of collocation points.

interval are plotted as function of number of collocation points it is clear that if we chose the number of collocation points larger enough the confidence interval begin small enough.
‘

6. CONCLUSION

In this work, we developed and analyzed a stochastic collocation method combined with a finite difference scheme to solve transient heat equations with uncertain diffusion coefficients and Robin-type boundary conditions. We established stability conditions and derived error estimates that reflect the impact of input uncertainties on the solution. The method enables efficient uncertainty quantification by converting the stochastic problem into a set of deterministic ones solved at collocation points. Numerical results confirm the theoretical findings and demonstrate the effectiveness of the proposed approach in capturing the influence of random inputs on heat transfer behavior. In the future, we aim to develop a more efficient solver by combining the stochastic collocation method with machine learning techniques, with the goal of accurately predicting the physical values of uncertain parameters.

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