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SOME RESULTS OF GENERALIZATION BILATERAL LAPLACE TRANSFORM

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Abstract. In this paper, we formulate the N_F^α-Bilateral Laplace transform on time scales, which is a generalization of the α-conformable Laplace transform. The existence theorem is proven, as well as the linearity, absolute and uniform convergence, the N_F^α-Bilateral transform of integral, and the N_F^α-Bilateral transform of derivative.

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1. Introduction

In order to combine the two methods of dynamic modeling: differential and difference equations. Stefan Hilger first proposed the time scale theory in his doctoral thesis in 1988. For more developments on the time scale, we refer to [15, 5, 8, 12].

Integral transforms on time scales are generalizations of classical integral transforms, like the Laplace, Sumudu, and Fourier transforms [16, 2, 4], designed to handle equations that are a mix of continuous and discrete dynamics. These transforms are defined on arbitrary "time scales." They are used to solve dynamic equations and integral equations by simplifying them into algebraic forms. Conformable integral transforms on time scales are an important focus of this work, such as the Conformable Fourier Transform on time scales studied in [12]

$$\mathcal{F}(x,s)=\int_{-\infty}^{+\infty} f(t)\mathbb{E}^{\sigma}{}_{\ominus_\alpha ix}(t,s)\Delta^\alpha t \tag{1.1}$$

And other transformations as in the following references [6, 1, 19].
Integral transforms on time scales have the same influence for solving various ordinary, partial-dynamic and integro-dynamic equations with given initial conditions [9, 10, 11, 14, 1, 17, 18].

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This work is organized as follows. In Section 2, we introduce the basic definitions and properties of delta N_F^α-derivative, N_F^α-circle plus and minus, and N_F^α-exponential function. In Section 3, we give a new definition of Bilateral Laplace transform:
In the first result (Theorem 3.8), we show that the new transform is absolute convergence, and in the second result (Theorem 3.10), we prove that the new transform is uniform convergence. In section 4, We prove some results of this new Bilateral Laplace transform.

2. Preliminaries and Tools

Let $\mathbb{T}$ be a time scale with delta differentiation operator Δ and forward leap operator σ, respectively. Let $F : \mathbb{T} \times (0,1] \to (0,\infty)$ be a given rising function with respect to the first argument, and let $\alpha \in (0,1]$.

Definition 2.1. Assume that $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$. The delta N_F^α-derivative of f at t noted by $f_F^{(\alpha)}(t)$ is defined by

$$\Big|\big(f(\sigma(t)) - f(s)\big)F(t,\alpha) - f_F^{(\alpha)}(t)(\sigma(t)-s)\Big| \leq \epsilon|\sigma(t)-s|.$$

In the neighbourhood $U \subset \mathbb{T}$ of t.

Remark 2.2. If f is differentiable at t, then

$$f^\sigma(t) := f(\sigma(t)) = f(t) + \frac{\mu(t)}{F(t,\alpha)} f_F^{(\alpha)}(t)$$

Theorem 2.3. *Let $t \in \mathbb{T}^k$ and assume that $f : \mathbb{T} \to \mathbb{R}$ is a function. f is continuous at t if it is N_F^α-differentiable at t.*

Proof. By Remark, we have For $s \in U \cap]t-\varepsilon, t+\varepsilon[$

$$\begin{aligned}
|f(t) - f(s)| &= \left| f(t) - \frac{\mu(t)}{F(t,\alpha)} f_F^{(\alpha)}(t) - f(s) \right| \\
&= \left| f(t) - f(s) - \frac{1}{F(t,\alpha)} f_F^{(\alpha)}(t)[\sigma(t) - t] \right| \\
&= \left| f(t) - f(s) - \frac{1}{F(t,\alpha)} f_F^{(\alpha)}(t)[\sigma(t) - s + s - t] \right| \\
&= \frac{1}{F(t,\alpha)} \left| \Big([f(t) - f(s)]F(t,\alpha) - f_F^{(\alpha)}(t)[\sigma(t)-s] + f_F^{(\alpha)}(t)[t-s] \Big) \right| \\
&\leq \frac{1}{F(t,\alpha)} \Big(\Big|[f(t) - f(s)]F(t,\alpha) - f_F^{(\alpha)}(t)[\sigma(t)-s]\Big| + |f_F^{(\alpha)}(t)||t-s| \Big) \\
&\leq \varepsilon \frac{1 + |f_F^{(\alpha)}(t)|}{F(t,\alpha)}.
\end{aligned}$$

This completes the proof. $\square$

Theorem 2.4. *Let* $t \in \mathbb{T}^k$ *and assume that* $f : \mathbb{T} \to \mathbb{R}$ *is a function. If t is right-dense, then f is* N_F^{α}*-differentiable at t if and only if the limit* $\lim_{s\to t} \frac{f(t)-f(s)}{t-s} F(t,\alpha)$ *is a finite number. In this case,*

$$f_F^{(\alpha)}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} F(t,\alpha). \tag{2.1}$$

Proof. Let $\varepsilon > 0$ be given. Since f is N_F^{α}-differentiable at t, there is U of t such that

$$\left| [f(\sigma(t)) - f(s)]F(t,\alpha) - f_F^{(\alpha)}(t)(\sigma(t) - s) \right| \le \varepsilon |\sigma(t) - s|$$

Since $\sigma(t) = t$

$$\left| [f(t) - f(s)]F(t,\alpha) - f_F^{(\alpha)}(t)(t - s) \right| \le \varepsilon |t - s|$$

Then

$$\left| \frac{f(t)-f(s)}{t-s} F(t,\alpha) - f_F^{(\alpha)}(t) \right| \le \varepsilon$$

Thus

$$f_F^{(\alpha)}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} F(t,\alpha).$$

Conversely, let's say that t is right-dense and that L is the limit on the right side of (2.1). Then, there is a V_t such that

$$\left| \big(f(t) - f(s)\big)F(t,\alpha) - L(t-s) \right| \le \varepsilon |t - s|, \forall s \in V.$$

Because t is right-dense ($\sigma(t) = t$)

$$\left| \big(f(\sigma(t)) - f(s)\big)F(t,\alpha) - L(\sigma(t)-s) \right| \le \varepsilon |\sigma(t) - s|, \forall s \in V_t.$$

Then f is N_F^{α}-differentiable at t and $f_F^{(\alpha)}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} F(t,\alpha)$. □

Using Theorem 2.4 and Definition 2.1, we get the following theorem:

Theorem 2.5. *If* f *is continuous at* t *and* t *is right-scattered, then* f *is* N_F^{α}*-differentiable at* t *with*

$$f^{(\alpha)}(t) = f^{\Delta}(t)F(t,\alpha).$$

With $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$.

Proof. Assume that f is continuous at t and t is right-scattered. By continuity,

$$\lim_{s\to t} \frac{f(\sigma(t))-f(s)}{\sigma(t)-s} F(t,\alpha) = \frac{f(\sigma(t))-f(t)}{\mu(t)} F(t,\alpha)$$

Hence, given $\varepsilon > 0$, there is a neighborhood V of t such that

$$\left| \frac{f(\sigma(t))-f(s)}{\sigma(t)-s} F(t,\alpha) - \frac{f(\sigma(t))-f(t)}{\mu(t)} F(t,\alpha) \right| \le \varepsilon$$

for all $s \in V$. It follows that

$$\left| \big(f(\sigma(t)) - f(s)\big)F(t,\alpha) - \frac{f(\sigma(t))-f(t)}{\mu(t)} F(t,\alpha)(\sigma(t)-s) \right| \le \varepsilon |\sigma(t) - s|$$

Thus, f is N_F^α-differentiable at t and

$$f_F^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} F(t,\alpha).$$

□

Theorem 2.6. *Assume $f; g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then:*

(1) *$f+g$ is conformable differentiable with $(f+g)_F^{(\alpha)}(t) = f_F^{(\alpha)}(t) + g_F^{(\alpha)}(t)$;*
(2) *fg is conformable differentiable with $(fg)_F^{(\alpha)}(t) = f_F^{(\alpha)}(t)g(\sigma(t)) + f(t)g_F^{(\alpha)}$;*
(3) *$\frac{1}{f}$ is conformable differentiable with $(\frac{1}{f})_F^{(\alpha)} = -\frac{f_F^{(\alpha)}}{ff^\sigma}$ for which $ff^\sigma \neq 0$;*
(4) *$\frac{f}{g}$ is conformable differentiable with $(\frac{f}{g})_F^{(\alpha)} = \frac{f_F^{(\alpha)}(t)g(t) - f(t)g_F^{(\alpha)}(t)}{g(t)g(\sigma(t))}$, $gg^\sigma \neq 0$.*

Theorem 2.7. *Let $f : \mathbb{T} \to \mathbb{R}$ such that $f(t) = t$. Then*

$$f_F^{(\alpha)}(t) = t_F^{(\alpha)} = F(t,\alpha).$$

Proof. We have

$$f(\sigma(t)) = f(t) + \frac{\mu(t)}{F(t,\alpha)} f_F^{(\alpha)}(t)$$

then

$$\sigma(t) = t + \frac{\mu(t)}{F(t,\alpha)} f_F^{(\alpha)}(t)$$

If $\mu(t) \neq 0$;

$$f_F^{(\alpha)}(t) = \frac{\sigma(t) - t}{\mu(t)} F(t,\alpha) = F(t,\alpha)$$

If $\mu(t) \neq 0$ (t is right-dense);

$$f_F^{(\alpha)}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} F(t,\alpha) = \lim_{s \to t} \frac{s - t}{s - t} F(t,\alpha) = F(t,\alpha)$$

□

Theorem 2.8. *We have*

(1) *$(t^2)_F^{(\alpha)} = (\sigma(t) + t)\, F(t,\alpha)$;*
(2) *$(t^n)_F^{(\alpha)} = \left(t^{n-1} + \sigma^{n-1}(t) + \sum_{j=1}^{n-2} \sigma^{n-1-j}(t) t^j\right) F(t,\alpha), n \geq 3$.*

Proof. 1. From Theorem 2.7 and Theorem 2.6

$$(t^2)_F^{(\alpha)} = (t.t)_F^{(\alpha)} = t_F^{(\alpha)}\sigma(t) + t t_F^{(\alpha)} = (\sigma(t) + t)\, F(t,\alpha)$$

2. By recurrence, Theorem 2.7 and Theorem 2.6

$$\begin{aligned}
(t^{n+1})_F^{(\alpha)} &= (t^n.t)_F^{(\alpha)} = (t^n)_F^{(\alpha)}\sigma(t) + t^n t_F^{(\alpha)} \\
&= \left(t^{n-1} + \sigma^{n-1}(t) + \sum_{j=1}^{n-2} \sigma^{n-1-j}(t) t^j \right) F(t,\alpha)\sigma(t) + t^n F(t,\alpha) \\
&= \left(t^{n-1}\sigma(t) + \sigma^{n}(t) + \sum_{j=1}^{n-2} \sigma^{n-j}(t) t^j \right) F(t,\alpha) + t^n F(t,\alpha) \\
&= \left(t^{n} + \sigma^{n}(t) + \sum_{j=1}^{n-1} \sigma^{n-j}(t) t^j \right) F(t,\alpha).
\end{aligned}$$

□

Definition 2.9. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Then, the delta N_F^α-integral of f, for $\alpha \in (0,1]$, is defined by

$$\int f(t)\Delta_F^\alpha t = \int f(t)F(t,\alpha)\Delta t.$$

With

$$\mathcal{R}_F^\alpha = \{f : \mathbb{T} \to \mathbb{R}/1 + \mu(t)f(t)F(t,\alpha) \neq 0, \quad t \in \mathbb{T}\}$$

and

$$\mathcal{R}_{F+}^\alpha = \{f : \mathbb{T} \to \mathbb{R}/1 + \mu(t)f(t)F(t,\alpha) > 0, \quad t \in \mathbb{T}\}.$$

Definition 2.10. In $\mathcal{R}_F^\alpha$ 'N_F^α-circle plus' addition $\oplus_{F,\alpha}$ is defined as below

$$(h_1 \oplus_{F,\alpha} h_2)(t) = h_1(t) + h_2(t) + \mu(t)h_1(t)h_2(t)F(t,\alpha), \text{ for all } t \in \mathbb{T}^\kappa.$$

The set $\mathcal{R}_F^\alpha$ forms an abelian group under $\oplus_{F,\alpha}$.
For $h \in \mathcal{R}_F^\alpha$, the inverse of h is given as

$$\ominus_{F,\alpha} h = \frac{-h(t)}{1 + \mu(t)h(t)F(t,\alpha)}, \text{ for all } t \in \mathbb{T}^\kappa.$$

Remark 2.11.

$$1 + \mu(t) \ominus_{F,\alpha} hF(t,\alpha) = \frac{1}{1 + \mu(t)h(t)F(t,\alpha)} = -\frac{\ominus_{F,\alpha} h}{h(t)}.$$

Definition 2.12. For $\mathcal{R}_F^\alpha$, 'N_F^α-circle minus' subtraction $\ominus_{F,\alpha}$ is defined as

$$(h_1 \ominus_{F,\alpha} h_2)(t) = \frac{h_1(t) - h_2(t)}{1 + \mu(t)h_2(t)F(t,\alpha)}, \text{ for all } t \in \mathbb{T}^\kappa.$$

Definition 2.13. Let $g \in \mathcal{R}_F^\alpha$, then the N_F^α-exponential function is defined by

$$\mathbb{E}_{F g}(t,s) = exp\Big(\int_s^t \xi_{\mu(\tau)}\big(g(\tau)F(\tau,\alpha)\big)\Delta\tau \Big), \text{ for all } t,s \in \mathbb{T}. \tag{2.2}$$

Using the concept of the cylindrical transformation, Equation 2.2 can be rewritten as

$$\mathbb{E}_{F g}(t,s) = exp\Big(\int_s^t \frac{1}{\mu(\tau)} Log(1 + \mu(\tau)g(\tau)F(\tau,\alpha))\Delta\tau \Big), \text{ for all } t,s \in \mathbb{T}. \tag{2.3}$$

Theorem 2.14. *If $f \in C_{rd}(\mathbb{T})$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)F(\tau,\alpha)\Delta\tau = \mu(t)f(t)F(t,\alpha).$$

Proof. Let $f \in C_{rd}(\mathbb{T})$, then there exist an antiderivative F^{α}: $F^{(\alpha)}(t) = f(t)F(t,\alpha)$. Then

$$\begin{aligned}\int_t^{\sigma(t)} f(\tau)F(\tau,\alpha)\Delta\tau &= F(\sigma(t)) - F(t)\\ &= (\sigma(t)-t)F^{(\alpha)}(t)\\ &= \mu(t)f(t)F(t,\alpha).\end{aligned}$$

This completes the proof. □

Theorem 2.15. *Let $f, g \in \mathcal{R}^{\alpha}_{F},$, for all $s,t,r \in \mathbb{T}$ and*

(1) $\mathbb{E}_{F0}(t,s) = \mathbb{E}_{Ff}(t,t) = 1$;
(2) $\mathbb{E}_{Ff}(t,s)\mathbb{E}_{Ff}(s,r) = \mathbb{E}_{Ff}(t,r)$;
(3) $\mathbb{E}_{Ff}(t,s) = \frac{1}{\mathbb{E}_{Ff}(s,t)} = \mathbb{E}_{F\ominus_{F,\alpha}f}(s,t)$;
(4) $\mathbb{E}_{Ff}(\sigma(t),s) = (1+\mu(t)f(t)F(t,\alpha))\,\mathbb{E}_{Ff}(t,s)$;
(5) $\mathbb{E}_{F\ominus_{F,\alpha}f}(\sigma(t),s) = \frac{1+\mu(t)f(t)F(t,\alpha)}{\mathbb{E}_{Ff}(t,s)}$;
(6) $\mathbb{E}^{\Delta}_{F\,f}(t,s) = f(t)\mathbb{E}_{Ff}(t,s)F(t,\alpha)$;
(7) $\mathbb{E}^{(\alpha)}_{F\ \ f}(t,s) = f(t)\mathbb{E}_{Ff}(t,s)F^2(t,\alpha)$;
(8) $\mathbb{E}^{\Delta}_{F\ominus_{F,\alpha}f}(t,t_0) = \ominus_{F,\alpha}f(t)\mathbb{E}_{F\ominus_{F,\alpha}f}(t,t_0)F(t,\alpha)$.

Proof. Let $f, g \in \mathcal{R}^{\alpha}_{F},$, for all $s,t,r \in \mathbb{T}$. Then
1. $\mathbb{E}_{F0}(t,s) = \mathbb{E}_{Ff}(t,t) = e^0 = 1$.
2.

$$\begin{aligned}&\mathbb{E}_{Ff}(t,s)\mathbb{E}_{Ff}(s,r)\\ &= exp\Big(\int_s^t \frac{1}{\mu(\tau)}Log(1+\mu(\tau)f(\tau)F(\tau,\alpha))\Delta\tau + \int_r^s \frac{1}{\mu(\tau)}Log(1+\mu(\tau)f(\tau)F(\tau,\alpha))\Delta\tau\Big)\\ &= \int_r^t \frac{1}{\mu(\tau)}Log(1+\mu(\tau)f(\tau)F(\tau,\alpha))\Delta\tau\mathbb{E}_{Ff}(t,r).\end{aligned}$$

3.

$$\begin{aligned}\mathbb{E}_{Ff}(t,s) &= exp\Big(\int_s^t \frac{1}{\mu(\tau)}Log(1+\mu(\tau)f(\tau)F(\tau,\alpha))\Delta\tau\Big)\\ &= exp\Big(-\int_t^s \frac{1}{\mu(\tau)}Log(1+\mu(\tau)f(\tau)F(\tau,\alpha))\Delta\tau\Big)\\ &= \frac{1}{\mathbb{E}_{Ff}(s,t)}.\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}_{F\ominus_{F,\alpha}f}(s,t) &= exp\Big(\int_s^t \frac{1}{\mu(\tau)} Log\big(1+\mu(\tau)\ominus_{F,\alpha} f(\tau)F(\tau,\alpha)\big)\Delta\tau\Big)\\
&= exp\Big(\int_s^t \frac{1}{\mu(\tau)} Log\big(1-\frac{f(\tau)}{\mu(\tau)f(\tau)F(\tau,\alpha)})\Delta\tau\Big)\\
&= exp\Big(\int_s^t \frac{1}{\mu(\tau)} Log(\frac{1}{1+\mu(\tau)f(\tau)F(\tau,\alpha)})\Delta\tau\Big)\\
&= exp\Big(-\int_t^s \frac{1}{\mu(\tau)} Log(1+\mu(\tau)f(\tau)F(\tau,\alpha))\Delta\tau\Big)\\
&= \frac{1}{\mathbb{E}_{F\,f}(s,t)}.
\end{aligned}$$

Then

$$\mathbb{E}_{F\,f}(t,s) = \frac{1}{\mathbb{E}_{F\,f}(s,t)} = \mathbb{E}_{F\ominus_{F,\alpha}f}(s,t).$$

4. From (2)

$$\mathbb{E}_{F\,f}(\sigma(t),s) = \mathbb{E}_{F\,f}(\sigma(t),t)\mathbb{E}_{F\,f}(t,s)$$

and by Theorem 2.14

$$\mathbb{E}_{F\,f}(\sigma(t),s) = \big(1+\mu(t)f(t)F(t,\alpha)\big)\mathbb{E}_{F\,f}(t,s).$$

5. From (3) and (4).
6. From (4), we have

$$\begin{aligned}
\mathbb{E}^{\Delta}_{F\,f}(t,s) &= \frac{\mathbb{E}_{F\,f}(\sigma(t),s)-\mathbb{E}_{F\,f}(t,s)}{\mu(t)}\\
&= \frac{(1+\mu(t)f(t)F(t,\alpha))\,\mathbb{E}_{F\,f}(t,s)-\mathbb{E}_{F\,f}(t,s)}{\mu(t)}\\
&= f(t)\mathbb{E}_{F\,f}(t,s)F(t,\alpha).
\end{aligned}$$

7. From (6) and 2.14

$$\mathbb{E}^{(\alpha)}_{F\ \ f}(t,s) = \mathbb{E}^{\Delta}_{F\,f}(t,s)F(t,\alpha) = f(t)\mathbb{E}_{F\,f}(t,s)F^2(t,\alpha).$$

8. We have

$$\begin{aligned}
\mathbb{E}^{\Delta}_{F\ominus_{F,\alpha}f}(t,t_0) &= \frac{\mathbb{E}_{F\ominus_{F,\alpha}f}(\sigma(t),t_0)-\mathbb{E}_{F\ominus_{F,\alpha}f}(t,t_0)}{\mu(t)}\\
&= \frac{\mathbb{E}_{F\ominus_{F,\alpha}f}(\sigma(t),t)\mathbb{E}_{F\ominus_{F,\alpha}f}(t,t_0)-\mathbb{E}_{F\ominus_{F,\alpha}f}(t,t_0)}{\mu(t)}\\
&= \frac{\Big(\mathbb{E}_{F\ominus_{F,\alpha}f}(\sigma(t),t)-1\Big)\mathbb{E}_{F\ominus_{F,\alpha}f}(t,t_0)}{\mu(t)}\\
&= \ominus_{F,\alpha}f(t)\mathbb{E}_{F\ominus_{F,\alpha}f}(t,t_0)F(t,\alpha).
\end{aligned}$$

□

3. The Delta N_F^α- Bilateral Laplace Transform

Suppose that $\inf \mathbb{T} = -\infty$, $\sup \mathbb{T} = \infty$. For $s \in \mathbb{T}$, define

$$\begin{aligned} \mu_*(s) &= \inf_{t\in[s,\infty)} \mu(t), \\ \mu^*(s) &= \sup_{t\in(-\infty,s]} \mu(t), \\ \overline{\mu}(s) &= \inf_{t\in(-\infty,s]} \mu(t). \end{aligned}$$

For $s, t \in \mathbb{T}$ and $\lambda \in \mathcal{R}_F^\alpha(\mathbb{T})$, set

$$M_\lambda(t,s) = \int_t^s \frac{F(\tau,\alpha)}{1+\lambda\mu(\tau)}\Delta\tau.$$

and

$$M(t) = M_0(0,t) = \int_0^t F(\tau,\alpha)\Delta\tau. \tag{3.1}$$

Lemma 3.1. *Let $s \in \mathbb{T}$, $\lambda \in \mathcal{R}_{F+}^\alpha((-\infty,s])$. Then*

(1) $M_\lambda^\Delta(t,s) < 0$ *for all* $t \in (-\infty,s)$
(2) $\lim_{t\to-\infty} M_\lambda(t,s) = \infty$.

Proof. (1) According to the function M_λ definition, it follows

$$\begin{aligned} M_\lambda^\Delta(t,s) &= -\frac{F(t,\alpha)}{1+\lambda\mu(t)} \\ &< 0, \quad t \in (-\infty,s]. \end{aligned}$$

(2) We will consider two cases.

(a) Let $\sup_{t\in(-\infty,s]} \mu(t) < \infty$. Then

$$\begin{aligned} 1+\lambda\mu(t) &\leq 1+|\lambda|\mu(t) \\ &\leq 1+|\lambda| \sup_{t\in(-\infty,s]} \mu(t), \quad t \in (-\infty,s]. \end{aligned}$$

Hence,

$$\begin{aligned} M_\lambda(t,s) &= \int_t^s \frac{F(\tau,\alpha)}{1+\lambda\mu(\tau)}\Delta\tau \\ &\geq \int_t^s \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \end{aligned}$$

(i) Let $s > 0$. Then

$$\int_t^s \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau$$

$$
\begin{aligned}
&= \int_t^0 \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau + \int_0^s \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&\geq \int_t^0 \frac{F(t,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau + \int_0^s \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&= \frac{F(t,\alpha)(-t)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)} + \int_0^s \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}
$$

(ii) Let $s \leq 0$. Then

$$
\begin{aligned}
\int_t^s \frac{F(\tau,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau &\geq \int_t^s \frac{F(t,\alpha)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)}\Delta\tau \\
&= \frac{F(t,\alpha)(s-t)}{1+|\lambda|\sup_{t\in(-\infty,s]}\mu(t)} \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}
$$

(b) Suppose that $\sup_{t\in(-\infty,s]}\mu(t) = \infty$. Because $\lambda \in \mathcal{R}^{\alpha}_{F+}((-\infty,s])$, we have that $\lambda \geq 0$. If $\lambda = 0$, then

$$
M_\lambda(t,s) = \int_t^s F(\tau,\alpha)\Delta\tau.
$$

(i) Let $s > 0$. Then

$$
\begin{aligned}
M_\lambda(t,s) &= \int_t^0 F(\tau,\alpha)\Delta\tau + \int_0^s F(\tau,\alpha)\Delta\tau \\
&\geq F(t,\alpha)(-t) + \int_0^s F(\tau,\alpha)\Delta\tau \\
&\to \infty, \quad \text{as} \quad t\to-\infty.
\end{aligned}
$$

(ii) Let $s \leq 0$. Then

$$\begin{aligned} M_\lambda(t,s) &\geq \int\limits_t^s F(t,\alpha)\Delta\tau \\ &= F(t,\alpha)(s-t) \\ &\to \infty, \quad \text{as} \quad t \to -\infty. \end{aligned}$$

Assume that $\lambda > 0$. Since $\sup_{t\in(-\infty,s]} \mu(t) = \infty$, there exists a decreasing and divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset (-\infty, s]$ of right-scattered points such that $\{\mu(\xi_k)F(\xi_k,\alpha)\}_{k\in\mathbb{N}}$ is a divergent sequence. Hence, we get

$$\begin{aligned} M_\lambda(t,s) &= \int_t^s \frac{F(\tau,\alpha)}{1+\lambda\mu(\tau)}\Delta\tau \\ &\geq \sum_{\sigma(\xi_k)\geq t, \xi_k\leq s} \int_{\xi_k}^{\sigma(\xi_k)} F(\tau,\alpha)\Delta\tau \\ &= \sum_{\sigma(\xi_k)\geq t, \xi_k\leq s} \mu(\xi_k)F(\xi_k,\alpha) \\ &\to \infty, \quad \text{as} \quad t\to-\infty. \end{aligned}$$

□

Theorem 3.2 (Decay of the Generalized Exponential Function)**.** *Let* $s \in \mathbb{T}$, $\lambda \in \mathcal{R}^\alpha_{F+}((-\infty,s])$ *and*

$$\overline{\mathbb{C}}_{\mu^*(s)}(\lambda) = \left\{z \in \mathbb{C} : Re_{\mu^*(s)}(z) < \lambda\right\}.$$

Then, for any $z \in \overline{\mathbb{C}}_{\mu^*(s)}(\lambda)$, *we have the following properties.*

(1)
$$\left|\mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}z}(t,s)\right| \leq \mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}Re_{\mu^*(s)}(z)}(t,s), \quad t\in(-\infty,s],$$

(2)
$$\lim_{t\to-\infty} \mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}Re_{\mu^*(s)}(z)}(t,s) = 0,$$

(3) $\lim_{t\to-\infty} \mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}z}(t,s) = 0.$

Proof. Let $\Psi_h(z,\lambda)$ be the following function

$$\Psi_h(z,\lambda) = \begin{cases} \frac{1}{h}\log\left|\frac{1+h\lambda}{1+hz}\right| & \text{if} \quad h>0 \\ \lambda - \mathrm{Re}(z) & \text{if} \quad h=0. \end{cases}$$

(1) We have

$$\Psi_h(z,\lambda) = \Psi_h(\mathrm{Re}_h(z),\lambda)$$

and

$$\left|\mathbb{E}^F{}_{\lambda\ominus_{F,\alpha} z}(t,s)\right| \leq \mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)}(t,s), \quad t\in(-\infty,s].$$

(2) For $z\in\overline{\mathbb{C}}_{\mu^*(s)}(\lambda)$, we have

$$\begin{aligned}\lambda\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z) &= \frac{\lambda-\mathrm{Re}_{\mu^*(s)}(z)}{1+\mu(t)\mathrm{Re}_{\mu^*(s)}(z)F(t,\alpha)}\\ &> \frac{\lambda-\mathrm{Re}_{\mu^*(s)}(z)}{1+\lambda\mu(t)F(t,\alpha)}\\ &> 0, \quad t\in(-\infty,s],\end{aligned}$$

and

$$\begin{aligned}1+\mu(t)\left(\lambda\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)\right)F(t,\alpha) &= \frac{1+\lambda\mu(t)F(t,\alpha)}{1+\mu(t)\mathrm{Re}_{\mu^*(s)}(z)F(t,\alpha)}\\ &> \frac{1+\lambda\mu(t)F(t,\alpha)}{1+\lambda\mu(t)F(t,\alpha)}\\ &= 1, \quad t\in(-\infty,s].\end{aligned}$$

Thus,

$$\lambda\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)\in\mathcal{R}^{\alpha}_{F+}((-\infty,s])$$

and

$$\begin{aligned}\mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)}(t,s) &= e^{\left(\lambda-\mathrm{Re}_{\mu^*(s)}(z)\right)\int_s^t \frac{F(\tau,\alpha)}{1+\mu(\tau)\mathrm{Re}_{\mu^*(s)}(z)F(\tau,\alpha)}\Delta\tau}\\ &= e^{\left(\mathrm{Re}_{\mu^*(s)}(z)-\lambda\right)M_{\mathrm{Re}_{\mu^*(s)}(z)F(\tau,\alpha)}(t,s)}\\ &\to 0, \quad \text{as} \quad t\to-\infty.\end{aligned}$$

(3) By 1 and 2, we obtain

$$\begin{aligned}\left|\mathbb{E}^F{}_{\lambda\ominus_{F,\alpha} z}(t,s)\right| &\leq \mathbb{E}^F{}_{\lambda\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)}(t,s)\\ &\to 0, \quad \text{as} \quad t\to-\infty.\end{aligned}$$

This completes the proof.

□

Definition 3.3. Assume that $s\in\mathbb{T}$, $f:\mathbb{T}\to\mathbb{R}$ is regulated. The definition of the conformable bilateral Laplace transform of f is

$$\mathcal{L}_{F,\alpha}{}^b(f)(z,s) = \int_{-\infty}^{\infty} f(t)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s)\Delta^\alpha_F t$$

for $z \in \mathbb{C}$ for which $1+\mu(t)zF(t,\alpha) \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral exists.

Definition 3.4. Let $s \in \mathbb{T}$. A function $f \in \mathcal{C}_{rd}(\mathbb{T})$ has conformable double exponential of order (β,γ) on $\mathbb{T}$ (or of conformable double exponential order (β,γ)) if

(1) $\beta \in \mathcal{R}^\alpha_{F+}([s,\infty))$, $\gamma \in \mathcal{R}^\alpha_{F+}((-\infty,s])$,
(2) $\exists K_\beta$, K_γ:

$$\begin{aligned} |f(t)| &\leq K_\beta \mathbb{E}^F{}_\beta(t,s), \quad t \in [s,\infty), \\ |f(t)| &\leq K_\gamma \mathbb{E}^F{}_\gamma(t,s), \quad t \in (-\infty,s]. \end{aligned}$$

Example 3.5. Let $\gamma > 0$. Consider the function

$$f(t) = \mathbb{E}^F{}_\gamma(t,s), \quad t \in \mathbb{T}.$$

If $s \leq t$, then $\beta = \gamma$. If $s > t$, then $0 \leq f(t) \leq 1$ and $\mathbb{E}^F{}_{\ominus_{F,\alpha}\mathcal{L}_{F,\alpha}\gamma}(t,s) > 1$. Thus,

$$f(t) \leq \mathbb{E}^F{}_{\ominus_{F,\alpha}\gamma}(t,s).$$

Therefore f is of double exponential order $(\gamma, \ominus_{F,\alpha}\gamma)$.

Lemma 3.6. *Let $s \in \mathbb{T}$ and $f \in \mathcal{C}_{rd}(\mathbb{T})$ be a function of conformable double exponential order (β,γ). Then*

(1) $\lim_{t\to\infty} f(t)\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s) = 0, \forall z \in \mathbb{C}_{\mu_*(s)}(\beta)$.
(2) $\lim_{t\to-\infty} f(t)\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s) = 0, \forall z \in \overline{\mathbb{C}}_{\mu^*(s)}(\gamma)$.

Proof. Since f is of double exponential order (β,γ), there exist positive constants K_β, K_γ such that

$$\begin{aligned} |f(t)| &\leq K_\beta \mathbb{E}^F{}_\beta(t,s), \quad t \in [s,\infty), \\ |f(t)| &\leq K_\gamma \mathbb{E}^F{}_\gamma(t,s), \quad t \in (-\infty,s]. \end{aligned}$$

By Theorem 3.2, we have

$$\begin{aligned} \left| f(t)\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s) \right| &\leq K_\gamma \left| \mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}z}(t,s) \right| \\ &\leq K_\gamma \mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)}(t,s) \\ &\to 0, \quad \text{as} \quad t \to -\infty. \end{aligned}$$

□

For $z \in \mathbb{C}$, denote

$$\overline{\overline{\mu}}(s) = \mu^*(s) \quad \text{if} \quad \mathrm{Re}_{\overline{\mu}(s)}(z) \leq 0$$

and

$$\overline{\overline{\mu}}(s) = \overline{\mu}(s) \quad \text{if} \quad \mathrm{Re}_{\overline{\mu}(s)}(z) > 0.$$

Definition 3.7. Let $s \in \mathbb{T}$, $\beta \in \mathcal{R}^{\alpha}_{F+}([s,\infty))$, $\gamma \in \mathcal{R}^{\alpha}_{F+}((-\infty,s])$. We say that (s,β,γ) is an admissible triple if

$$\begin{aligned}\mathbb{C}_{s,\beta,\gamma} &= \Big\{ z \in \mathbb{C} : \mathrm{Re}_{\mu^*(s)}(z) < \gamma, \quad \mathrm{Re}_{\mu_*(s)}(z) > \beta, \\ &\qquad 1 + \overline{\overline{\mu}}(s)\mathrm{Re}_{\overline{\mu}(s)}(z) F(s,\alpha) \neq 0 \Big\}.\end{aligned}$$

Theorem 3.8. *Let $f \in \mathcal{C}_{rd}(\mathbb{T})$ is double exponential of order (β,γ), and let (s,β,γ) be an admissible triple. Then on $\mathbb{C}_{s,\beta,\gamma}$, ${\mathcal{L}_{F,\alpha}}^b(f)(\cdot,s)$ exists and converges absolutely.*

Proof. Since f is of double exponential order (β,γ), there exist positive constants K_β, K_γ such that

$$\begin{aligned}|f(t)| &\leq K_\beta {\mathbb{E}^F}_\beta(t,s), \quad t \in [s,\infty), \\ |f(t)| &\leq K_\gamma {\mathbb{E}^F}_\gamma(t,s), \quad t \in (-\infty,s].\end{aligned}$$

Next, for $t \leq s$, we have

$$\begin{aligned}|1 + \mu(t) z F(t,\alpha)| &= 1 + \mu(t)\mathrm{Re}_{\mu(t)}(z) F(t,\alpha) \\ &\geq 1 + \overline{\overline{\mu}}(s)\mathrm{Re}_{\overline{\mu}(s)}(z) F(t,\alpha).\end{aligned}$$

For $z \in \mathbb{C}_{s,\beta,\gamma}$, we find

$$\begin{aligned}\left|{\mathcal{L}_{F,\alpha}}^b(f)(z,s)\right| &= \left| \int_s^\infty f(\tau){\mathbb{E}^F}_{\ominus_{F,\alpha} z}(\sigma(\tau),s)\Delta^\alpha_F \tau + \int_{-\infty}^s f(\tau){\mathbb{E}^F}_{\ominus_{F,\alpha} z}(\sigma(\tau),s)\Delta^\alpha_F \tau \right| \\ &\leq \int_s^\infty \left| f(\tau){\mathbb{E}^F}_{\ominus_{F,\alpha} z}(\sigma(\tau),s)\right| \Delta^\alpha_F \tau + \int_{-\infty}^s \left| f(\tau){\mathbb{E}^F}_{\ominus_{F,\alpha} z}(\sigma(\tau),s)\right| \Delta^\alpha_F \tau \\ &= J_1 + J_2.\end{aligned}$$

We have

$$J_1 \leq \frac{K_\beta}{\mathrm{Re}_{\mu_*(s)}(z) - \beta}, \quad z \in \mathbb{C}_{s,\alpha,\gamma}.$$

Consider the integral

$$J = \int_t^s \left| f(\tau){\mathbb{E}^F}_{\ominus_{F,\alpha} z}(\sigma(\tau),s)\right| \Delta^\alpha_F \tau, \quad t \leq s.$$

Using Theorem 3.2, 1, 2, we obtain

$$
\begin{aligned}
J &\leq K_\gamma \int_t^s \frac{\left|\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}z}(\tau,s)\right|}{|1+\mu(\tau)zF(\tau,\alpha)|}\Delta_F^\alpha\tau\\
&= K_\gamma \int_t^s \frac{|\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}z}(\tau,s)|}{1+\mu(\tau)\mathrm{Re}_{\mu(\tau)}(z)F(\tau,\alpha)}\Delta_F^\alpha\tau\\
&\leq K_\gamma \int_t^s \frac{\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu(\tau)}(z)}(\tau,s)}{1+\mu(\tau)\mathrm{Re}_{\mu(\tau)}(z)F(\tau,\alpha)}\Delta_F^\alpha\tau\\
&= K_\gamma \int_t^s \frac{1}{\gamma-\mathrm{Re}_{\mu(\tau)}(z)}\mathbb{E}^{F(\alpha)}{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu(\tau)}(z)}(\tau,s)\Delta_F^\alpha\tau\\
&\leq \frac{K_\gamma}{\gamma-\mathrm{Re}_{\mu(\tau)}(z)}\left(1-\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu(\tau)}(z)}(t,s)\right)
\end{aligned}
$$

Therefore

$$
J_2 \leq \frac{K_\gamma}{\gamma-\mathrm{Re}_{\mu^*(s)}(z)}
$$

and

$$
\left|\mathcal{L}_{F,\alpha}{}^b(f)(z,s)\right| \leq \frac{K_\beta}{\mathrm{Re}_{\mu_*(s)}(z)-\beta}+\frac{K_\gamma}{\gamma-\mathrm{Re}_{\mu^*(s)}(z)}.
$$

Consequently the conformable bilateral Laplace transform of the function f converges absolutely on $\mathbb{C}_{s,\beta,\gamma}$. This completes the proof. □

Corollary 3.9. *Let (s,β,γ) be an admissible triple, $f\in\mathcal{C}_{rd}(\mathbb{T})$ is of conformable double exponential order (β,γ). Then*

$$
\lim_{|z|\to\infty}\mathcal{L}_{F,\alpha}{}^b(f)(z,s)=0.
$$

Proof. Note that $|z|\to\infty$ implies $\mathrm{Re}_{\mu_*(s)}(z)\to\infty$, $\mathrm{Re}_{\mu^*(s)}(z)\to\infty$ and $\mathrm{Re}_{\overline{\mu}(s)}(z)\to\infty$. □

Theorem 3.10. *Given an admissible triple (s,β,γ), $f\in\mathcal{C}_{rd}(\mathbb{T})$ is conformable double exponential of order (α,γ). The uniform convergence of $\mathcal{L}_{F,\alpha}{}^b(f)(\cdot,s)$ in $\mathbb{C}_{s,\beta,\gamma}$ follows.*

Proof. Let $\epsilon>0$ be arbitrarily chosen. We have that there exists an $r\in[s,\infty)$ such that

$$
\left|\int_t^\infty f(\tau)\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(\sigma(\tau),s)\Delta_F^\alpha\tau\right|\leq\epsilon,\quad t\in[r,\infty),\quad z\in\mathbb{C}_{s,\alpha,\gamma}.
$$

Let $t \in (-\infty, s]$ and $a \leq t$. We get

$$\begin{aligned}\left|\int_a^t f(\tau)\mathbb{E}^F{}_{\ominus_{F,\alpha} z}(\sigma(\tau), s)\Delta_F^\alpha \tau\right| &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)}\left(\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu(\tau)}(z)}(t,s) - \mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu(\tau)}(z)}(a,s)\right)\\ &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)}\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu(\tau)}(z)}(t,s)\\ &\leq \frac{K_\gamma}{\gamma - \mathrm{Re}_{\mu^*(s)}(z)}\mathbb{E}^F{}_{\gamma\ominus_{F,\alpha}\mathrm{Re}_{\mu^*(s)}(z)}(t,s).\end{aligned}$$

Thus, there exists an $r_1 \in (-\infty, s]$, such that

$$\left|\int_{-\infty}^t f(\tau)\mathbb{E}^F{}_{\ominus_{F,\alpha} z}(\sigma(\tau), s)\Delta_F^\alpha \tau\right| \leq \epsilon, \quad t \in (-\infty, r_1], \quad z \in \mathbb{C}_{s,\beta,\gamma}.$$

This completes the proof. □

Theorem 3.11. *Let $f \in \mathcal{C}_{rd}([s,\infty))$ be a conformable double exponential of order (α_1, γ_1), and let (s, α_1, γ_1) be an admissible triple. Assume that ${\mathcal{L}_{F,\alpha}}^b(f)(\cdot, s)$ on $\mathbb{C}_{s,\alpha_1,\gamma_1}$ has a limited number of regressive poles of finite order. $\{z_1, z_2, \ldots, z_n\}$ and $\widetilde{F}_{\mathbb{R}}^b(z)$ is the bilateral Laplace transform of the function $\widetilde{f}$ on $\mathbb{R}$, which is equivalent to the transform ${\mathcal{L}_{F,\alpha}}^b(f)(z,s)$ of f on $\mathbb{T}$.If*

$$\int_{c-i\infty}^{c+i\infty} \left|\widetilde{F}_{\mathbb{R}}^b(z)\right| |dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^n res_{z=z_i} \mathbb{E}^F{}_z(t,s){\mathcal{L}_{F,\alpha}}^b(f)(z,s)$$

has conformable bilateral Laplace transform ${\mathcal{L}_{F,\alpha}}^b(f)(z,s)$ for all $z \in \mathbb{C}_{s,\alpha_1,\gamma_1}\bigcap\mathbb{C}$.

Proof. Without loss of generality, suppose that $s = 0$. We have that ${\mathcal{L}_{F,\alpha}}^b(f)(\cdot)$ converges uniformly on $\mathbb{C}_{s,\alpha_1,\gamma_1}\bigcap\mathbb{C}$ and hence, it is analytic in this region. Next, we have that

$$\lim_{|z|\to\infty} {\mathcal{L}_{F,\alpha}}^b(f)(z) = 0.$$

Let C be the collection of the bilateral Laplace transforms over $\mathbb{R}$, D be the collection of the conformable bilateral Laplace transforms over $\mathbb{T}$, i.e., $C = \left\{\widetilde{F}_{\mathbb{R}}^b(z)\right\}$, $D = \left\{{\mathcal{L}_{F,\alpha}}^b(f)(z)\right\}$, where

$$\begin{aligned}\widetilde{F}_{\mathbb{R}}^b(z) &= G(z)e^{-z\tau},\\ \\ {\mathcal{L}_{F,\alpha}}^b(f)(z) &= G(z)\mathbb{E}^F{}_{\ominus_{F,\alpha} z}(\tau, 0)\end{aligned}$$

where $\tau \in \mathbb{T}$ is a constant and G a rational function. Suppose that $C_{p-c0}(\mathbb{R},\mathbb{R})$ denotes the space of piecewise continuous functions of exponential order. The space of right-hand piecewise continuous functions of conformal double exponential type, where the exponential function coincides with a conformal Hilger exponential function, is denoted $C_{prd-ez}(\mathbb{T},\mathbb{R})$. Each of the functions $\theta, \gamma, \theta^{-1}, \gamma^{-1}$

associates the functions involving the time-scale conformal exponential with the continuous exponential, and vice versa. For example, γ represents the function

$$\widetilde{F}^b_{\mathbb{R}}(z) = \frac{e^{-za}}{z}$$

to the function

$$\mathcal{L}_{F,\alpha}{}^b(f)(z) = \frac{\mathbb{E}^F{}_{\ominus_{F,\alpha} z}(a,0)}{z},$$

while γ^{-1} maps $\mathcal{L}_{F,\alpha}{}^b(f)(z)$ back to $\widetilde{F}^b_{\mathbb{R}}(z)$ in the obvious manner. If the representation of $\mathcal{L}_{F,\alpha}{}^b(f)(z)$ is independent of the exponential, that is $\tau = 0$, then γ and its inverse γ^{-1} will act as the identity. For example,

$$\gamma\left(\frac{1}{1+z^2}\right) = \gamma^{-1}\left(\frac{1}{1+z^2}\right) = \frac{1}{1+z^2}.$$

The map θ will set the continuous exponential function to the time scale conformable exponential function in the following manner: if we write $\widetilde{f} \in C_{p-c0}(\mathbb{R}, \mathbb{R})$ as

$$\widetilde{f}(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} e^{zt} \widetilde{F}^b_{\mathbb{R}}(z),$$

then

$$\theta\left(\widetilde{f}(t)\right) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} \mathbb{E}^F{}_z(t,0) \mathcal{L}_{F,\alpha}{}^b(f)(z).$$

To go from $\widetilde{F}^b_{\mathbb{R}}(z)$ to $\mathcal{L}_{F,\alpha}{}^b(f)(z)$, we simply switch expressions involving the continuous exponential in $\widetilde{F}^b_{\mathbb{R}}$ with the time scale conformable exponential giving $\mathcal{L}_{F,\alpha}{}^b(f)(z)$ as was done for γ and its inverse θ^{-1} will then act on $g \in C_{prd-ez}(\mathbb{T}, \mathbb{R})$,

$$g(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} \mathbb{E}^F{}_z(t,0) G_{\mathbb{T}}(z)$$

as

$$\theta^{-1}(g(t)) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} e^{zt} \widetilde{G}_{\mathbb{R}}(z).$$

For example, we know from the continuous result that we may express the unit step function $\widetilde{u}_a(t)$ on $\mathbb{R}$ as

$$\widetilde{u}_a(t) = \mathrm{Res}_{z=0} e^{zt} \frac{e^{-az}}{z},$$

so that if $a \in \mathbb{T}$, then

$$\theta\left(\widetilde{u}_a(t)\right) = \mathrm{Res}_{z=0} \mathbb{E}^F{}_z(t,0) \frac{\mathbb{E}^F{}_{\ominus_{F,\alpha} z}(a,0)}{z}.$$

Now, for a given time scale conformable bilateral Laplace transform $\mathcal{L}_{F,\alpha}{}^b(f)(z)$, we begin by mapping to $\widetilde{F}^b_{\mathbb{R}}(z)$ via γ^{-1}. We have that the inverse of $\widetilde{F}^b_{\mathbb{R}}(z)$ exists for all z with $z \in \mathbb{C}_{s,\alpha_1,\gamma_1} \bigcap \mathbb{C}$ and it is given by

$$\widetilde{f}(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} e^{zt} \widetilde{F}^b_{\mathbb{R}}(z).$$

Applying θ to $\widetilde{f}(t)$ to retrieve the time scale function, we get

$$f(t) = \sum_{i=1}^{n} \mathrm{Res}_{z=z_i} \mathbb{E}^F{}_z(t,0) \mathcal{L}_{F,\alpha}{}^b(f)(z),$$

whereby

$$\left(\gamma \circ \widetilde{F}^b_{\mathbb{R}} \circ \theta^{-1}\right)(f(t)) = \mathcal{L}_{F,\alpha}{}^b(f)(z).$$

This completes the proof. □

Theorem 3.12. *Suppose that the functions $f, g : \mathbb{T} \to \mathbb{R}$ have the same conformal bilateral Laplace transform and satisfy Theorem 3.11, then $f = g$ a.e.*

Proof. Suppose that

$$\int_{-\infty}^{\infty} \mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s) f(t) \Delta^\alpha_F t = \int_{-\infty}^{\infty} \mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s) g(t) \Delta^\alpha_F t.$$

Hence, $h = f - g$ has conformable bilateral Laplace transform zero and $h \in \ker \mathcal{L}_{F,\alpha}{}^b$. If we denote by $\mathcal{L}_{F,\alpha}{}^{b-1}$ the inversion of $\mathcal{L}_{F,\alpha}{}^b$, then

$$\begin{aligned} \mathcal{L}_{F,\alpha}{}^{b-1} \circ \mathcal{L}_{F,\alpha}{}^b(h) &= \mathcal{L}_{F,\alpha}{}^{b-1}(0) \\ &= 0 \\ &= h. \end{aligned}$$

Therefore $f = g$ a.e. This completes the proof. □

4. Bilateral Laplace Transform

Theorem 4.1. *Let $f, g : \mathbb{T} \to \mathbb{R}, \lambda, \mu \in \mathbb{C}$. Then*

$$\mathcal{L}_{F,\alpha}{}^b(\lambda f + \mu g)(z,s) = \lambda \mathcal{L}_{F,\alpha}{}^b(f)(z,s) + \mu \mathcal{L}_{F,\alpha}{}^b(g)(z,s).$$

For all $z \in \mathcal{R}^\alpha_F$.

Proof.

$$\begin{aligned} \mathcal{L}_{F,\alpha}{}^b(\lambda f + \mu g)(z,s) &= \int_{-\infty}^{\infty} [\lambda f(t) + \mu g(t)] \mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s) \Delta^\alpha_F t \\ &= \lambda \int_{-\infty}^{\infty} f(t) \mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s) \Delta^\alpha_F t + \mu \int_{-\infty}^{\infty} g(t) \mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s) \Delta^\alpha_F t \\ &= \lambda \mathcal{L}_{F,\alpha}{}^b(f)(z,s) + \mu \mathcal{L}_{F,\alpha}{}^b(g)(z,s). \end{aligned}$$

□

Theorem 4.2. *Assume that $H(t)$ is regulated function such that $H(t)=\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau$ such that $\lim_{t\to\pm\infty} F(t)\ominus_{F,\alpha}z(t,s)=0$. Then*

$$\mathcal{L}_{F,\alpha}{}^b\left(\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau\right)(z,s)=\frac{1}{z}\mathcal{L}_{F,\alpha}{}^b f(t)(z,s).$$

Proof. By Theorem 2.15

$$\begin{aligned}
\mathcal{L}_{F,\alpha}{}^b\left(\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau\right)(z,s) &= \int_{-\infty}^{\infty}\left(\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau\right)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= \int_{-\infty}^{\infty}\left(\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau\right)[1+\mu(t)F(t,\alpha)\ominus_{F,\alpha}z]\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= \int_{-\infty}^{\infty}\left(\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau\right)[\frac{1}{1+\mu(t)F(t,\alpha)z}]\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= -\frac{1}{z}\int_{-\infty}^{\infty}\left(\int_0^t f(\tau)F(\tau,\alpha)\Delta\tau\right)(\mathbb{E}^F)^{(\alpha)}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= \frac{1}{z}\int_{-\infty}^{\infty}\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= \frac{1}{z}\mathcal{L}_{F,\alpha}{}^b f(t)(z,s).
\end{aligned}$$

□

Theorem 4.3. *Let $f:\mathbb{T}\to\mathbb{R}$ such that $\lim_{t\to\pm\infty} f(t)\ominus_{F,\alpha}z(t,s)=0$. Then*

$$\mathcal{L}_{F,\alpha}{}^b\Big(\frac{1}{F^2(t,\alpha)}f^{(\alpha)}(t)\Big)(z,s)=z\mathcal{L}_{F,\alpha}{}^b f(t)(z,s).$$

For all $z\in\mathcal{R}_F^\alpha$.

Proof. By Theorem 2.15

$$\begin{aligned}
\mathcal{L}_{F,\alpha}{}^b\Big(\frac{1}{F^2(t,\alpha)}f^{(\alpha)}(t)\Big)(t)(z,s) &= \int_{-\infty}^{\infty}\frac{1}{F^2(t,\alpha)}f^{(\alpha)}(t)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= -\int_{-\infty}^{\infty} f(t)\ominus_{F,\alpha}z\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= z\int_{-\infty}^{\infty} f(t)\frac{\mathbb{E}^F{}_{\ominus_{F,\alpha}z}(t,s)}{1+\mu(t)zF(t,\alpha)}\Delta_F^\alpha t\\
&= z\int_{-\infty}^{\infty} f(t)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(t,s)\Delta_F^\alpha t\\
&= z\mathcal{L}_{F,\alpha}{}^b f(t)(z,s).
\end{aligned}$$

□

And by recurrence we have the following theorem.

Theorem 4.4. *Let $f:\mathbb{T}\to\mathbb{R}$ such that $\lim_{t\to\pm\infty} f^{(\alpha^k)}(t)\ominus_{F,\alpha}z(t,s)=0, k\in\{0,1,...,n-1\}$ Then*

$$\mathcal{L}_{F,\alpha}{}^b\Big(\frac{1}{F^{2n}(t,\alpha)}f^{(\alpha^n)}(t)\Big)(z,s)=z^n\mathcal{L}_{F,\alpha}{}^b f(t)(z,s).$$

where $f^{(\alpha^n)} = f^{(\alpha)} f^{(\alpha^{n-1})}$, $z \in \mathcal{R}_F^\alpha$ *and* $n \in \mathbb{N}$.

Theorem 4.5. *Let* $f \in C_{rd}(\mathbb{T})$ *be a function double exponential of order* (α, β). *Then, we have*

$$\frac{d}{dz}\mathcal{L}^b_{F,\alpha} f(t)(z,s) = -\mathcal{L}^b_{F,\alpha}\left(f(t)M^\sigma_{zF(\tau,\alpha)}\right)(z,s).$$

Proof. We have

$$\begin{aligned}
\frac{d}{dz}\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s) &= \frac{d}{dz} exp\left(-\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} Log[1 - \frac{\mu(\tau)F(\tau,\alpha)z}{1+\mu(\tau)F(\tau,\alpha)z}\Delta\tau\right);\\
&= \frac{d}{dz} exp\left(-\int_s^{\sigma(t)} \frac{1}{\mu(\tau)} Log[1 + \mu(\tau)F(\tau,\alpha)z\Delta\tau\right);\\
&= -\int_s^{\sigma(t)} \frac{F(\tau,\alpha)}{1+\mu(\tau)zF(\tau,\alpha)}\Delta\tau \mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s);\\
&= -M^\sigma_{zF(\tau,\alpha)}(t,s)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dz}\mathcal{L}_{F,\alpha}{}^b f(t)(z,s) =& \frac{d}{dz}\int_{-\infty}^{+\infty} f(t)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s)\Delta^\alpha_F t;\\
=& \int_{-\infty}^{+\infty} f(t)\frac{d}{dz}\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s)\Delta^\alpha_F t;\\
=& -\int_{-\infty}^{+\infty} f(t)M^\sigma_{zF(\tau,\alpha)}(t,s)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s)\Delta^\alpha_F t;\\
=& -\mathcal{L}^b_{F,\alpha}\left(f(t)M^\sigma_{zF(\tau,\alpha)}\right)(z,s).
\end{aligned}$$

□

Theorem 4.6. *Let* $f \in C_{rd}(\mathbb{T})$ *be a function double exponential of order* (α, β) *and* $M(t)$ *defined in (3.1). Then*

$$\mathcal{L}^b_{F,\alpha}[f * g](t)(z,s) = \mathcal{L}^b_{F,\alpha} f(t)(z,s)\ {}^s\mathcal{L}^b_{F,\alpha} g(t)(z,s).$$

Where ${}^s\mathcal{L}^b_{F,\alpha} g(t)(z,s) = \int_s^{+\infty} g\Big(M^{-1}\big(M(t) - M(s)\big)\Big)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha} z}(t,s)\Delta^\alpha_F t$ *and*

$$(f * g)(t) = \int_{-\infty}^{t} f(\tau)g\Big(M^{-1}\big(M(t) - M(\tau)\big)\Big)\Delta^\alpha_F \tau.$$

Proof. We have

$$\mathcal{L}^b_{F,\alpha}\big(f*g(t)\big)(z,s)=\int_{-\infty}^{+\infty}\big(f*g\big)(t)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(t,s)\Delta^\alpha_F t$$
$$=\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{t}f(\tau)g\Big(M^{-1}\big(M(t)-M(\tau)\big)\Big)\Delta^\alpha_F\tau\right)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(t,s)\Delta^\alpha_F t$$
$$=\int_{-\infty}^{+\infty}f(\tau)\left(\int_{\tau}^{+\infty}g\Big(M^{-1}\big(M(t)-M(\tau)\big)\Big)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(t,s)\Delta^\alpha_F t\right)\Delta^\alpha_F\tau$$
$$=\int_{-\infty}^{+\infty}f(\tau)\left(\int_{0}^{+\infty}g(u)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}\Big(M^{-1}\big(M(u)+M(\tau)\big),s\Big)\Delta^\alpha_F u\right)\Delta^\alpha_F\tau$$
$$=\int_{-\infty}^{+\infty}f(\tau)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(\tau,s)\Delta^\alpha_F\tau\left(\int_{0}^{+\infty}g(u)\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}\Big(M^{-1}\big(M(u)+M(\tau)\big),\tau\Big)\Delta^\alpha_F u\right)$$
$$=\mathcal{L}^b_{F,\alpha}(f)(z,s)\int_{\tau}^{+\infty}g(M^{-1}\big(M(v)-M(\tau)\big))\mathbb{E}^{F^\sigma}{}_{\ominus_{F,\alpha}z}(v,\tau)\Delta^\alpha_F v$$
$$=\mathcal{L}^b_{F,\alpha}(f)(z,s)^s\mathcal{L}^b_{F,\alpha}(g)(z,s).$$

This completes the proof. □

5. Conclusion

A new definition of N^α_F-Bilateral Laplace transform has been given. Many results relating to the classical Laplace and conformable Laplace case have been obtained and demonstrated in the N^α_F-Bilateral Laplace case.

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