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SOME RESULTS OF THE EXISTENCE AND BEHAVIOR OF SOLUTIONS FOR p-LAPLACIAN PARABOLIC-TYPE EQUATIONS ON NETWORKS

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Abstract. In this paper, we consider a class of discrete p-Laplacian nonlinear parabolic equations with mixed boundary conditions. First, we establish the local existence and uniqueness of bounded solutions under suitable assumptions on the nonlinearity. We then prove a comparison principle and a strict comparison principle in the linear diffusion case, which provide key insights into the ordering of solutions. Furthermore, we show that solutions remain positive or vanish in finite time, depending on the parameters involved. Regarding blow-up behavior, we prove that solutions blow up in finite time for parameter values beyond a defined threshold, under a specific condition imposed on the nonlinearity. Finally, by removing this condition, we show that there is no finite-time blow-up of the solution in the linear diffusion case under some assumptions on the nonlinearity and the parameter range.

Keywords. Discrete p-Laplacian operator, Global existence, Blow-up, Finite Graphs
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1. Introduction

In this paper, we consider the following discrete parabolic equation involving the p-Laplacian operator:

$$\begin{cases} u_t(x,t) - \Delta_{p,\omega} u(x,t) + \lambda |u(x,t)|^{p-2} u(x,t) = f(u(x,t)), \quad (x,t) \in S \times (0,+\infty), \\ \mu(x) \dfrac{\partial u}{\partial_p n}(x,t) + \sigma(x) |u(x,t)|^{p-2} u = 0, \quad (x,t) \in \partial S \times [0,+\infty), \\ u(x,0) = u_0(x), \quad x \in \bar{S}, \end{cases} \tag{1.1}$$

where S is a network with a boundary ∂S, $p > 1$, λ is a real number, σ and μ are non-negative functions defined on the boundary ∂S such that $\mu(x) + \sigma(x) > 0$, and f is a real-valued function.

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A network is a structure in which nodes are connected by edges, and it can be naturally represented as a connected graph whose vertices correspond to nodes and edges to connections. The analysis of such networks has attracted considerable attention in recent years, particularly in studying discrete weighted Laplacian operators Δ_ω and $\Delta_{p,\omega}$ (see [17, 14, 15, 18, 16, 19, 20, 13]).

Several related works have explored parabolic equations on discrete networks:

In [2], the authors studied

$$\begin{cases} u_t(x,t) - \Delta_\omega u(x,t) = u(x,t)^q, \quad (x,t) \in S \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in S, \end{cases}$$

focusing on the asymptotic behavior of nontrivial, nonnegative solutions. They showed that the long-time behavior depends critically on the sign of $q-1$.

In [5], the authors examined the problem

$$\begin{cases} u_t(x,t) - \Delta_\omega u(x,t) = |u(x,t)|^{q-1} u(x,t), \quad (x,t) \in S \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \bar{S}. \end{cases}$$

They established local existence for $q > 0$, global existence for $0 < q \leq 1$, and blow-up when $q > 1$.

In [8], the problem

$$\begin{cases} u_t(x,t) - \Delta_\omega u(x,t) = \psi(t) f(u(x,t)), \quad (x,t) \in S \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in S, \end{cases}$$

was considered. By identifying a critical set dependent on the function ψ, they demonstrated that if f belongs to this critical set, then solutions blow up in finite time for any initial data. Otherwise, the solutions may either be global or blow up, depending on the initial data u_0. Also, in [21], the authors studied this problem in the special case when $\psi \equiv 1$, they proved local existence and blow-up results of solutions to Fujita-type equations.

In [11], the authors investigated the following semilinear parabolic equations on graphs

$$\begin{cases} u_t(x,t) - \Delta_\omega u(x,t) = \psi(t) f(u(x,t)), \quad (x,t) \in S \times (0,t^\star), \\ \mu(x) \dfrac{\partial u}{\partial n}(x,t) + u(x,t) = 0, \quad (x,t) \in \Gamma_1 \times [0,t^\star), \\ \dfrac{\partial u}{\partial n}(x,t) = 0, \quad (x,t) \in \Gamma_2 \times [0,t^\star), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \bar{S}, \end{cases}$$

and they discussed the necessary and sufficient conditions for the existence and nonexistence of global solutions by introducing the so-called minorant method. Furthermore, they showed that no global solution exists for any initial data if and only if the function f satisfies

$$\int_0^\infty \psi(t) \frac{f(\epsilon \|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty} dt = +\infty,$$

for each $\epsilon > 0$, where $(S(t))_{t\geq 0}$ is the heat semigroup on networks with the corresponding boundary conditions.

For the nonlinear version involving $\Delta_{p,\omega}$, the authors in [4] investigated equation

$$\begin{cases} u_t(x,t) - \Delta_{p,\omega}u(x,t) = \lambda|u(x,t)|^{q-1}u(x,t), \quad (x,t) \in S \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \bar{S}. \end{cases}$$

where $q > 0$, $p > 1$, and $\lambda > 0$. They demonstrated global existence for $0 < q \leq 1$ or $1 < q < p-1$, and blow-up for $1 < p < q+1$, $q > 1$, and $\max_{x\in S} u_0(x) > \left(\frac{\omega_0}{\lambda}\right)^{\frac{1}{q-p+1}}$, where $\omega_0 = \max_{x\in S} \sum_{y\in\bar{S}} \omega(x,y)$.

For $\lambda = 0$ and $p > 1$ in (1.1), the authors in [7] investigated blow-up phenomena using the condition

$$(C_p) : \alpha F(s) \leq sf(s) + \beta s^p + \gamma, \ \ s > 0$$

where $\gamma > 0$, $\alpha > p$, and $0 \leq \beta \leq \frac{\alpha-p}{p}\lambda_{1,p}$. Here, $\lambda_{1,p}$ denotes the first eigenvalue of the discrete $-\Delta_{p,\omega}$.

Throughout this paper, we consider the following condition on f:

$$(f_1): \quad f : \mathbb{R} \to \mathbb{R} \text{ is a locally Lipschitz continuous function.}$$

Motivated by the aforementioned works, in the present manuscript, we establish the local existence and uniqueness of solutions to equation (1.1) for $\lambda \in \mathbb{R}$ under condition (f_1) on f. Additionally, for $\lambda \in \mathbb{R}$ and $p \in (1,2]$, or $\lambda \geq 0$ and $p > 1$, we show that the solution is either positive or vanishes in finite time whenever $u_0 \geq 0$. Furthermore, using the modified condition $(C_p)'$ of (C_p), which will be introduced in Section 3, we prove that the solution blows up in finite time when $\lambda \geq -\lambda_{1,p}$. Finally, we obtain the global phenomena that occur for any nonnegative initial data under suitable conditions.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary tools and definitions, followed by the study of local existence and comparison principles. In Section 3, we present and prove the main results concerning global existence and blow-up phenomena.

2. Local existence and Comparison Principles

In this section, we begin with definitions of graph-theoretic notions frequently used throughout this paper (see [1, 12] for more details). We then address the local existence and comparison principle, respectively.

For a graph $G := G(V,E)$, we mean finite sets V of vertices (or nodes) with a set E of two-element subsets of V (whose elements are called edges). The set of vertices and edges of a graph G are sometimes denoted by $V(G)$ and $E(G)$, or simply V and E, respectively. Conventionally, we denote by $x \in V$ or $x \in G$ the facts that x is a vertex in G.

A graph G is said to be simple if it has neither multiple edges nor loops, and G is said to be connected if, for every pair of vertices x and y, there exists a sequence (called a path) of vertices

$$x = x_0,\ x_1, \cdots, x_n = y,$$

such that x_j and x_{j+1} are connected by an edge (called adjacent) for $j = 1, \cdots, n$.

A graph $S := S(V', E')$ is called to be a subgraph of G, if $V' \subset V$ and $E' \subset E$.

A weight on a graph G is a function $\omega : V \times V \to [0, \infty)$ satisfying for $(x, y) \in V \times V$

(1) $\omega(x, y) = 0$ if and only if $x = y$ or $x \nsim y$,
(2) $\omega(x, y) = \omega(y, x)$ if $x \sim y$.

Here, $x \sim y$ means that two vertices x and y are connected(adjacent) by an edge in E.

A graph associated with a weight is said to be a weight graph or a network.

For a subgraph S of a graph G, the (vertex) boundary ∂S of S is the set of all vertices $x \in V \setminus S$ but is adjacent to some vertex in S; that is,

$$\partial S = \{x \in V \setminus S : x \sim y \text{ for some } y \in S\}.$$

We denote by $\bar{S}$, the graph whose vertices and edges are in both S and ∂S.

Throughout this paper, all subgraphs S and $\bar{S}$ in our concern are assumed to be simple and connected.

For a function $g : \bar{S} \to \mathbb{R}$, the discrete p-Laplacian on S is defined by

$$\Delta_{p,\omega} g(x) := \sum_{y \in \bar{S}} |g(y) - g(x)|^{p-2}(g(y) - g(x))\omega(x, y), \quad x \in S,$$

and the discrete p-normal derivative $\dfrac{\partial g}{\partial_p n}$ on ∂S is defined by

$$\frac{\partial g}{\partial_p n}(x) := \sum_{y \in S} |g(x) - g(y)|^{p-2}(g(x) - g(y))\omega(x, y), \quad x \in \partial S.$$

Lemma 2.1. *(See [3]) Let $p > 1$. For functions $g, h : \bar{S} \to \mathbb{R}$ the discrete p-Laplacian $\Delta_{p,\omega}$ satisfies*

$$\sum_{x \in \bar{S}} g(x)(-\Delta_{p,\omega} h(x)) = \frac{1}{2} \sum_{x,y \in \bar{S}} |h(y) - h(x)|^{p-2}(h(y) - h(x))(g(y) - g(x))\omega(x, y).$$

In particular, when $g = h$, we have

$$\sum_{x \in \bar{S}} h(x)(-\Delta_{p,\omega} h(x)) = \frac{1}{2} \sum_{x,y \in \bar{S}} |h(y) - h(x)|^{p}\omega(x, y).$$

Lemma 2.2. *(See [7, 9]) For all $p > 1$, there exists $\lambda_{1,p} > 0$ and a function $\phi_0(x) > 0, x \in S \cup \Gamma$, such that*

$$\begin{cases} -\Delta_{p,\omega}\phi_0(x) = \lambda_{1,p}|\phi_0(x)|^{p-2}\phi_0(x), \quad x \in S, \\ \mu(x)\dfrac{\partial \phi_0(x)}{\partial_p n} + \sigma(x)|\phi_0(x)|^{p-2}\phi_0(x) = 0, \quad x \in \partial S \end{cases}$$

where $\Gamma = \{x \in \partial S : \mu(x) > 0\}$ *and* σ, μ *are two positive functions defined on* ∂S *such that* $\sigma(x) + \mu(x) > 0$ *for* $x \in \partial S$*. Moreover,* $\lambda_{1,p}$ *is characterized by*

$$\lambda_{1,p} = \min_{h \in \Sigma} \frac{\frac{1}{2}\sum\limits_{x,y\in\bar{S}} |h(y) - h(x)|^p \omega(x,y) + \sum\limits_{x\in S} \frac{\sigma(x)}{\mu(x)} |h(x)|^p}{\sum\limits_{x\in S} |h(x)|^p},$$

here, $\Sigma = \{h : \bar{S} \to \mathbb{R} : h \not\equiv 0 \text{ in } S, h = 0 \text{ on } \partial S \setminus \Gamma\}$.

Before to start with the local existence, we introduce the set for a compact interval I:

$$C(\bar{S} \times I) := \left\{g : \bar{S} \times I \to \mathbb{R} : g(x, \cdot) \in C(I) \text{ for all } x \in \bar{S}\right\}.$$

Also, we need the following modified version of the Arzelà-Ascoli theorem(see [10]).

Theorem 2.3. *Let* F *be a compact subset of* $\mathbb{R}$ *and* $\bar{S}$ *be a network. Consider a Banach space* $C(\bar{S} \times F)$ *equipped with the maximum norm* $\|g\|_{\bar{S},F} := \max\limits_{(x,t)\in\bar{S}\times F} |g(x,t)|$*. Then a subset* E *of* $C(\bar{S} \times F)$ *is relatively compact if* E *is uniformly bounded on* $\bar{S} \times F$ *and equicontinuous on* F *for each* $x \in \bar{S}$*.*

Now, we are in the position to prove the local existence.

Theorem 2.4. *Let* f *satisfy* (f_1)*, then, for sufficiently small positive numbers* T*, Eq.* (1.1) *has a unique bounded solution* u *such that for any* $x \in \bar{S}$*,* $u(x, \cdot)$ *is continuous and differentiable on* $[0, T]$*.*

Proof. Let

$$C(\bar{S} \times [0,T]) := \left\{g : \bar{S} \times [0,T] \to \mathbb{R} : g(x, \cdot) \in C([0,T]) \text{ for all } x \in \bar{S}\right\},$$

with the maximum norm $\|g\| := \|g\|_{\bar{S},[0,T]} = \max\limits_{(x,t)\in\bar{S}\times[0,T]} |g(x,t)|$, where

$$T = \frac{\|u_0\|}{(4\|u_0\|)^{p-1}\big(\omega_0 + |\lambda|\big) + 4M\,\|u_0\|}$$

and $\omega_0 := \max\limits_{x\in\bar{S}} \sum\limits_{y\in\bar{S}} \omega(x,y)$.

Now consider the subspace

$$B_T := \left\{g \in C(\bar{S} \times [0,T]) : \|g\| \le 2\|u_0\|\right\}$$

of a Banach space $C(\bar{S} \times [0,T])$.

From [10], it si well know that B_T is closed.

Next, for $(x,t) \in S \times [0,T]$ define an operator $\Psi : B_T \to B_T$ by

$$\Psi(u)(x,t) := u_0 + \int_0^t [\Delta_{p,\omega} u(x,s) - \lambda |u(x,s)|^{p-2} u(x,s) + f(u(x,s))] ds.$$

In view of the definition of T, we see that the operator Ψ is well-defined. Now, we will show that Ψ is continuous and $\Psi(B_T)$ is relatively compact. Let $u, v \in B_T$

and $t \in [0,T]$, then, by the main value, there exists $C_1, C_2 > 0$ depending on $p, \|u_0\|$ such that

$$\begin{aligned}
|\Psi(u)-\Psi(v)| &= \left|\int_0^t [\Delta_{p,\omega}u - \Delta_{p,\omega}v - \lambda(|u|^{p-2}u - |v|^{p-2}v) + (f(u)-f(v))]ds\right| \\
&\leq \int_0^t \left[|\Delta_{p,\omega}u - \Delta_{p,\omega}v| + |\lambda| \left||u|^{p-2}u - |v|^{p-2}v\right| + |(f(u)-f(v))|\right] ds \\
&\leq \int_0^t \left(C_1 2^{p-1}\|u-v\|^{p-1}\omega_0 + C_2|\lambda|\|u-v\|^{p-1} + M\|u-v\|\right) ds,
\end{aligned}$$

here, M is the Lipschitz constant on the compact interval $[-2\|u_0\|, 2\|u_0\|]$. Therefore we get the continuity of Ψ. In order to show that $\Psi(B_T)$ is relatively compact, we use Theorem 2.3 it suffices to show that $\Psi(B_T)$ is uniformly bounded on $\bar{S}\times[0,T]$ and equicontinuous on $[0,T]$. It is clear that $\Psi(B_T)$ uniformly bounded on $\bar{S}\times[0,T]$, since $\Psi(B_T)\subset B_T$. Furthermore, for any $x \in \bar{S}$, $t_1, t_2 \in [0,T]$, and $u \in B_T$ it follows that

$$|\Psi(u)(x,t_1) - \Psi(u)(x,t_2)| \leq \left[2^{p-1}(4\|u_0\|)^{p-1}\omega_0 C_1 + |\lambda|(4\|u_0\|)^{p-1}C_2 + 4M\|u_0\|\right]|t_1 - t_2|,$$

which indicates that $\Psi(B_T)$ is equicontinuous on $[0,T]$. Hence by the Schauder fixed point theorem there is a function $u \in B_T$ satisfying $\Psi(u) = u$.

On the other hand, the function

$$g(\delta) := \sum_{x\in S} \mu(y)|\delta - u(x,t)|^{p-2}(\delta - u(x,t))\omega(x,y) + \sigma(y)|\delta|^{p-2}\delta$$

for $y \in \partial S, t \in [0,T]$, and $\delta \in \mathbb{R}$ is continuous bijective. Then, there exists a unique $\delta \in \mathbb{R}$ such that $g(\delta) = 0$. Next, we set

$$v(x,t) := \begin{cases} u(x,t), \text{ for } (x,t) \in S\times[0,T], \\ \delta, \text{ for } (x,t)\in \partial S \times [0,T]. \end{cases}$$

Then, v satisfies the boundary condition $\mu(x)\dfrac{\partial v}{\partial_p n}(x,t) + \sigma(x)|v(x,t)|^{p-2}v(x,t) = 0, x \in \partial S$. Thus, v solve the equation (1.1). Moreover, v is bounded and continuous on $\bar{S}\times[0,T]$, and for each $x \in \bar{S}$, $v(x,\cdot)$ is differentiable in $(0,T)$.

Now, we prove the uniqueness of the local solution. To this end, we assume that there exist two solutions, $u(x,t) := u(x)$ and $v(x,t) := v(x)$. Let $w := u - v$. Then, we have:

$$\sum_{x\in S} u_t(x)w(x) = \sum_{x\in\bar{S}} \Delta_{p,\omega}u(x)w(x) + \sum_{x\in\partial S}\frac{\partial u(x)}{\partial_p n}w(x) - \sum_{x\in S}\lambda|u(x)|^{p-2}u(x)w(x) + \sum_{x\in\bar{S}} f(u(x))w(x)$$

and

$$\sum_{x\in S} v_t(x)w(x) = \sum_{x\in\bar{S}} \Delta_{p,\omega}v(x)w(x) + \sum_{x\in\partial S}\frac{\partial v(x)}{\partial_p n}w(x) - \sum_{x\in S}\lambda|v(x)|^{p-2}v(x)w(x) + \sum_{x\in\bar{S}} f(v(x))w(x).$$

Then,

$$\begin{aligned}\sum_{x\in S} w_t(x)w(x) &= \sum_{x\in\bar{S}}(\Delta_{p,\omega}u(x)-\Delta_{p,\omega}v(x))w(x)+\sum_{x\in\partial S}(\frac{\partial u(x)}{\partial_p n}-\frac{\partial v(x)}{\partial_p n})w(x)\\ &-\lambda\sum_{x\in S}(|u(x)|^{p-2}u(x)-|v(x)|^{p-2}v(x))w(x)+\sum_{x\in\bar{S}}(f(u(x))-f(v(x)))w(x)\\ &=\sum_{x\in\bar{S}}(\Delta_{p,\omega}u(x)-\Delta_{p,\omega}v(x))w(x)-\sum_{x\in\Gamma}\frac{\sigma(x)}{\mu(x)}(|u(x)|^{p-2}u(x)-|v(x)|^{p-2}v(x))w(x)\\ &-\lambda\sum_{x\in S}(|u(x)|^{p-2}u(x)-|v(x)|^{p-2}v(x))w(x)+\sum_{x\in\bar{S}}(f(u(x))-f(v(x)))w(x)\\ &\le\sum_{x\in\bar{S}}(\Delta_{p,\omega}u(x)-\Delta_{p,\omega}v(x))w(x)-\sum_{x\in\Gamma}\frac{\sigma(x)}{\mu(x)}(|u(x)|^{p-2}u(x)-|v(x)|^{p-2}v(x))w(x)\\ &+|\lambda|\sum_{x\in S}(|u(x)|^{p-2}u(x)-|v(x)|^{p-2}v(x))w(x)+\sum_{x\in\bar{S}}(f(u(x))-f(v(x)))w(x).\end{aligned}$$

Next, using the monotonicity of the operator $\Delta_{p,\omega}$(see [6]) and the function $h(s)=\dfrac{1}{p}|s|^p,\ p>1$, obtain:

$$\frac{d}{dt}\left[\sum_{x\in S}w^2(x)\right]\le 2(C|\lambda|+M)\sum_{x\in S}w^2(x),$$

since $f(s)=|s|^{p-2}s$ is locally Lipschitz on each compact interval in $\mathbb{R}$, where $C,M>0$ are constants obtained from the Lipschitz condition. Hence, Gronwall's Lemma asserts that:

$$\begin{aligned}\sum_{x\in S}w^2(x)&\le\left[\sum_{x\in S}w_0^2(x)\right]e^{-2(C|\lambda|+M)t}\\ &=0,\quad (x,t)\in S\times[0,T),\end{aligned}$$

since, $w_0(x)=0$. The proof now is complete. □

Now, we are ready to prove the following two types of comparison principles.

Theorem 2.5. *(Comparison principle) Let $T>0$(T may be ∞), $p\in(1,2]$, and f satisfying (f_1). Suppose that real-valued functions $g(x,t),h(x,t)$ are continuous in $[0,T)$ and differentiable on $(0,T)$ for all $x\in\bar{S}$ and satisfy*

$$\begin{cases} g_t-\Delta_{p,\omega}g+\lambda|g|^{p-2}g-f(g)\ge h_t-\Delta_{p,\omega}h+\lambda|h|^{p-2}h-f(h), & x\in S, t\in(0,T),\\ \mu(x)\dfrac{\partial g}{\partial_p n}+\sigma(x)|g|^{p-2}g\ge\mu(x)\dfrac{\partial h}{\partial_p n}+\sigma(x)|h|^{p-2}h, & x\in\partial S, t\in[0,T),\\ g(x,0):=g_0(x)\ge h(x,0):=h_0(x), & x\in\bar{S}.\end{cases}\tag{2.1}$$

Then $g(x,t)\ge h(x,t)$ for all $(x,t)\in\bar{S}\times[0,T)$.

Proof. Let $T_1\in(0,T)$ and $\tilde{g},\tilde{h}:\bar{S}\times[0,T_1]\to\mathbb{R}$ be the functions defined by

$$\tilde{g}(x,t):=e^{-\alpha t}g(x,t)\ \text{ and }\ \tilde{h}(x,t):=e^{-\alpha t}h(x,t),$$

where $\alpha > 0$ can be chosen latter. Then, by (2.1) it follows that

$$\tilde{g}_t - \tilde{h}_t - e^{\alpha(p-2)t}(\Delta_{p,\omega}\tilde{g} - \Delta_{p,\omega}\tilde{h}) + \lambda e^{\alpha(p-2)t}(|\tilde{g}|^{p-2}\tilde{g} - |\tilde{h}|^{p-2}\tilde{h}) - e^{-\alpha t}(f(g) - f(h)) + \alpha(\tilde{g} - \tilde{h}) \geq 0, \tag{2.2}$$

for all $(x,t) \in S \times (0, T_1]$.

By the mean value theorem, for each $(x,t) \in S \times [0, T_1]$, there exists $c(x,t) := c$ lying between $\tilde{g}(x,t)$ and $\tilde{h}(x,t)$ such that

$$|\tilde{g}|^{p-2}\tilde{g} - |\tilde{h}|^{p-2}\tilde{h} = (p-1)|c|^{p-2}(\tilde{g} - \tilde{h}).$$

Then, (2.2) becomes

$$\tilde{g}_t - \tilde{h}_t - e^{\alpha(p-2)t}(\Delta_{p,\omega}\tilde{g} - \Delta_{p,\omega}\tilde{h}) + (\alpha + \lambda(p-1)|c|^{p-2}e^{\alpha(p-2)t})(\tilde{g} - \tilde{h}) - e^{-\alpha t}(f(g) - f(h)) \geq 0, \tag{2.3}$$

for all $(x,t) \in S \times (0, T_1]$.

Let $(x_0, t_0) \in \bar{S} \times [0, T_1]$ such that $(\tilde{g} - \tilde{h})(x_0, t_0) = \min\limits_{(x,t)\in\bar{S}\times[0,t_1]} (\tilde{g} - \tilde{h})(x,t)$, since $\tilde{g} - \tilde{h}$ is continuous on the compact set $\bar{S} \times [0, T_1]$. Then, it remains to show that $(\tilde{g} - \tilde{h})(x_0, t_0) \geq 0$. By contradiction, if it false, we have

$$\tilde{g}(x, t_0) - \tilde{g}(x_0, t_0) \geq \tilde{h}(x, t_0) - \tilde{h}(x_0, t_0), \quad x \in \bar{S}. \tag{2.4}$$

From

$$\mu(x)\frac{\partial g}{\partial_p n} + \sigma(x)|g|^{p-2}g \geq \mu(x)\frac{\partial h}{\partial_p n} + \sigma(x)|h|^{p-2}h, \quad (x,t) \in \partial S \times [0, T_1],$$

we get

$$\mu(x)\frac{\partial \tilde{g}}{\partial_p n} + \sigma(x)|\tilde{g}|^{p-2}\tilde{g} \geq \mu(x)\frac{\partial \tilde{h}}{\partial_p n} + \sigma(x)|\tilde{h}|^{p-2}\tilde{h}, \quad (x,t) \in \partial S \times [0, T_1],$$

i.e.,

$$\begin{aligned} 0 \leq \mu(x)\sum_{y\in S}&\Big[|\tilde{g}(x,t_0) - \tilde{g}(y,t_0)|^{p-2}(\tilde{g}(x,t_0) - \tilde{g}(y,t_0)) - |\tilde{h}(x,t_0) - \tilde{h}(y,t_0)|^{p-2}(\tilde{h}(x,t_0) - \tilde{h}(y,t_0))\Big]\,\omega(x,y) \\ &+ \sigma(x)(|\tilde{g}|^{p-2}\tilde{g} - |\tilde{h}|^{p-2}\tilde{h}), \ x \in \partial S. \end{aligned} \tag{2.5}$$

The main value theorem implies the existence of $c_1(x,y,t_0) := c_1$ lying between $\tilde{g}(x,t_0) - \tilde{g}(y,t_0)$ and $\tilde{h}(x,t_0) - \tilde{h}(y,t_0)$ such that

$$\begin{aligned} &|\tilde{g}(x,t_0) - \tilde{g}(y,t_0)|^{p-2}(\tilde{g}(x,t_0) - \tilde{g}(y,t_0)) - |\tilde{h}(x,t_0) - \tilde{h}(y,t_0)|^{p-2}(\tilde{h}(x,t_0) - \tilde{h}(y,t_0)) = \\ &\qquad (p-1)|c_1|^{p-2}[\tilde{g}(x,t_0) - \tilde{g}(y,t_0) - \tilde{h}(x,t_0) + \tilde{h}(y,t_0)]. \end{aligned} \tag{2.6}$$

Combinig (2.5) and (2.6), we obtain

$$\begin{aligned} 0 \leq \mu(x)\sum_{y\in S}&(p-1)|c_1|^{p-2}\Big[\tilde{g}(x,t_0) - \tilde{g}(y,t_0) - \tilde{h}(x,t_0) + \tilde{h}(y,t_0)\Big]\,\omega(x,y) \\ &+ \sigma(x)(p-1)|c|^{p-2}(\tilde{g}(x,t_0) - \tilde{h}(x,t_0)), \quad x \in \partial S. \end{aligned} \tag{2.7}$$

Hence, if $x_0 \in \partial S$ and $\sigma(x_0) > 0$, it follows from (2.4) that the equation (2.7) is negative, which leads to a contradiction. If $x_0 \in \partial S$ and $\sigma(x_0) = 0$, it follows from (2.4) and (2.5) that

$$\tilde{g}(x_0,t_0) - \tilde{g}(y,t_0) - \tilde{h}(x_0,t_0) + \tilde{h}(y,t_0) = 0, \quad \forall\, y \in S,$$

that is,

$$(\tilde{g}-\tilde{h})(x_0,t_0)=(\tilde{g}-\tilde{h})(x,t_0), \forall\, x\in S,$$

thus we may assume that $x_0\in S$, and by (2.1) we get $t_0>0$.

Note that, (2.6) hold true for $x,y\in S$, then by (2.4), we deduce that

$$\Delta_{p,\omega}\tilde{g}(x_0,t_0)\geq \Delta_{p,\omega}\tilde{h}(x_0,t_0), \tag{2.8}$$

and from the differentiability of $(\tilde{g}-\tilde{h})(x,\cdot)$ on $[0,T_1]$ for any $x\in\bar{S}$ we conclude that

$$(\tilde{g}_t-\tilde{h}_t)(x_0,t_0)\leq 0. \tag{2.9}$$

On the other hand, since f is locally Lipschitz on $\mathbb{R}$, then there exists $M>0$ such that

$$|f(\tilde{g}(x_0,t_0))-f(\tilde{h}(x_0,t_0))|\leq M|\tilde{g}(x_0,t_0)-\tilde{h}(x_0,t_0)|,\quad \tilde{g},\tilde{h}\in[-\beta,\beta],$$

where $\beta=\max\limits_{(x,t)\in\bar{S}\times[0,T_1]}\{|u(x,t)|,|v(x,t)|\}$. Then,

$$\begin{aligned}(\alpha+\lambda(p-1)|c|^{p-2}e^{\alpha(p-2)t_0})&(\tilde{g}(x_0,t_0)-\tilde{h}(x_0,t_0))-e^{-\alpha t_0}(f(g(x_0,t_0))-f(h(x_0,t_0)))\\ &\leq[\alpha+\lambda(p-1)|c|^{p-2}e^{\alpha(p-2)t_0}-M](\tilde{g}(x_0,t_0)-\tilde{h}(x_0,t_0)).\end{aligned} \tag{2.10}$$

Now, we choose $\alpha>0$ sufficiently large so that $\alpha+\lambda(p-1)|c|^{p-2}e^{\alpha(p-2)t_0}-M>0$, since $\alpha+\lambda(p-1)|c|^{p-2}e^{\alpha(p-2)t_0}\to+\infty$ as $\alpha\to+\infty$.

Combining (2.8)-(2.10), we obtain (2.3) is negative, which leads to a contradiction.

Thus, $(\tilde{g}-\tilde{h})(x_0,t_0)\geq 0$ and from (2.1), we get $(\tilde{g}-\tilde{h})(x,t)\geq 0,\forall\,(x,t)\in S\times[0,T_1]$. The proof is now complete, since $T_1<T$ is arbitrary. □

Theorem 2.6. *(Strictly Comparison Principle) Assume that $p=2$ in Theorem 2.5. If there exists $x_1\in S$ such that $g_0(x_1)>h_0(x_1)$, then $g(x,t)>h(x,t),\forall\,(x,t)\in S\cup\Gamma\times(0,T)$.*

Proof. By Theorem 2.5, it is well known that $g(x,t)\geq h(x,t)$ for all $(x,t)\in S\cup\Gamma\times(0,T)$. Next, for $(x,t)\in\bar{S}\times[0,T_1]$, we define $\phi(x,t):=(g-h)(x,t)$.

From (2.1), we have

$$\phi_t(x_1,t)-\Delta_\omega\phi(x_1,t)-(f(g(x_1,t))-f(h(x_1,t)))\geq 0,\quad \forall\, t\in(0,T_1]. \tag{2.11}$$

Then,

$$\begin{aligned}\phi_t(x_1,t)&\geq\Delta_\omega\phi(x_1,t)+(f(g(x_1,t))-f(h(x_1,t)))\\ &\geq-\gamma\phi(x_1,t),\quad t\in(0,T_1],\end{aligned}$$

where $\gamma:=\left(\sum\limits_{x\in\bar{S}}\omega(x_1,x)+M\right)>0$.

Hence,

$$\phi(x_1,t)\geq\phi_0(x_1)e^{-\gamma t}>0,\quad t\in[0,T_1], \tag{2.12}$$

since $\phi_0(x_1):=\phi(x_1,0)>0$.

Note that, $\phi(x,t)\geq 0$ for all $(x,t)\in S\cup\Gamma\times(0,T_1]$, then it remains to show that $\phi(x,t)\neq 0$ for all $(x,t)\in S\cup\Gamma\times(0,T_1]$. Otherwise, there exists $(x_0,t_0)\in S\cup\Gamma\times(0,T_1]$ such that $\phi(x_0,t_0)=0$.

If $x_0 \in \Gamma$, we get by the boundary condition that

$$\begin{aligned} 0 &\leq \mu(x_0) \sum_{x \in S} (\phi(x_0, t_0) - \phi(x, t_0))\omega(x_0, x) + \sigma(x_0)\phi(x_0, t_0) \\ &= -\mu(x_0) \sum_{x \in S} \phi(x, t_0)\omega(x_0, x) \\ &\leq 0, \end{aligned}$$

since $\phi(x,t) \geq \phi(x_0, t_0) = 0$. Then, $\phi(x_0, t_0) = \phi(x, t_0)$ for all $x \in S$. This contradict (2.12).

If $x_0 \in S$, we get from $\phi(x,t) \geq \phi(x_0, t_0)$ that

$$\phi_t(x_0, t_0) \leq 0, \text{ and } \Delta_\omega \phi(x_0, t_0) \geq 0.$$

It follows from (2.1) that

$$\begin{aligned} 0 &\geq \phi_t(x_0, t_0) - \Delta_\omega \phi(x_0, t_0) \\ &\geq -\lambda \phi(x_0, t_0) + (f(g(x_0, t_0)) - f(h(x_0, t_0))) \\ &\geq -(\lambda + M)\phi(x_0, t_0) \\ &= 0. \end{aligned}$$

Hence $\Delta_\omega \phi(x_0, t_0) = 0$ and $\phi(x, t_0) = 0$ for all $x \sim x_0$. Now, for any $x \in \bar{S}$, there exists a path

$$x_0 = y_1, y_2, \cdots, y_n = x,$$

such that $y_j \sim y_{j+1}$ for $j = 1, \cdots, n$, since $\bar{S}$ is connected. By applying the same argument as before inductively we see that $\phi(x, t_0) = 0$ for all $x \in \bar{S}$, which contradicts to (2.12).

Hence $\phi(x,t) > 0$ for all $(x,t) \in S \cup \Gamma \times (0, T_1]$ and $g(x,t) > h(x,t)$ for any $(x,t) \in S \cup \Gamma \times (0, T)$ since $T_1 < T$ is arbitrary. This ends the proof. □

Remark 2.7. For $\lambda \geq 0$, Theorem 2.5 and 2.6 hold true for $p > 1$ and $p \geq 2$ respectively.

By applying the comparison principle, we establish that the solution to problem (1.1) exhibits a dichotomous behavior: it either remains positive for all time or extinctive in finite time whenever $u_0 \geq 0$.

Theorem 2.8. *Let f satisfies (f_1) with $f(0) = 0$ and $u_0 \geq 0$. If $p \in (1, 2]$ with $\lambda \in \mathbb{R}$ or $\Big(p > 1, \lambda \geq 0\Big)$, then u is nonnegative and we have either*

(1) *$u(x,t) > 0$ for all $(x,t) \in [0, T)$,*
or
(2) *there exists $t_0 \in (0, T)$ such that*

$$u > 0 \text{ in } S \times [0, t_0), \text{ and } u = 0 \text{ in } S \times [t_0, T).$$

Proof. First of all, we note that the uniqueness of solution to (1.1) assert that the solution is trivial whenever $u_0 = 0$.

Hence, we may assume that u_0, is non-negative and non-trivial. Then by the Comparison principle in Theorem 2.5, we have $u(x,t) \geq 0$ for all $(x,t) \in S \times [0, T)$.

Suppose that there exists $(x_0, t_0) \in S \times [0, T)$ such that $u(x_0, t_0) = 0$. Then, $u_t(x_0, t_0) = 0$, since u is a C^1 nonnegative function.

Therefor,

$$\Delta_{p,\omega} u(x_0, t_0) = 0 \text{ and } \sum_{y \sim x_0} |u(y, t_0)|^{p-1} \omega(y, x_0) = 0.$$

That is, $u(y, t_0) = 0$ for all $y \sim x_0$. Since S is connected, applying the above argument inductively, we deduce that $u(x, t_0) = 0$ for all $x \in S$. Hence, taking t_0 as the initial time in (1.1), we obtain u is a trivial solution, which completes the proof. □

3. Global and Blow-up results

In this section, we will discuss the global existence and blow up of the solution to equation (1.1) under certain conditions given by f and the parameter λ, and analyze their asymptotic behavior by means of a differential inequality techniques.

Now, we deal with the finite time blow-up of solution with the condition:

$$(C_p)' : \alpha F(s) \leq s f(s) + \beta |s|^p + \gamma, \;\; s \in \mathbb{R}$$

where $\gamma > 0, \alpha > \max\{p, 2\}$, $0 \leq \beta \leq \dfrac{\alpha - p}{p} \lambda_{1,p}$ and $F(t) = \int_0^t f(s) ds$.

Before, we start with the definition of the blow-up and we state our result.

Definition 3.1. (Blow-up) We say that a solution u to equation (1.1) blows up at finite time T if there exist $x \in S$ such that $|u(x, t)| \to \infty$ as $t \to \infty$, or, equivalently, $\sum_{x \in S} |u(x, t)| \to \infty$ as $t \to \infty$.

Theorem 3.2. *Assume that f satisfies (f_1) and (C_p) for $p > 1$ and $\lambda \geq -\lambda_{1,p}$. Then the solution to (1.1) blow up in finite time provided that the initial data satisfying*

$$\frac{1}{2p} \sum_{x,y \in \bar{S}} |u_0(y) - u_0(x)|^p \omega(x, y) + \frac{1}{p} \sum_{x \in \Gamma} \frac{\sigma(x)}{\mu(x)} |u_0(x)|^p + \frac{\lambda}{p} \sum_{x \in S} |u_0(x)|^p - \sum_{x \in S} (F(u_0(x)) - \gamma) < 0.$$

Proof of Theorem 3.2. For $t \in [0, T)$, we define the following two functionals by

$$J_p(t) := -\frac{1}{2p} \sum_{x,y \in \bar{S}} |u(y,t) - u(x,t)|^p \omega(x,y) - \frac{1}{p} \sum_{x \in \Gamma} \frac{\sigma(x)}{\mu(x)} |u(x,t)|^p - \frac{\lambda}{p} \sum_{x \in S} |u(x,t)|^p + \sum_{x \in S} [F(u(x,t)) - \gamma],$$

and

$$I(t) := \sum_{x \in S} u^2(x, t).$$

Then,

$$\begin{aligned}
J_p'(t) &= -\frac{1}{2}\sum_{x,y\in\bar{S}} |u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))(u_t(y,t)-u_t(x,t))\omega(x,y)\\
&\quad -\sum_{x\in\Gamma}\frac{\sigma(x)}{\mu(x)}|u(x,t)|^{p-2}u(x,t)u_t(x,t)-\lambda\sum_{x\in S}|u(x,t)|^{p-2}u(x,t)u_t(x,t)+\sum_{x\in S} f(u(x,t))u_t\\
&= \sum_{x\in\bar{S}}\Delta_{p,\omega}u(x,t)u_t(x,t)+\sum_{x\in\partial S}\frac{\partial u}{\partial_p n}(x,t)u_t(x,t)-\lambda\sum_{x\in S}|u(x,t)|^{p-2}u(x,t)u_t(x,t)+\sum_{x\in S} f(u(x,t))u_t\\
&= \sum_{x\in S}\Delta_{p,\omega}u(x,t)u_t(x,t)-\lambda\sum_{x\in S}|u(x,t)|^{p-2}u(x,t)u_t(x,t)+\sum_{x\in S} f(u(x,t))u_t\\
&= \sum_{x\in S}u_t^2(x,t)\geq 0.
\end{aligned}$$

Hence, it follows from the assumption on the initial data u_0 that $J_p(t) > 0$.

On the other hand, we have

$$\begin{aligned}
I'(t) &= 2\sum_{x\in S}u(x,t)u_t(x,t)\\
&= 2\sum_{x\in S}[\Delta_{p,\omega}u(x,t)-\lambda|u(x,t))|^{p-2}u(x,t)+f(u(x,t))]u\\
&\geq 2\alpha J_p(t)+2\left(\frac{\alpha}{p}-1\right)\left(\frac{1}{2}\sum_{x,y\in\bar{S}}|u(y,t)-u(x,t)|^p\omega(x,y)+\sum_{x\in\Gamma}\frac{\sigma(x)}{\mu(x)}|u(x,t)|^p\right)\\
&\quad +2\left[\lambda\left(\frac{\alpha}{p}-1\right)-\beta\right]\sum_{x\in S}|u(x,t)|^p\\
&\geq 2\alpha J_p(t)+2\left[(\lambda+\lambda_{1,p})\left(\frac{\alpha}{p}-1\right)-\beta\right]\sum_{x\in S}|u(x,t)|^p\\
&\geq 2\alpha J_p(t). \qquad (3.1)
\end{aligned}$$

Multiplying (3.1) by $I'(t) > 0$ and using the Schwarz inequality, (3.1) we obtain

$$\begin{aligned}
\frac{\alpha}{2}J_p(t)I'(t) &\leq \left(\frac{1}{2}I'(t)\right)^2\\
&= \left(\sum_{x\in S}u(x,t)u_t(x,t)\right)^2\\
&\leq \left(\sum_{x\in S}u^2(x,t)\right)\left(\sum_{x\in S}u_t^2(x,t)\right)\\
&= I(t)J_p'(t).
\end{aligned}$$

That is

$$\frac{d}{dt}\left[I^{-\frac{\alpha}{2}}(t)J_p(t)\right]\geq 0. \qquad (3.2)$$

Again multiplying (3.1) by $I^{-\frac{\alpha}{2}}(t)$, it follows from (3.2) that

$$\begin{aligned} I^{-\frac{\alpha}{2}}(t)I'(t) &\geq 2\alpha I^{-\frac{\alpha}{2}}(t)J_p(t) \\ &\geq 2\alpha I^{-\frac{\alpha}{2}}(0)J_p(0), \end{aligned}$$

Solving the differential inequality, we get

$$I(t) \geq \left[\frac{1}{I^{1-\frac{\alpha}{2}}(0)-(\alpha-2)I^{-\frac{\alpha}{2}}(0)J_p(0)t}\right]^{\frac{2}{\alpha-2}}.$$

Hence the function I cannot remain finite for all $t \in [0,T)$. In other words, the solution u blows up in finite time in the sense of Definition 3.1, since the function $K(s) = s^p$ is superadditive for $p > 1$. □

For the linear case(i.e., $p = 2$), we have the following results

Theorem 3.3. *(Non blowup in finite time) If $p = 2, u_0 \geq 0$, and f be a nonnegative function satisfies (f_1), there holds*

1. *if $\lambda < -\omega_M$, then u exists globally and $\lim\limits_{t\to+\infty}\sum\limits_{x\in S} u(x,t) = +\infty$;*
2. *if $\lambda > -\omega_m$, then u exists globally and $\lim\limits_{t\to+\infty}\sum\limits_{x\in S} u(x,t) = 0$;*
3. *if $-\omega_M \leq \lambda \leq -\omega_m$, there is no finite time blowup.*

Where $\omega_m = \min\limits_{x\in S}\sum\limits_{y\in\partial S}\omega(x,y)$ and $\omega_M = \max\limits_{x\in S}\sum\limits_{y\in\partial S}\omega(x,y)$.

Proof of Theorem 3.3. Assume that $p = 2$. Note that for $x \in \partial S$ and $t \in [0,T)$, we have

$$\begin{aligned} \frac{\partial u}{\partial n}(x,t) &= \sum_{y\in S}(u(x,t)-u(y,t))\omega(x,y) \\ &= \sum_{y\in\bar{S}}(u(x,t)-u(y,t))\omega(x,y) - \sum_{y\in\partial S}(u(x,t)-u(y,t))\omega(x,y), \end{aligned}$$

then,

$$\begin{aligned} \sum_{x\in\partial S}\frac{\partial u}{\partial n}(x,t) &= \sum_{x\in\partial S}\sum_{y\in\bar{S}}(u(x,t)-u(y,t))\omega(x,y) - \sum_{x\in\partial S}\sum_{y\in\partial S}(u(x,t)-u(y,t))\omega(x,y) \\ &= \sum_{x\in\partial S}\sum_{y\in\bar{S}}(u(x,t)-u(y,t))\omega(x,y) \\ &= -\sum_{x\in\partial S}\Delta_\omega u(x,t). \end{aligned}$$

Therefore, summing (1.1) over S and by the fact that $u(x,t) \geq 0$, we get

$$\begin{aligned}
\sum_{x\in S} u_t(x,t) &= \sum_{x\in \bar{S}} \Delta_\omega u(x,t) - \sum_{x\in\partial S} \Delta_\omega u(x,t) - \lambda \sum_{x\in S} u(x,t) + \sum_{x\in S} f(u(x,t)) \\
&= \sum_{x\in\partial S} \frac{\partial}{\partial n} u(x,t) - \lambda \sum_{x\in S} u(x,t) + \sum_{x\in S} f(u(x,t)) \\
&\geq \sum_{x\in\partial S} \frac{\partial}{\partial n} u(x,t) - \lambda \sum_{x\in S} u(x,t) \\
&= \sum_{x\in\partial S} \sum_{y\in S} (u(x,t) - u(y,t))\omega(x,y) - \lambda \sum_{x\in S} u(x,t) \\
&\geq - \sum_{x\in\partial S} \sum_{y\in S} u(y,t)\omega(x,y) - \lambda \sum_{x\in S} u(x,t) \\
&= - \sum_{x\in S} \left[u(x,t) \sum_{y\in\partial S} \omega(x,y) \right] - \lambda \sum_{x\in S} u(x,t). \qquad (3.3)
\end{aligned}$$

From (3.3), we get

$$-(\omega_M + \lambda) \sum_{x\in S} u(x,t) \leq \sum_{x\in S} u_t(x,t) \leq -(\omega_m + \lambda) \sum_{x\in S} u(x,t).$$

By solving the differential inequality above, there hold

$$\left[\sum_{x\in S} u_0 \right] e^{-(\lambda+\omega_M)t} \leq \sum_{x\in S} u(x,t) \leq \left[\sum_{x\in S} u_0 \right] e^{-(\lambda+\omega_m)t}, \quad t \in [0,T). \qquad (3.4)$$

Hence, one can conclude the assertions $(1)-(3)$ from (3.4). This ends the proof. □

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