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LOCKING-FREE FINITE ELEMENT APPROXIMATIONS FOR A PARAMETER-DEPENDENT VARIATIONAL INEQUALITY

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Abstract. We present two robust finite element approximations for the unilateral contact problem of a Timoshenko beam. The two numerical approximations are based on conforming methods with dual Lagrange multipliers. The first uses a suitable stable pair of finite element spaces. The second is a stabilized formulation that allows the use of standard low-order finite element spaces. A priori and a residual posteriori error estimates are derived. Comprehensive numerical tests that validate and illustrate our approach are also given.

Keywords. Variational inequalities, Finite element method, a priori estimates, a posteriori estimate
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1. Introduction

The Timoshenko-Ehrenfest and the Euler-Bernoulli models are the most currently used for beam numerical computations. Timoshenko-Ehrenfest beam theory is a much newer theory than the Euler-Bernoulli beam theory, and as such, Timoshenko beam theory is considered to be an improved or more accurate theory, especially when the aspect ratio of the beam is small i.e thick beams. Euler-Bernoulli beam theory tends to be good for long, slender beams, that is, beams whose length is much greater than their height, and Euler-Bernoulli beam theory loses accuracy as the thickness-to-length ratio increases.
Despite its apparent simplicity, the numerical approximation of this problem can be challenging, particularly when low-order finite element methods are employed. In particular, care must be taken to avoid the so-called shear locking phenomenon, which occurs when the beam thickness parameter becomes small. Many applications in solid and fluid mechanics have demonstrated the superiority of mixed schemes, particularly when a very small parameter causes locking in displacement-based discretizations. However, adopting a mixed formulation alone is not sufficient to resolve the issue: the discrete approximation spaces must be chosen with care to avoid shear locking and prevent spurious modes. We will present an example of a finite element pair that satisfies the Babushka-Brezzi inf-sup condition, but the scheme is not locking-free.
In the equality case, several approaches to overcome shear locking have been developed. Let us mention some of these methods. Arnold and Brezzi introduced the partial selective reduced integration in [2] to mitigate spurious modes. This approach uses two distinct integration schemes: one for the bending component and another for the shear and membrane components. Interdependent Interpolation Element (see [23]). Based on

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exact solutions, avoids locking completely, making it highly effective for structural analysis. Friedman and Kosmatka [15] proposed a two-node Timoshenko beam finite element in which the transverse displacement and rotation are approximated by cubic and quadratic Lagrange polynomials, respectively. While this element has been employed in various applications, its applicability remains restricted, as the associated shape functions depend on the material properties. Caillerie et al. [11] introduced a Timoshenko finite element formulation designed to improve efficiency, with shape functions that do not depend on the material properties. Brenner [8], presented an analysis of multigrid methods applied to parameter-dependent problems. Within a unified theoretical framework, it is shown that the contraction numbers of the proposed algorithms remain uniformly bounded away from one, regardless of the parameter values and the mesh level. Isogeometric collocation methods were well-established in [13], where it is shown that locking-free solutions can be obtained for the mixed method independently of the degree of approximation. The virtual element method, which leads to a locking-free formulation for the Timoshenko problem, was examined by [27]. It is shown that the virtual element method leads to a straightforward locking-free formulation. Approaches based on Helmholtz decomposition type offer a very efficient way to avoid locking (see [21]). It is based first on adding a new unknown, say, γ, which represents the shear stress and then writing it as the sum of two functions, $\gamma = \phi' + k$, with $\phi \in H_0^1(\Omega)$ and $k = \bar{\gamma}$.

A natural question that arises when we deal with variational inequalities rather than variational equations is: Does any finite element method that converges for variational equations also converge for variational inequalities? Another question we address in this paper is the locking issue for parameter-dependent variational inequalities, which, to the best of our knowledge, has not yet been considered.

For the inequality-constrained case, many approaches are used to well-approximate the solution of variational inequalities (see [22] and [5] for instance).
Recently, in [17] and [18], the authors introduced new mixed and stabilized methods for the obstacle problem of an elastic membrane and for a Kirchhoff plate. The considered problems are parameter-independent variational inequalities. In [17] and [18], the approach consists of reformulating the variational inequality into a mixed then to a compact formulation, then using an adaptive method based on residual a posteriori error estimators.
A posteriori error estimators are computable quantities used for adaptive algorithms and error control in the numerical solution of partial differential equations, and they are required to be reliable and (locally) efficient (see, e.g., [14], [26], [16]).
There exist several types of a posteriori error estimators, (see [24], [7]). In [26], an efficient and reliable indicator was derived for the obstacle problem of an elastic membrane; the approach uses a novel Galerkin functional. Within the framework of the hypercircle method, a posteriori error estimates for the obstacle problem were derived in [7] using a Prager–Synge–type theorem together with an equilibration procedure. The averaging technique was considered in [3], where it is stressed that the averaging estimator is not only reliable but also efficient for sufficiently smooth solutions.

This paper examines mixed and stabilized methods for the locking phenomenon in a parameter-dependent variational inequality model.
This paper is organized as follows. The second section is devoted to deriving the variational inequality and the corresponding complementarity system. We prove the existence, uniqueness, and regularity of the solution, and we investigate the locking phenomenon when the thickness approaches zero using the lower-order finite element method. A mixed

reformulation of the problem is proposed in Section 3. The proposed mixed formulation is locking-free if the pair of finite element spaces is well chosen. In Section 4, a new compact reformulation is used. Section 5 is devoted to the finite element approximation of the compact reformulation. A posteriori error analysis of residual type is presented in Section 6. In Section 7, a stabilized formulation is presented. This formulation can be used for any conforming pair of finite element subspaces. Comprehensive numerical simulations are presented in Section 8.

We conclude this introduction by fixing some notation used throughout the paper. The standard norm and seminorm of the Sobolev space $H^s(\Omega)$, with $s \geq 0$, are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. We write $A \lesssim B$ to indicate the estimate $A \leq C\,B$, where C denotes a generic constant independent of A and B; in particular, C is independent of the mesh size h and of the beam thickness ε, which is assumed to be strictly positive.

2. The continuous variational inequality

We consider an undisplaced beam that occupies the region

$$\Omega^\varepsilon = (0,\ell) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$$

in the (x,y)-plane and is subject to a vertical body force $\varepsilon^2 f(x)$ which is independent of y. (The factor ε^2 assures the energy has a finite, non-zero limit as $\varepsilon \to 0$). The beam is over a rigid obstacle $\psi \in C^0(0,\ell)$, $\psi(0) \leq 0$ and $\psi(\ell) \leq 0$ (see Fig. 1).

Note that the Timoshenko model may be derived from the classical theory of plane linear

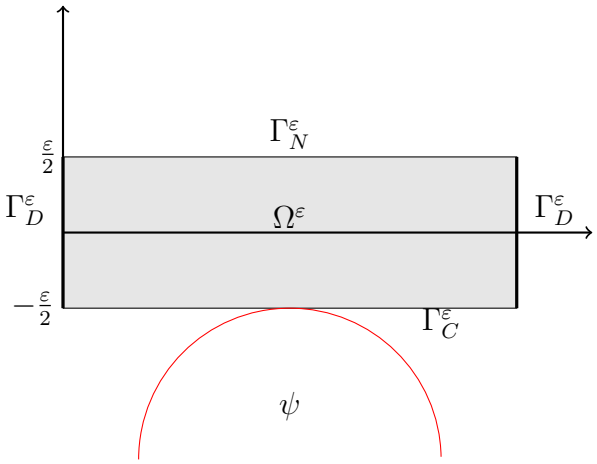


FIGURE 1. A Timoshenko's beam over a rigid obstacle

elasticity by dimensional reduction (see [1]). Using the Mindlin assumptions:

i) The vertical displacement u_2^ε is independent of y.
ii) The mid line $y = 0$ is not displaced horizontally.
iii) The vertical fibers $x = \text{const}$ remain linear after displacement.

In addition, we assume that there is no membrane deformation, i.e, pure bending/shear case. Mathematically, this implies that u_1^ε and u_2^ε have the special form

$$\begin{cases} u_1^\varepsilon(x,y) = -y\theta_\varepsilon(x) \\ u_2^\varepsilon(x,y) = w_\varepsilon(x) \end{cases} \tag{2.1}$$

Then, the Timoshenko model can be seen as a minimization problem of the following functional,

$$J((\zeta, v)) = \frac{1}{2}\left(\int_0^\ell (\zeta')^2 \, dx + \varepsilon^{-2}\int_0^\ell (v'-\zeta)^2 \, dx\right) - \langle f, v\rangle$$

over the following closed and convex set:

$$\mathcal{K} := \{(\zeta, v) \in V; v(x) \geq \psi(x), \text{ for all } x \in [0,\ell]\}$$

where $V = H_0^1([0,\ell]) \times H_0^1([0,\ell])$, here, $H_0^1([0,\ell])$ denotes the usual Sobolev space. Hence, the variational inequality reads

$$\begin{cases} \text{Find } (\theta_\varepsilon, w_\varepsilon) \in \mathcal{K} \\ \boldsymbol{a}((\theta_\varepsilon, w_\varepsilon); (\zeta - \theta_\varepsilon, v - w_\varepsilon)) \geq \langle f, v - w_\varepsilon\rangle, \quad \forall (\zeta, v) \in \mathcal{K} \end{cases} \tag{2.2}$$

where

$$\boldsymbol{a}((\theta, w); (\zeta, v)) = \int_0^\ell \theta'\zeta' \, dx + \varepsilon^{-2}\int_0^\ell (w'-\theta)(v'-\zeta) \, dx$$

Note that the problem (2.2) is equivalent to minimizing the functional $J((\zeta, v))$ over the closed convex set $\mathcal{K}$.

2.1. **Existence and Uniqueness.** The existence and the uniqueness of the solution of (2.2) follow immediately, by applying Stampacchia's theorem (see [9] for instance).

Lemma 2.1. *The bilinear form $\boldsymbol{a}(\cdot,\cdot)$ is coercive on the Hilbert space V.*

Proof. See [1]. □

Theorem 2.2. *For the problem (2.2) there exists a unique solution.*

Proof. It suffices to apply Stampacchia's theorem (see [9]). □

2.2. **Regularity of the solution of (2.2).** It is straightforward to check that the problem (2.2) reads, at least formally, as the following complementarity system:

$$\begin{cases} -\theta'' - \varepsilon^{-2}(w'-\theta) = 0, & \text{a.e in } (0,\ell) \\ -\varepsilon^{-2}(w'-\theta)' \geq f, & \text{a.e in } (0,\ell) \\ w - \psi \geq 0, & \text{a.e in } (0,\ell) \\ (-\varepsilon^{-2}(w'-\theta)' - f)(w-\psi) = 0, & \text{a.e in } (0,\ell) \\ \theta(0) = \theta(\ell) = 0, & \\ w(0) = w(\ell) = 0. & \end{cases} \tag{2.3}$$

Theorem 2.3. *Let (θ, w) be a solution of the complementarity system (2.3). If $f \in L^2(0,\ell)$ and $\psi \in H^2(0,\ell)$, then $(\theta, w) \in (H^2(0,\ell))^2$.*

Proof. The first equation in the system (2.3) and the fact that θ and w are a priori in $H^1(0,\ell)$ imply that

$$-\theta'' + \varepsilon^{-2}\theta \in L^2(0,\ell)$$

Then by the elliptic regularity we have $\theta \in H^2(0,\ell)$.
For every nonnegative function $\phi \in C^2(\Omega)$ with $\operatorname{supp}\phi \subset B_\epsilon(\xi)$, we have $w + \phi \geq \psi$ and,

the second equation of the complementarity system implies that

$$\begin{aligned}
0 &\leq \varepsilon^{-2}\int_{B_\epsilon(\xi)}(w'-\theta)\phi'\mathrm{d}x - \int_{B_\epsilon(\xi)} f\phi\mathrm{d}x \\
&= \varepsilon^{-2}\int_{\xi-\epsilon}^{\xi}(w'-\theta)\phi'\mathrm{d}x - \int_{\xi-\epsilon}^{\xi} f\phi\mathrm{d}x + \varepsilon^{-2}\int_{\xi}^{\xi+\epsilon}(w'-\theta)\phi'\mathrm{d}x - \int_{\xi}^{\xi+\epsilon} f\phi\mathrm{d}x \\
&= \varepsilon^{-2}\left[w'\phi\right]_{\xi-\epsilon}^{\xi} - \varepsilon^{-2}\int_{\xi-\epsilon}^{\xi} w''\phi\mathrm{d}x - \int_{\xi-\epsilon}^{\xi} f\phi\mathrm{d}x - \varepsilon^{-2}\int_{\xi}^{\xi+\epsilon}\psi''\phi\mathrm{d}x - \int_{\xi}^{\xi+\epsilon} f\phi\mathrm{d}x \\
&+ \varepsilon^{-2}\left[\psi'\phi\right]_{\xi}^{\xi+\epsilon} - \varepsilon^{-2}\int_{B_\epsilon(\xi)}\theta\phi' \\
&= -\varepsilon^{-2}\int_{\xi}^{\xi+\epsilon}\psi''\phi\mathrm{d}x - \int_{\xi}^{\xi+\epsilon} f\phi\mathrm{d}x + \varepsilon^{-2}\left(w'\left(\xi\right)-\psi'\left(\xi\right)\right)\phi\left(\xi\right) - \varepsilon^{-2}\int_{\xi}^{\xi+\epsilon}\theta\phi'.
\end{aligned}$$

Then,

$$\left(w'\left(\xi\right)-\psi'\left(\xi\right)\right)\phi\left(\xi\right) \geq \int_{\xi}^{\xi+\epsilon}\varepsilon^2(\varepsilon^{-2}\psi''+f)\phi\mathrm{d}x + \int_{\xi}^{\xi+\epsilon}\theta\phi'.$$

Since $\phi \geq 0$ is arbitrary, by taking the limit $\epsilon \to 0$, we find that

$$w'\left(\xi\right) \geq \psi'\left(\xi\right) \tag{2.4}$$

The function $\tilde{w} = w - \psi$ is nonnegative continuous and satisfies :

$$\tilde{w}\left(\xi\right) = 0 \quad \text{and} \quad \tilde{w}'\left(\xi\right) \geq 0, \quad (\text{by}(2.4)).$$

Then ξ is a local minimum for the function $\tilde{w}$. Thus, we must have $\tilde{w}'\left(\xi\right) = 0$, i.e.,

$$w'\left(\xi\right) = \psi'\left(\xi\right)$$

which implies that w' is continuous at ξ, since ψ' is continuous. □

2.3. **Regularity of the solution of the limit problem.** In the context of the penalty method, it is standard to show that the solution $(\theta_\varepsilon, w_\varepsilon)$ of (2.2) converges strongly as ε goes to 0 to the solution of the obstacle problem of Euler-Bernoulli beam. It is possible to show that solutions of problem (2.2) obey higher regularity properties. If the data are sufficiently smooth, namely, $f \in L^2(\Omega)$ and $\psi \in H^3(\Omega)$, then continuity of the second derivative of the solution, i.e., $u \in H^3(\Omega)$, can be proved (see [12] or [10]).

Proposition 2.4. *Let $f \in L^2(0,\ell)$ and $\psi \in H^3(0,\ell)$. Then the solution w_0 of limit problem, belongs to $H^3(0,\ell)$.*

2.4. **Locking phenomena for Timoshenko's model.** It is standard to prove that for fixed ε, θ_h and w_h converge to θ and w in both L^2 and H^1 at the approximation theoretic optimal rate with respect to h. Such convergence, however it is not uniform with respect to the thickness ε. The pair $V_h^1 \times V_h^1$ does not satisfy the necessary approximation properties uniformly in ε. A simple calculation using the same arguments as in [6] for the equality case, leads to, for h fixed,

$$\lim_{\varepsilon\to 0}\|(\theta_h, w_h)\|_V = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\|(\theta_\varepsilon, w_\varepsilon)\|_V \neq 0.$$

As a result, we conclude that there is a locking phenomenon for the choice $V_h^1 \times V_h^1$ for very small values of ε.

3. A mixed formulation and its Finite element approximation

Because of the difficulty in constructing finite element spaces that incorporate the required constraints, problems with inequality constraints and locking phenomena are commonly cited as motivations for favoring mixed formulations over primal displacement-based formulations. In order to avoid the locking phenomena previously explained, and to rewrite the inequality constraint in a new form easy to implement, we first use a mixed formulation which is based on introducing a new Lagrange multiplier λ belonging to the closed set (cone) Λ defined by,

$$\Lambda := \{\mu \in H^{-1}(0,\ell), \langle \mu, \varphi \rangle \geq 0, \quad \forall \varphi \geq 0, \varphi \in C_0^\infty(0,\ell)\}$$

The mixed formulation consists of :

$$\begin{cases} \text{Seek} \quad (\theta, w, \lambda) \in V \times \Lambda \\ \boldsymbol{a}((\theta, w), (\zeta, v)) + \boldsymbol{b}((\zeta, v), \lambda) = F((\zeta, v)), \qquad \forall (\zeta, v) \in V \\ \boldsymbol{b}((\theta, w), \mu - \lambda) \leq G(\mu - \lambda), \qquad \forall \mu \in \Lambda \end{cases} \tag{3.1}$$

where

$$\boldsymbol{b}((\zeta, v), \mu) = -\langle \mu, v \rangle, \quad F((\zeta, v)) = \langle f, v \rangle, \text{ and } G(\mu) = -\langle \mu, \psi \rangle.$$

It is straightforward to check that the following inf-sup condition

$$\sup_{(\zeta,v)\in V} \frac{\boldsymbol{b}((\zeta, v), \mu)}{\|(\zeta, v)\|_1} \gtrsim \|\mu\|_{-1} \quad \forall \mu \in Q, \tag{3.2}$$

is satisfied. Indeed, it is a direct consequence of the definition of the bilinear form $\boldsymbol{b}(\cdot,\cdot)$ and the dual norm $\|\cdot\|_{-1}$, and the coercivity of $\boldsymbol{a}(\cdot,\cdot)$ on $V \times V$ is already proved. Hence, the mixed problem admits a unique solution (see [17],[19]).
Let $\mathcal{I}_h$ be a quasi-uniform discretization of $(0,\ell)$. We consider two finite-dimensional spaces V_h and Q_h such that $V_h \subset V$, $Q_h \subset Q = H^{-1}(0,\ell)$.
Moreover, we define the set:

$$\Lambda_h := \{\mu_h \in Q_h; \quad \mu_h \geq 0\} \subset \Lambda.$$

The discrete problem reads:

$$\begin{cases} \text{Find} \ (\theta_h, w_h, \lambda_h) \in V_h \times \Lambda_h \text{ such that} \\ \boldsymbol{a}((\theta_h, w_h), (\zeta_h, v_h)) + \boldsymbol{b}((\zeta_h, v_h), \lambda_h) = F((\zeta_h, v_h)), \qquad \forall (\zeta_h, v_h) \in V_h \\ \boldsymbol{b}((\theta_h, w_h), \mu_h - \lambda_h) \leq G(\mu_h - \lambda_h), \qquad \forall \mu_h \in \Lambda_h. \end{cases} \tag{3.3}$$

In order to get a well-posed discrete problem (3.3), the spaces V_h and Q_h must have an additional property. The desired property is known as *the discrete inf-sup condition*:

$$\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h \rangle}{\|v_h\|_1} \gtrsim \|\mu_h\|_{-1} \quad \forall \mu_h \in Q_h \tag{3.4}$$

Note that mixed methods are subject to possible instabilities which do not arise for primal formulations. For the existence, uniqueness, and convergence of the discrete solution, the spaces must also satisfy some rather subtle stability conditions. In other words, an arbitrary conforming choice of pair (V_h, Q_h) does not necessarily satisfy the inequality (3.4) and therefore, the corresponding discrete problem fails to give a good approximation of the original problem, and may lead to a singular system.

3.1. **A pair leading to a singular system.** First, we define

$$V_h^k = \{v_h \in H_0^1(0,\ell); \quad v_h|_{[x_i,x_{i+1}]} \in \mathbb{P}^k, \quad k \geq 1\}, \quad V_0^h = \{v_h \in L^2(0,\ell); \quad v_h|_{[x_i,x_{i+1}]} \in \mathbb{P}^0\}$$

Then, if we consider the choice $V_h = V_h^1 \times V_h^1$, $Q_h = V_h^0$. This yields a singular system, because the kernel of the discrete divergence operator contains spurious modes. Hence, the condition (3.4) is not verified.

4. A compact formulation

Let us now introduce the Hilbert space $\mathcal{H} = V \times Q$, for which we define the norm

$$\| (\zeta, v, \mu) \|_{\mathcal{H}} = (\|\zeta\|_1^2 + \|v\|_1^2 + \|\mu\|_{-1}^2)^{1/2}$$

We define the bilinear form $\mathscr{A} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by:

$$\mathscr{A}((\zeta, v, \mu), (\xi, z, \rho)) = \boldsymbol{a}((\zeta, v), (\xi, z)) - \langle \mu, z \rangle - \langle \rho, v \rangle$$

and the linear form $\mathscr{L} : \mathcal{H} \to \mathbb{R}$ by:

$$\mathscr{L}((\xi, z, \rho)) = (f, z) - \langle \rho, \psi \rangle$$

Theorem 4.1. *Let (θ, w, λ) be the solution of the mixed problem, then it solves the following compact problem:*

$$\begin{cases} \text{Find } (\theta, w, \lambda) \in V \times \Lambda \\ \mathscr{A}((\theta, w, \lambda), (\zeta, v, \mu - \lambda)) \leq \mathscr{L}((\zeta, v, \mu - \lambda)) \quad \forall (\zeta, v, \mu) \in V \times \Lambda \end{cases} \tag{4.1}$$

Proof. First, problem (3.1) can be written as,

$$\begin{cases} \text{Find } (\theta, w, \lambda) \in V \times \Lambda \\ \boldsymbol{a}((\theta, w), (\zeta, v)) - \langle \lambda, v \rangle = (f, v), & \forall (\zeta, v) \in V \\ -\langle \mu - \lambda, w \rangle \leq -\langle \mu - \lambda, \psi \rangle & \forall \mu \in \Lambda \end{cases}$$

By, summation, we obtain,

$$\boldsymbol{a}((\theta, w), (\zeta, v)) - \langle \lambda, v \rangle - \langle \mu - \lambda, w \rangle \leq (f, v) - \langle \mu - \lambda, \psi \rangle$$

Thus

$$\mathscr{A}((\theta, w, \lambda), (\zeta, v, \mu - \lambda)) \leq \mathscr{L}((\zeta, v, \mu - \lambda))$$

□

Now, we need to prove the inf-sup condition for the form $\mathscr{A}$.

Proposition 4.2. *For all $(\zeta, v, \mu) \in V \times \Lambda$, there exists $z \in H_0^1(0,\ell)$ such that:*

$$\mathscr{A}((\zeta, v, \mu); (\zeta, z, -\mu)) \gtrsim \| (\zeta, v, \mu) \|_{\mathcal{H}}^2 \tag{4.2}$$

$$\|z\|_1 \lesssim \|v\|_1 + \|\mu\|_{-1} \tag{4.3}$$

Proof. Let $\mu \in Q$ be arbitrary. We consider the following auxiliary problem:

$$\begin{cases} \text{Find } q \in H_0^1(0,\ell) \\ a(q,p) + (q,p) = \left\langle \varepsilon^{-2}\mu, p \right\rangle, \quad \forall p \in H_0^1(0,\ell) \end{cases} \tag{4.4}$$

Here, $a(p,q) = \int_0^\ell p'q'dx$. It standard to show that this problem admits a unique solution by the Lax–Milgram theorem. Taking $p = q$, we get

$$\|q\|_1^2 = \varepsilon^{-2} \langle \mu, q \rangle$$

Then,

$$\|q\|_1 \leq \varepsilon^{-2} \|\mu\|_{-1}$$

Besides

$$\|\mu\|_{-1} = \sup_{p\in H_0^1(0,\ell)} \frac{\langle \mu, p\rangle}{\|p\|_1} = \sup_{p\in H_0^1(0,\ell)} \frac{\varepsilon^2\left(a(q,p)+(q,p)\right)}{\|p\|_1} \leq \varepsilon^2\|q\|_1$$

Thus

$$\varepsilon^{-2}\|\mu\|_{-1} = \|q\|_1 \tag{4.5}$$

- We put $z = v - \varepsilon^2 q$, using Young's, Cauchy-Schwarz inequalities

$$\begin{aligned}
\mathscr{A}((\zeta,v,\mu);(\zeta,z,-\mu)) &= \mathscr{A}((\zeta,v,\mu);(\zeta,v-\varepsilon^2 q,-\mu)) \\
&= \boldsymbol{a}((\zeta,v),(\zeta,v-\varepsilon^2 q)) - \langle \mu, v-\varepsilon^2 q\rangle + \langle \mu, v\rangle \\
&= a(\zeta,\zeta) + \varepsilon^{-2} b((\zeta,v),(\zeta,v-\varepsilon^2 q)) + \varepsilon^2\langle \mu, q\rangle \\
&= a(\zeta,\zeta) + \varepsilon^{-2} b((\zeta,v),(\zeta,v)) - b((\zeta,v),(0,q)) + \varepsilon^2\langle \mu, q\rangle \\
&= \|\zeta'\|_0^2 + \varepsilon^{-2}\|v'-\zeta\|_0^2 - \int_0^\ell (v'-\zeta)q'\,dx + \varepsilon^2\langle \mu, q\rangle \\
&\geq \|\zeta'\|_0^2 + \varepsilon^{-2}\|v'-\zeta\|_0^2 - \|v'-\zeta\|_0^2\|q'\|_0^2 + \varepsilon^2\langle \mu, q\rangle \\
&\geq \|\zeta'\|_0^2 + \varepsilon^{-2}\|v'-\zeta\|_0^2 - \frac{\varepsilon^{-2}}{2}\|v'-\zeta\|_0^2 - \frac{\varepsilon^2}{2}\|q'\|_0^2 + \varepsilon^4\|q\|_1^2 \\
&\geq \|\zeta'\|_0^2 + \frac{\varepsilon^{-2}}{2}\|v'-\zeta\|_0^2 + \frac{\varepsilon^4}{2}\|q\|_1^2 \\
&= \frac{1}{2}\boldsymbol{a}((\zeta,v),(\zeta,v)) + \frac{\varepsilon^4}{2}\|q\|_1^2
\end{aligned}$$

Utilizing (4.5) and Lemma 2.1

$$\begin{aligned}
\mathscr{A}((\zeta,v,\mu);(\zeta,z,-\mu)) &\gtrsim \left(\|\zeta\|_1^2 + \|v\|_1^2\right) + \frac{1}{2}\|\mu\|_{-1}^2 \\
&\gtrsim \left(\|\zeta\|_1^2 + \|v\|_1^2 + \|\mu\|_{-1}^2\right)
\end{aligned}$$

- Again, using (4.5), we obtain,

$$\|z\|_1 = \|v-\varepsilon^2 q\|_1 \leq \|v\|_1 + \varepsilon^2\|q\|_1 = \|v\|_1 + \|\mu\|_{-1}$$

□

Proposition 4.3. *For all $(\zeta,v,\mu)\in V\times\Lambda$, we have,*

$$\sup_{(\xi,z,\rho)\in V\times Q} \frac{\mathscr{A}((\zeta,v,\mu),(\xi,z,\rho))}{\|(\xi,z,\rho)\|_{\mathcal{H}}} \gtrsim \|(\zeta,v,\mu)\|_{\mathcal{H}} \tag{4.6}$$

Proof. We proved that,

$$\frac{\mathscr{A}\left((\zeta,v,\mu);(\zeta,z,-\mu)\right)}{\|\left(\zeta,v,\mu\right)\|_{\mathcal{H}}} \gtrsim \|\left(\zeta,v,\mu\right)\|_{\mathcal{H}}$$

In addition we have, i.e.,

$$\frac{1}{\|\left(\zeta,z,-\mu\right)\|_{\mathcal{H}}} \gtrsim \frac{1}{\|\left(\zeta,v,\mu\right)\|_{\mathcal{H}}}$$

Then

$$\frac{\mathscr{A}\left((\zeta,v,\mu);(\zeta,z,-\mu)\right)}{\|\left(\zeta,v,-\mu\right)\|_{\mathcal{H}}} \gtrsim \|\left(\zeta,v,\mu\right)\|_{\mathcal{H}}$$

Thus

$$\sup_{(\xi,z,\rho)\in V\times Q}\frac{\mathscr{A}((\zeta,v,\mu),(\xi,z,\rho))}{\|(\xi,z,\rho)\|_{\mathcal{H}}}\gtrsim\|(\zeta,v,\mu)\|_{\mathcal{H}}$$

□

5. Finite element approximations

In the linear equation case, it is well-known that the well-posedness of the variational problem, as well as the stability and the approximation properties of the discrete problem, In the linear case, it is well known that the well-posedness of the variational problem, as well as the stability and approximation properties of the discrete problem, are guaranteed by Brezzi-Babuška conditions; these conditions were adapted to extend the well-posedness of elliptic variational inequalities (see [19], [17]).

5.1. Stable formulations: Inf-sup condition.

We consider the following discrete problem

$$\begin{cases}\text{Find } (\theta_h,w_h,\lambda_h)\in V_h\times\Lambda_h\\ \mathscr{A}((\theta_h,w_h,\lambda_h),(\zeta_h,v_h,\mu_h-\lambda_h))\leq\mathscr{L}((\zeta_h,v_h,\mu_h-\lambda_h)),\ \forall(\zeta_h,v_h,\mu_h)\in V_h\times\Lambda_h\end{cases}\tag{5.1}$$

where $V_h=V_h^\zeta\times V_h^v$ and Λ_h are the same defined in the section 3.
In this section we assume that V_h and Q_h are such that [1]

$$\sup_{(\zeta_h,v_h)\in V_h}\frac{\langle\mu_h,v_h\rangle}{\|v_h\|_1}\gtrsim\|\mu_h\|_{-1}\quad\forall\mu_h\in Q_h\tag{5.2}$$

Proposition 5.1. *For all $(\zeta_h,v_h,\mu_h)\in V_h\times\Lambda_h$, there exists $z_h\in V_h^v$ such that:*

$$\mathscr{A}\left((\zeta_h,v_h,\mu_h);(\zeta_h,z_h,-\mu_h)\right)\gtrsim\|\left(\zeta_h,v_h,\mu_h\right)\|_{\mathcal{H}}^2\tag{5.3}$$

$$\|z_h\|_1\lesssim\|v_h\|_1+\|\mu_h\|_{-1}\tag{5.4}$$

Proof. Let μ_h be an arbitrary element in Q_h. We define the following auxiliary problem

$$\begin{cases}\text{Find } q_h\in V_h^v\\ a(q_h,p_h)+(q_h,p_h)=\langle\mu_h,p_h\rangle,\quad\forall p_h\in V_h^v\end{cases}\tag{5.5}$$

It is clear that problem admits a unique solution by the Lax–Milgram theorem. Taking $p_h=q_h$, we get

$$\|q_h\|_1^2\leq\langle\mu_h,q_h\rangle$$

Thus,

$$\|q_h\|_1\leq\|\mu_h\|_{-1}$$

Besides, from the condition (5.2) we have

$$\|\mu_h\|_{-1}\lesssim\sup_{p_h\in V_h^v}\frac{\langle\mu_h,p_h\rangle}{\|p_h\|_1}=\sup_{p_h\in V_h^v}\frac{a(q_h,p_h)+(q_h,p_h)}{\|p_h\|_1}\leq\|q_h\|_1$$

Thus

$$\exists C>0,\quad C\|\mu_h\|_{-1}\leq\|q_h\|_1\leq\|\mu_h\|_{-1}\tag{5.6}$$

[1] We will give examples of pair (V_h,Q_h) satisfy the condition (5.2) in the end of this section.

We put $z_h = v_h - q_h$, using (5.6), Young's, Cauchy-Schwarz inequalities and Lemma (2.1) we obtain,

$$\begin{aligned}
\mathscr{A}((\zeta_h, v_h, \mu_h); (\zeta_h, z_h, -\mu_h)) &= \mathscr{A}((\zeta_h, v_h, \mu_h); (\zeta_h, v_h - q_h, -\mu_h)) \\
&= \boldsymbol{a}((\zeta_h, v_h), (\zeta_h, v_h - q_h)) - \langle \mu_h, v_h - q_h \rangle + \langle \mu_h, q_h \rangle \\
&= a(\zeta_h, \zeta_h) + \varepsilon^{-2} b((\zeta_h, v_h), (\zeta_h, v_h - q_h)) + \langle \mu_h, q_h \rangle \\
&\geq a(\zeta_h, \zeta_h) + b((\zeta_h, v_h), (\zeta_h, v_h - q_h)) + \langle \mu_h, q_h \rangle \\
&= a(\zeta_h, \zeta_h) + b((\zeta_h, v_h), (\zeta_h, v_h)) - b((\zeta_h, v_h), (0, q_h)) \\
&+ \langle \mu_h, q_h \rangle
\end{aligned}$$

This amounts to write,

$$\begin{aligned}
\mathscr{A}((\zeta_h, v_h, \mu_h); (\zeta_h, z_h, -\mu_h)) &= \|\zeta_h'\|_0^2 + \|v_h' - \zeta_h\|_0^2 - \int_0^\ell (v_h' - \zeta_h) q_h' \, dx + \langle \mu_h, q_h \rangle \\
&\geq \|\zeta_h'\|_0^2 + \|v_h' - \zeta\|_0^2 - \|v_h' - \zeta_h\|_0^2 \|q_h'\|_0^2 + \langle \mu_h, q_h \rangle \\
&\geq \|\zeta_h'\|_0^2 + \|v_h' - \zeta\|_0^2 - \frac{1}{2}\|v_h' - \zeta\|_0^2 - \frac{1}{2}\|q_h'\|_0^2 + \|q_h\|_1^2 \\
&= \|\zeta_h'\|_0^2 + \frac{1}{2}\|v_h' - \zeta\|_0^2 + \frac{1}{2}\|q_h\|_1^2 \\
&\geq \frac{1}{2}\left(\|\zeta_h'\|_0^2 + \|v_h' - \zeta\|_0^2\right) + \frac{1}{2}\|q_h\|_1^2 \\
&\gtrsim \left(\|\zeta_h\|_1^2 + \|v_h\|_1^2\right) + \frac{1}{2}C^2\|\mu_h\|_{-1}^2 \\
&\gtrsim \left(\|\zeta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2\right).
\end{aligned}$$

Again applying (5.6), yields,

$$\|z_h\|_1 = \|v_h - q_h\|_1 \leq \|v_h\|_1 + \|q_h\|_1 \leq \|v_h\|_1 + \|\mu_h\|_{-1}.$$

□

Proposition 5.2. *For all $(\zeta_h, v_h, \mu_h) \in V_h \times \Lambda_h$, we have,*

$$\sup_{(\xi_h, z_h, \rho_h) \in V_h \times Q_h} \frac{\mathscr{A}((\zeta_h, v_h, \mu_h), (\xi_h, z_h, \rho_h))}{\|(\xi_h, z_h, \rho_h)\|_{\mathcal{H}}} \gtrsim \|(\zeta_h, v_h, \mu_h)\|_{\mathcal{H}} \tag{5.7}$$

Proof. We have shown that,

$$\mathscr{A}\left((\zeta_h, v_h, \mu_h); (\zeta_h, z_h, -\mu_h)\right) \gtrsim \| \left(\zeta_h, v_h, \mu_h\right) \|_{\mathcal{H}}^2$$

i.e,

$$\frac{\mathscr{A}\left((\zeta_h, v_h, \mu_h); (\zeta_h, z_h, -\mu_h)\right)}{\| \left(\zeta_h, v_h, \mu_h\right) \|_{\mathcal{H}}} \gtrsim \| \left(\zeta_h, v_h, \mu_h\right) \|_{\mathcal{H}}$$

On the other hand, we have,

$$\| \left(\zeta_h, z_h, -\mu_h\right) \|_{\mathcal{H}}^2 = \|\zeta_h\|_1^2 + \|z_h\|_1^2 + \|\mu_h\|_{-1}^2 \lesssim \|\zeta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2 = \| \left(\zeta_h, v_h, \mu_h\right) \|_{\mathcal{H}}^2$$

i.e,

$$\frac{1}{\| \left(\zeta_h, z_h, -\mu_h\right) \|_{\mathcal{H}}} \gtrsim \frac{1}{\| \left(\zeta_h, v_h, \mu_h\right) \|_{\mathcal{H}}}$$

Then,

$$\frac{\mathscr{A}\left((\zeta_h, v_h, \mu_h); (\zeta_h, z_h, -\mu_h)\right)}{\| \left(\zeta_h, v_h, -\mu_h\right) \|_{\mathcal{H}}} \gtrsim \| \left(\zeta_h, v_h, \mu_h\right) \|_{\mathcal{H}}.$$

Thus,

$$\sup_{(\xi_h,z_h,\rho_h)\in V_h\times Q_h} \frac{\mathscr{A}((\zeta_h,v_h,\mu_h),(\xi_h,z_h,\rho_h))}{\|(\xi_h,z_h,\rho_h)\|_{\mathcal{H}}} \gtrsim \|(\zeta_h,v_h,\mu_h)\|_{\mathcal{H}}$$

□

Both propositions were proven under the assumption (5.2), which is not necessarily satisfied for arbitrary subspaces V_h, Q_h.

Lemma 5.3. *The finite element spaces $V_h = V_h^1 \times V_h^1$ and $Q_h = V_h^0$ can not satisfy the condition (5.2).*

Proof. See the proof of (3.1). □

Lemma 5.4. *The finite element spaces $V_h = V_h^1 \times V_h^2$, $Q_h = V_h^0$ satisfy the condition (5.2).*

Proof. See the proof of (5.3) below. □

We can generalize the result of the previous lemma:

Remark 5.5. For integer $k \geq 1$, the choice $V_h = V_h^k \times V_h^{k+1}$, $Q_h = V_h^{k-2}$, leads to stable system.

5.2. **A priori error estimation.** In this subsection, we present an a priori error estimate.

Theorem 5.6. *Let (θ, w, λ) be the solution of the continuous problem (4.1), and $(\theta_h, w_h, \lambda_h)$ be the solution of the discrete problem (5.1). Then, the following error estimate :*

$$\|\theta-\theta_h\|_1 + \|w-w_h\|_1 + \|\lambda-\lambda_h\|_{-1} \lesssim \inf_{(\zeta_h,v_h)\in V_h} (\|\theta-\zeta_h\|_1 + \|w-v_h\|_1) + \inf_{\mu_h\in\Lambda_h} (\|\lambda-\mu_h\|_{-1} + \sqrt{\langle w-\psi, \mu_h\rangle}) \quad (5.8)$$

holds true.

Proof. We have from (5.3)

$$
\begin{aligned}
&\|\theta_h-\zeta_h\|_1^2+\|w_h-v_h\|_1^2+\|\lambda_h-\mu_h\|_1^2\\
&\lesssim \mathscr{A}((\theta_h-\zeta_h,w_h-v_h,\lambda_h-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&=\mathscr{A}((\theta_h,w_h,\lambda_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))-\mathscr{A}((\zeta_h,v_h,\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&-\mathscr{A}((\theta,w,\lambda);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))+\mathscr{A}((\theta,w,\lambda);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&=\mathscr{A}((\theta_h,w_h,\lambda_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))-\mathscr{A}((\theta,w,\lambda);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&+\mathscr{A}((\theta-\zeta_h,w-v_h,\lambda-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&\leq \mathscr{L}((\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))-\mathscr{A}((\theta,w,\lambda);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&+\mathscr{A}((\theta-\zeta_h,w-v_h,\lambda-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&=(f,z_h)-\langle\mu_h-\lambda_h,\psi\rangle-\int_0^\ell\theta'(\theta_h-\zeta_h)'\,dx-\varepsilon^{-2}\int_0^\ell(w'-\theta)(z_h'-(\theta_h-\zeta_h))\,dx\\
&+\langle\lambda,z_h\rangle+\langle\mu_h-\lambda_h,w\rangle+\mathscr{A}((\theta-\zeta_h,w-v_h,\lambda-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))\\
&=(f,z_h)-\langle\mu_h-\lambda_h,\psi\rangle+\int_0^\ell\theta''(\theta_h-\zeta_h)\,dx+\varepsilon^{-2}\int_0^\ell(w'-\theta)'(z_h)\,dx\\
&+\varepsilon^{-2}\int_0^\ell(w'-\theta)(\theta_h-\zeta_h)\,dx+\langle\lambda,z_h\rangle+\langle\mu_h-\lambda_h,w\rangle\\
&+\mathscr{A}((\theta-\zeta_h,w-v_h,\lambda-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))
\end{aligned}
$$

Since, from (2.3), we have

$$
\begin{cases}\theta''+\varepsilon^{-2}(w'-\theta)=0\\ \varepsilon^{-2}(w'-\theta)'+\lambda+f=0\end{cases}\tag{5.9}
$$

Then,

$$
\begin{aligned}
&\|\theta_h-\zeta_h\|_1^2+\|w_h-v_h\|_1^2+\|\lambda_h-\mu_h\|_1^2\\
&\qquad\lesssim \mathscr{A}((\theta-\zeta_h,w-v_h,\lambda-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))+\langle\mu_h-\lambda_h,w-\psi\rangle
\end{aligned}
$$

We have $w-\psi\geq 0,\quad \lambda_h\geq 0$, so

$$
\begin{aligned}
&\|\theta_h-\zeta_h\|_1^2+\|w_h-v_h\|_1^2+\|\lambda_h-\mu_h\|_1^2\\
&\qquad\lesssim \mathscr{A}((\theta-\zeta_h,w-v_h,\lambda-\mu_h);(\theta_h-\zeta_h,z_h,\mu_h-\lambda_h))+\langle\mu_h,w-\psi\rangle\\
&\qquad\lesssim \big(\|\theta-\zeta_h\|_1+\|w-v_h\|_1+\|\lambda-\mu_h\|_{-1}\big)\big(\|\theta_h-\zeta_h\|_1+\|z_h\|_1+\|\lambda_h-\mu_h\|_{-1}\big)\\
&\qquad+\langle\mu_h,w-\psi\rangle\\
&\qquad\lesssim \|\theta-\zeta_h\|_1^2+\|w-v_h\|_1^2+\|\lambda-\mu_h\|_{-1}^2+\|\theta_h-\zeta_h\|_1^2+\|z_h\|_1^2+\|\lambda_h-\mu_h\|_{-1}^2\\
&\qquad+\langle\mu_h,w-\psi\rangle
\end{aligned}
$$

Because $\|z_h\|_1\lesssim\|\theta_h-\zeta_h\|_1+\|\lambda_h-\mu_h\|_{-1}$, we obtain

$$
\begin{aligned}
\|\theta_h-\zeta_h\|_1^2+\|w_h-v_h\|_1^2+\|\lambda_h-\mu_h\|_{-1}^2\lesssim\|\theta-\zeta_h\|_1^2+\|w-v_h\|_1^2+\|\lambda-\mu_h\|_{-1}^2\\
+\langle\mu_h,w-\psi\rangle\,.
\end{aligned}
$$

This implies that,

$$\|\theta_h-\zeta_h\|_1+\|w_h-v_h\|_1+\|\lambda_h-\mu_h\|_{-1}\lesssim\|\theta-\zeta_h\|_1+\|w-v_h\|_1+\|\lambda-\mu_h\|_{-1}+\sqrt{\langle\mu_h,w-\psi\rangle}$$

On the other hand, we have,

$$\|\theta-\theta_h\|_1+\|w-w_h\|_1+\|\lambda-\lambda_h\|_{-1}\leq\|\theta-\zeta_h\|_1+\|w-v_h\|_1+\|\lambda-\mu_h\|_{-1}+\|\theta_h-\zeta_h\|_1+\|w_h-v_h\|_1+\|\lambda_h-\mu_h\|_{-1}$$

Thus

$$\|\theta-\theta_h\|_1+\|w-w_h\|_1+\|\lambda-\lambda_h\|_{-1}\lesssim\|\theta-\zeta_h\|_1+\|w-v_h\|_1+\|\lambda-\mu_h\|_{-1}+\sqrt{\langle\mu_h,w-\psi\rangle}$$

Hence

$$\|\theta-\theta_h\|_1+\|w-w_h\|_1+\|\lambda-\lambda_h\|_{-1}\lesssim\inf_{(\zeta_h,v_h)\in V_h}(\|\theta-\zeta_h\|_1+\|w-v_h\|_1)+\inf_{\mu_h\in\Lambda_h}(\|\lambda-\mu_h\|_{-1}+\sqrt{\langle\mu_h,w-\psi,\rangle})$$

□

5.3. **A pair leading to a stable system.** Let us consider now the finite element spaces

$$V_h=V_h^1\times V_h^2,\quad Q_h=V_h^0:=\{\mu_h\in L^2(0,\ell);\quad \mu_h|_{[x_i,x_{i+1}]}\in\mathbb{P}^0\}. \tag{5.10}$$

We introduce the following discrete norm,

$$\|\mu_h\|_{-1,h}^2=\sum_{I_i\in\mathcal{I}_h}h_i^2\|\mu_h\|_{0,I_i}^2,\quad\forall\mu_h\in Q_h.$$

Lemma 5.7. *For the discrete spaces (5.10) the bilinear form $\boldsymbol{b}(\cdot,\cdot)$ satisfies a uniform inf-sup inequality, i.e, for any $\mu_h\in Q_h$ there exists a positive constant independent of h and t such that:*

$$\sup_{(\zeta_h,v_h)\in V_h}\frac{\boldsymbol{b}((\zeta_h,v_h),\mu_h)}{\|(\zeta_h,v_h)\|_1}\geq C_1\|\mu_h\|_{-1} \tag{5.11}$$

Proof. We define the function $b_i(x)$ by

$$b_i(x)=\begin{cases}(x-x_i)(x_{i+1}-x) & \text{if } x\in I_i\\ 0 & \text{otherwise}\end{cases}$$

and let $\mu_h\in Q_h$. Then

$$\|b_i\|_{L^1}=\frac{h_i^3}{6},\quad\|b_i'\|_{L^2}^2=\frac{h_i^3}{3},$$

We set $v_h|_{I_i}=b_i\mu_h$, so

$$\langle\mu_h,v_h\rangle=\langle\mu_h,\mu_hb_i\rangle=\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\mu_h^2b_i=\sum_{I_i\in\mathcal{I}_h}\frac{h_i^2}{6}\|\mu_h\|_{0,I_i}^2.$$

$$\|v_h\|_1^2=\|v_h'\|_0^2=\sum_{I_i\in\mathcal{I}_h}\|v_h'\|_{0,I_i}^2=\sum_{I_i\in\mathcal{I}_h}\|\mu_hb_i'\|_{0,I_i}^2=\sum_{I_i\in\mathcal{I}_h}\frac{h_i^2}{3}\|\mu_h\|_{0,I_i}^2$$

Then we get,

$$\|v_h\|_1=\left(\sum_{I_i\in\mathcal{I}_h}\frac{h_i^2}{3}\|\mu_h\|_{0,I_i}^2\right)^{1/2}$$

Thus

$$\frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} = \frac{\sqrt{3}}{6}\|\mu_h\|_{-1,h} \tag{5.12}$$

Hence,

$$\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \geq C_1\|\mu_h\|_{-1,h} \tag{5.13}$$

On the other hand, we have,

$$\|\mu_h\|_{-1} = \sup_{v\in H_0^1(0,L)} \frac{\langle \mu_h, v\rangle}{\|v\|_1}$$

we get,

$$\langle \mu_h, v\rangle \geq C_2\|\mu_h\|_{-1}\|v\|_1,$$

so, using Cauchy-Schwarz inequality, we obtain,

$$\begin{aligned}
\langle \mu_h, v_h\rangle &= \langle \mu_h, v_h - v\rangle + \langle \mu_h, v\rangle \\
&= \sum_{I_i\in\mathcal{I}_h} \int_{I_i} h_i(\mu_h)h_i^{-1}(v_h - v) + \langle \mu_h, v\rangle \\
&\geq -\left(\sum_{I_i\in\mathcal{I}_h} h_i^2\|\mu_h\|_{0,I_i}^2\right)^{1/2}\left(\sum_{I_i\in\mathcal{I}_h}\left(h_i^{-1}\|v_h - v\|_{0,I_i}\right)^2\right)^{1/2} + \langle \mu_h, v\rangle \\
&\geq -C_3\|\mu_h\|_{-1,h}\|v\|_1 + \langle \mu_h, v\rangle \\
&\geq (C_2\|\mu_h\|_{-1} - C_3\|\mu_h\|_{-1,h})\|v\|_1 \\
&\gtrsim (C_2\|\mu_h\|_{-1} - C_3\|\mu_h\|_{-1,h})\|v_h\|_1
\end{aligned}$$

Note that in the last inequality, we have used the fact that $\|v_h\|_1 \lesssim \|v\|_1$. But of course, to gather with $C_2\|\mu_h\|_{-1} - C_3\|\mu_h\|_{-1,h} \geq 0$, if $C_2\|\mu_h\|_{-1} - C_3\|\mu_h\|_{-1,h} < 0$ then (5.13) implies the desired inf-sup condition immediately. Then

$$\frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \geq -C_3\|\mu_h\|_{-1,h} + C_2\|\mu_h\|_{-1}$$

i.e.,

$$\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \geq -C_3\|\mu_h\|_{-1,h} + C_2\|\mu_h\|_{-1} \tag{5.14}$$

Using (5.12) and (5.14) and letting $\delta \in (0,1)$, we have

$$\begin{aligned}
\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} &= (1-\delta)\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} + \delta\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \\
&\geq C_1(1-\delta)\|\mu_h\|_{-1,h} + \delta\left(C_2\|\mu_h\|_{-1} - C_3\|\mu_h\|_{-1,h}\right) \\
&= \delta C_2\|\mu_h\|_{-1} + \left((1-\delta)C_1 - C_3\right)\|\mu_h\|_{-1,h}.
\end{aligned}$$

It suffices to take:

$$(1-\delta)C_1 - C_3\delta > 0 \text{ or } 0 < \delta < \frac{C_1}{C_1 + C_3},$$

to conclude that,

$$\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \gtrsim \|\mu_h\|_{-1}.$$

□

6. A posteriori analysis for the mixed formulation

In this section, we propose an a posteriori error indicator for the mixed formulation introduced in section 3. First of all, we define the local indicators:

$$\eta_i^{(1)} = h_i \|\varepsilon^{-2}(w_h'' - \theta_h') + f + \lambda_h\|_{0,I_i}$$
$$\eta_i^{(2)} = h_i \|\theta_h'' + \varepsilon^{-2}(w_h' - \theta_h)\|_{0,I_i}$$

We also define the global indicators:

$$\eta^2 = \sum_{I_i \in \mathcal{I}_h} \left(\left(\eta_i^{(1)}\right)^2 + \left(\eta_i^{(2)}\right)^2 \right) \tag{6.1}$$

$$S_m = \|\psi - w_h\|_1 + \sqrt{\langle \lambda_h, (\psi - w_h)_+ \rangle} \tag{6.2}$$

6.1. The reliability of a posteriori error indicator.

Lemma 6.1. *Let (θ, w, λ) be the solution of the continuous problem (4.1), and $(\theta_h, w_h, \lambda_h)$ be the solution of the discrete problem (5.1). Then, there exists $z \in H_0^1(0,\ell)$ such that*

$$\mathscr{A}((\theta - \theta_h, w - w_h, \lambda - \lambda_h); (\theta - \theta_h, z, \lambda_h - \lambda)) \gtrsim \|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2$$
$$\|z\|_1 \lesssim \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1}$$

Proof. In (4.2), (4.3), we take $(\zeta, v, \mu) = (\theta - \theta_h, w - w_h, \lambda - \lambda_h)$. □

Lemma 6.2. *Let z_h be the Lagrange interpolation of z. Then,*

$$0 \le \mathscr{L}((0, -z_h, 0)) - \mathscr{A}((\theta_h, w_h, \lambda_h); (0, -z_h, 0)). \tag{6.3}$$

Proof. It suffices to take $(\zeta_h, v_h, \mu_h) = (0, -z_h, \lambda_h)$, in the discrete problem (5.1). □

Proposition 6.3. *Let (θ, w, λ) be the solution of the continuous problem (4.1), and $(\theta_h, w_h, \lambda_h)$ be the solution of the discrete problem (5.1). Then,*

$$\|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2$$
$$\lesssim \mathscr{L}((\theta - \theta_h, z - z_h, \lambda_h - \lambda)) - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda)) \tag{6.4}$$

Proof.

$$\begin{aligned}
\|(\theta - \theta_h, w - w_h, \lambda - \lambda_h)\|_{\mathcal{H}}^2 &\lesssim \mathscr{A}((\theta - \theta_h, w - w_h, \lambda - \lambda_h); (\theta - \theta_h, z, \lambda_h - \lambda)) \\
&= \mathscr{A}((\theta, w, \lambda); (\theta - \theta_h, z, \lambda_h - \lambda)) \\
&\quad - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z, \lambda_h - \lambda)) \\
&\lesssim \mathscr{A}((\theta, w, \lambda); (\theta - \theta_h, z, \lambda_h - \lambda)) \\
&\quad - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z, \lambda_h - \lambda)) + \mathscr{L}(0, -z_h, 0) \\
&\quad - \mathscr{A}((\theta_h, w_h, \lambda_h); (0, -z_h, 0)) \\
&= \mathscr{A}((\theta, w, \lambda); (\theta - \theta_h, z, \lambda_h - \lambda)) \\
&\quad - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda)) + \mathscr{L}(0, -z_h, 0) \\
&= \mathscr{A}((\theta, w, \lambda); (\theta - \theta_h, z, \lambda_h - \lambda)) \\
&\quad - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda)) + \mathscr{L}(0, -z_h, 0) \\
&\quad - \mathscr{L}(\theta - \theta_h, -z, \lambda_h - \lambda) + \mathscr{L}(\theta - \theta_h, -z, \lambda_h - \lambda) \\
&\le \mathscr{L}(\theta - \theta_h, z - z_h, \lambda_h - \lambda) \\
&\quad - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda))
\end{aligned}$$

□

Lemma 6.4. *Let (θ, w, λ) be the solution of the continuous problem (4.1), and $(\theta_h, w_h, \lambda_h)$ be the solution of the discrete problem (5.1). Then, the following inequality holds:*

$$\langle \lambda_h - \lambda, w_h - \psi \rangle \leq \frac{\|\lambda - \lambda_h\|_{-1}^2}{2} + S_m^2 \tag{6.5}$$

Proof. If we take $\zeta_h = v_h = 0$ in the discrete problem (5.1) we get,

$$-\langle \mu_h - \lambda_h, w_h \rangle \leq -\langle \mu_h - \lambda_h, \psi \rangle \tag{6.6}$$

and for the choice $\mu_h = 0$, $\mu_h = 2\lambda_h$ in (6.6), we find that,

$$\langle \lambda_h, w_h - \psi \rangle = 0.$$

Then,

$$\begin{aligned}
\langle \lambda_h - \lambda, w_h - \psi \rangle &= \langle \lambda, \psi - w_h \rangle \\
&\leq \langle \lambda, (\psi - w_h)_+ \rangle \\
&= \langle \lambda - \lambda_h, (\psi - w_h)_+ \rangle + \langle \lambda_h, (\psi - w_h)_+ \rangle \\
&\leq \|\lambda - \lambda_h\|_{-1} \|(\psi - w_h)_+\|_1 + \langle \lambda_h, (\psi - w_h)_+ \rangle
\end{aligned}$$

Hence, by Young's inequality,

$$\begin{aligned}
\langle \lambda_h - \lambda, w_h - \psi \rangle &\leq \frac{\|\lambda - \lambda_h\|_{-1}^2}{2} + \frac{\|(\psi - w_h)_+\|_1^2}{2} + \langle \lambda_h, (\psi - w_h)_+ \rangle \\
&\leq \frac{\|\lambda - \lambda_h\|_{-1}^2}{2} + S_m^2
\end{aligned}$$

□

Theorem 6.5. *Let (θ, w, λ) be the solution of the continuous problem (4.1), and $(\theta_h, w_h, \lambda_h)$ be the solution of the discrete problem (5.1). Then, the following a posteriori reliability estimate holds:*

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \lesssim \eta + S_m \tag{6.7}$$

Proof. From 6.4, using Cauchy-Schwarz's and Young's inequalities

$$\begin{aligned}
&\|\theta-\theta_h\|_1^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2\\
&\lesssim \mathscr{L}((\theta-\theta_h,z-z_h,\lambda_h-\lambda))-\mathscr{A}((\theta_h,w_h,\lambda_h);(\theta-\theta_h,z-z_h,\lambda_h-\lambda))\\
&=(f,z-z_h)-\langle\lambda_h-\lambda,\psi\rangle-a(\theta_h,\theta-\theta_h)-\varepsilon^{-2}b((\theta_h,w_h);(\theta-\theta_h,z-z_h))+\langle\lambda_h,z-z_h\rangle\\
&+\langle\lambda_h-\lambda,w_h\rangle\\
&=(f,z-z_h)-\int_0^\ell \theta_h'(\theta-\theta_h)'\,dx-\varepsilon^{-2}\int_0^\ell (w_h'-\theta_h)(z-z_h)'\,dx\\
&+\varepsilon^{-2}\int_0^\ell (w_h'-\theta_h)(\theta-\theta_h)\,dx+\langle\lambda_h,z-z_h\rangle+\langle\lambda_h-\lambda,w_h-\psi\rangle\\
&=\sum_{I_i\in\mathcal{I}_h}(f,z-z_h)_{0,I_i}-\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\theta_h'(\theta-\theta_h)'dx-\varepsilon^{-2}\sum_{I_i\in\mathcal{I}_h}\int_{I_i}(w_h'-\theta_h)(z-z_h)'dx\\
&+\varepsilon^{-2}\sum_{I_i\in\mathcal{I}_h}\int_{I_i}(w_h'-\theta_h)(\theta-\theta_h)dx+\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\lambda_h(z-z_h)dx+\langle\lambda_h-\lambda,w_h-\psi\rangle\\
&=\sum_{I_i\in\mathcal{I}_h}\int_{I_i}h_i\big(\varepsilon^{-2}(w_h''-\theta_h')+f+\lambda_h\big)h_i^{-1}\big(z-z_h\big)dx\\
&+\sum_{I_i\in\mathcal{I}_h}\int_{I_i}h_i\big(\theta_h''+\varepsilon^{-2}(w_h'-\theta_h)\big)h_i^{-1}\big(\theta-\theta_h\big)dx+\langle\lambda_h-\lambda,w_h-\psi\rangle
\end{aligned}$$

Now, we may write,

$$\begin{aligned}
&\|\theta-\theta_h\|_1^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2\\
&\le\Big(\sum_{I_i\in\mathcal{I}_h}h_i^2\|\varepsilon^{-2}(w_h''-\theta_h')+f+\lambda_h\|_{0,I_i}^2\Big)^{1/2}\Big(\sum_{I_i\in\mathcal{I}_h}\big(h_i^{-1}\|z-z_h\|_{0,I_i}\big)^2\Big)^{1/2}\\
&+\Big(\sum_{I_i\in\mathcal{I}_h}h_i^2\|\theta_h''+\varepsilon^{-2}(w_h'-\theta_h)\|_{0,I_i}^2\Big)^{1/2}\Big(\sum_{I_i\in\mathcal{I}_h}\big(h_i^{-1}\|\theta-\theta_h\|_{0,I_i}\big)^2\Big)^{1/2}+\langle\lambda_h-\lambda,w_h-\psi\rangle\\
&\le\frac{1}{2}\sum_{I_i\in\mathcal{I}_h}\Big(\eta_i^{(1)}\Big)^2+\frac{1}{2}\|z\|_1^2+\frac{1}{2}\sum_{I_i\in\mathcal{I}_h}\Big(\eta_i^{(2)}\Big)^2+\frac{1}{2}\|\theta-\theta_h\|_1^2+\langle\lambda_h-\lambda,w_h-\psi\rangle
\end{aligned}$$

Then,

$$\begin{aligned}
\|\theta-\theta_h\|_1^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2&\le\eta^2+\|z\|_1^2+\langle\lambda_h-\lambda,w_h-\psi\rangle\\
&\le\eta^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2+\frac{\|\lambda-\lambda_h\|_{-1}^2}{2}+S_m^2
\end{aligned}$$

So,

$$\|\theta-\theta_h\|_1^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2\lesssim\eta^2+S_m^2$$

Hence,

$$\|\theta-\theta_h\|_1+\|w-w_h\|_1+\|\lambda-\lambda_h\|_{-1}\lesssim\eta+S_m$$

□

6.2. **The optimality of the a posteriori error indicator.** Let $f_h\in Q_h$ be the projection of f, and we define

$$osc_i(f)=h_i\|f-f_h\|_{0,I_i}\qquad osc(f)^2=\sum_{I_i\in\mathcal{I}_h}h_i^2\|f-f_h\|_{0,I_i}^2$$

The following propositions can be shown using standard techniques of residual-based a posteriori error estimates.

Proposition 6.6. *For all* $(\zeta_h, v_h, \mu_h) \in V_h \times Q_h$, *we have*

$$\varepsilon^2 h_i \|\varepsilon^{-2}\,(v_h'' - \zeta_h') + \mu_h + f\|_{0,I_i} \lesssim \|\theta - \zeta_h\|_{1,I_i} + \|w - v_h\|_{1,I_i} + \|\lambda - \mu_h\|_{-1} + osc_i(f) \tag{6.8}$$

Proposition 6.7. *For all* $(\zeta_h, v_h, \mu_h) \in V_h \times Q_h$, *we have*

$$\varepsilon^2 h_i \|\zeta_h'' + v_h' - \zeta_h\|_{0,I_i} \lesssim \|\theta - \zeta_h\|_{1,I_i} + \|w - v_h\|_{1,I_i} + \|\lambda - \mu_h\|_{-1} + osc_i(f) \tag{6.9}$$

Theorem 6.8. *Let* (θ, w, λ) *be the solution of the continuous problem (4.1), and* $(\theta_h, w_h, \lambda_h)$ *be the solution of the discrete problem (5.1). Then, the following a posteriori optimality estimate holds:*

$$\varepsilon^2 \eta \lesssim \|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} + osc(f) \tag{6.10}$$

Proof. If we take $(\zeta_h, v_h, \mu_h) = (\theta_h, w_h, \lambda_h)$ in (6.8), (6.9), we obtain

$$\begin{aligned}
\varepsilon^2 \eta_i^{(1)} &\lesssim \|\theta - \theta_h\|_{1,I_i} + \|w - w_h\|_{1,I_i} + \|\lambda - \lambda_h\|_{-1,I_i} + osc_i(f) \\
\varepsilon^2 \eta_i^{(2)} &\lesssim \|\theta - \theta_h\|_{1,I_i} + \|w - w_h\|_{1,I_i}
\end{aligned}$$

Hence

$$\varepsilon^2 \eta \lesssim \|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} + osc(f)$$

□

7. Stabilized formulation

In this section, we present a stabilized formulation for which stability and error estimates are valid for any reasonable conforming finite element pair (V_h, Q_h). It is shown in section 2.4 that the choice $V_h = V_h^1 \times V_h^1$ suffers from locking, and for the compact formulation, the choice $V_h = V_h^1 \times V_h^1$ and $Q_h = V_h^0$ leads to an ill-posed problem (see Lemma 5.3), despite the fact that this constitutes a conforming approximation.
Let $\alpha > 0$, we introduce the following bilinear and linear forms:

$$\begin{aligned}
\mathcal{S}_h((\sigma,\tau,\rho);(\zeta,v,\mu)) = h^2\varepsilon^4 \int_0^\ell \Big[&\big(\sigma'' + \varepsilon^{-2}(\tau' - \sigma)\big)\big(\zeta'' + \varepsilon^{-2}(v' - \zeta)\big) \\
&+ \big(\varepsilon^{-2}(\tau'' - \sigma') + \rho\big)\big(\varepsilon^{-2}(v'' - \zeta') + \mu\big)\Big]\, dx \\
\mathcal{L}_h((\zeta,v,\mu)) = -h^2\varepsilon^4 \int_0^\ell f\big(\varepsilon^{-2}(v'' - \zeta') + \mu\big)\, dx &
\end{aligned}$$

and then we define, the bilinear form $\mathscr{A}_h$ and the linear form $\mathscr{L}_h$

$$\begin{aligned}
\mathscr{A}_h((\sigma,\tau,\rho);(\zeta,v,\mu)) &= \mathscr{A}((\sigma,\tau,\rho);(\zeta,v,\mu)) - \alpha\mathcal{S}_h((\sigma,\tau,\rho);(\zeta,v,\mu)) \\
\mathscr{L}_h((\zeta,v,\mu)) &= \mathscr{L}((\zeta,v,\mu)) - \alpha\mathcal{L}_h((\zeta,v,\mu))
\end{aligned}$$

We consider the discrete problem

$$\begin{cases}
\text{Find } (\theta_h, w_h, \lambda_h) \in V_h \times \Lambda_h \\
\mathscr{A}_h((\theta_h, w_h, \lambda_h), (\zeta_h, v_h, \mu_h - \lambda_h)) \le \mathscr{L}_h((\zeta_h, v_h, \mu_h - \lambda_h)),\ \forall (\zeta_h, v_h, \mu_h) \in V_h \times \Lambda_h
\end{cases} \tag{7.1}$$

7.1. **Stability Analysis.** For the stability analysis, we need to the following inverse inequalities.

Lemma 7.1. There exists a positive constant C_I such that for any $\phi_h \in V_h^k, k \geq 1$

$$C_I \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\phi_h'\|_{0,I_i}^2 \leq \|\phi_h\|_0^2 \tag{7.2}$$

$$C_I \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\phi_h''\|_{0,I_i}^2 \leq \|\phi_h'\|_0^2 \tag{7.3}$$

Proof. See [14]. □

The following stability condition holds.

Theorem 7.2. Suppose that $0 < \alpha < \min\left(\frac{\varepsilon^{-2}}{2h^2+C_I^{-1}}, \frac{C_I}{2\varepsilon^4}\right)$. Then for any $(\zeta_h, v_h, \mu_h) \in V_h \times \Lambda_h$, there exits $(u_h, p_h) \in V_h$ such that

$$\mathscr{A}_h((\zeta_h, v_h, \mu_h); (u_h, p_h, -\mu_h)) \gtrsim \|\zeta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2 \tag{7.4}$$

$$\|u_h\|_1 + \|p_h\|_1 \lesssim \|\zeta_h\| + \|v_h\|_1 + \|\mu_h\|_{-1} \tag{7.5}$$

Proof. The proof is divided into several steps:

i) Using (7.3) and Young's inequality we obtain

$$\begin{aligned}
&\mathscr{A}_h((\zeta_h, v_h, \mu_h); (\zeta_h, v_h, -\mu_h))\\
=&\mathscr{A}((\zeta_h, v_h, \mu_h); (\zeta_h, v_h, -\mu_h)) - \alpha\mathcal{S}_h((\zeta_h, v_h, \mu_h); (\zeta_h, v_h, -\mu_h))\\
=&\boldsymbol{a}((\zeta_h, v_h); ((\zeta_h, v_h)) - \alpha\mathcal{S}_h((\zeta_h, v_h, \mu_h); (\zeta_h, v_h, -\mu_h))\\
=&\|\zeta_h'\|_0^2 + \varepsilon^{-2}\|v_h' - \zeta_h\|_0^2\\
&-\alpha h^2\left(\varepsilon^4 \int_0^\ell \left(\zeta_h'' + \varepsilon^{-2}(v_h' - \zeta_h)\right)^2 \, dx + \int_0^\ell (v_h'' - \zeta_h')^2 \, dx - \varepsilon^4 \int_0^\ell \mu_h^2 \, dx\right)
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathscr{A}_h((\zeta_h, v_h, \mu_h); (\zeta_h, v_h, -\mu_h))\\
&\geq \|\zeta_h'\|_0^2 + \varepsilon^{-2}\|v_h' - \zeta_h\|_0^2 - \alpha\varepsilon^4 h^2 \int_0^\ell \left(\zeta_h'' + \varepsilon^{-2}(v_h' - \zeta_h)\right)^2 \, dx\\
&-\alpha C_I^{-1} \int_0^\ell (v_h' - \zeta_h)^2 \, dx + \alpha\varepsilon^4 h^2 \int_0^\ell \mu_h^2\\
&\geq \|\zeta_h'\|_0^2 + \varepsilon^{-2}\|v_h' - \zeta_h\|_0^2 - 2\alpha\varepsilon^4 h^2\|\zeta_h''\|_0^2 - 2\alpha h^2\|v_h' - \zeta_h\|_0^2 - \alpha C_I^{-1}\|v_h' - \zeta_h\|_0^2\\
&+\alpha\varepsilon^4 h^2\|\mu_h\|_0^2\\
&\geq \|\zeta_h'\|_0^2 + \varepsilon^{-2}\|v_h' - \zeta_h\|_0^2 - 2\alpha C_I^{-1}\varepsilon^4\|\zeta_h'\|_0^2 - 2\alpha h^2\|v_h' - \zeta_h\|_0^2 - \alpha C_I^{-1}\|v_h' - \zeta_h\|_0^2\\
&+\alpha\varepsilon^4 h^2\|\mu_h\|_0^2\\
&\geq (1 - 2\alpha\varepsilon^4 C_I^{-1})\|\zeta_h'\|_0^2 + (\varepsilon^{-2} - 2\alpha h^2 - \alpha C_I^{-1})\|v_h' - \zeta_h\|_0^2 + \alpha\varepsilon^4 h^2\|\mu_h\|_0^2
\end{aligned}$$

Because, we have $0 < \alpha < \min\left(\frac{\varepsilon^{-2}}{2h^2+C_I^{-1}}, \frac{C_I}{2\varepsilon^4}\right)$, we have

$$1 - 2\alpha\varepsilon^4 C_I^{-1} > 0 \quad \text{and} \quad \varepsilon^{-2} - 2\alpha h^2 - \alpha C_I^{-1} > 0.$$

Hence,

$$\mathscr{A}_h((\zeta_h, v_h, \mu_h); (\zeta_h, v_h, -\mu_h)) \geq C_3(\|\zeta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1,h}^2) \tag{7.6}$$

where $C_3 = \min(\alpha\varepsilon^4, 1 - 2\alpha\varepsilon^4 C_I^{-1}, \varepsilon^{-2} - 2\alpha h^2 - \alpha C_I^{-1})$.

ii) We have from (5.14)

$$\sup_{(\zeta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \geq C_1\|\mu_h\|_{-1} - C_1\|\mu_h\|_{-1,h}$$

we deduce that $\forall \mu_h \in Q_h, \quad \exists(\rho_h, q_h) \in V_h$ such that

$$\|\mu_h\|_{-1} = \|(\rho_h, q_h)\|_V \tag{7.7}$$
$$\langle \mu_h, q_h\rangle \geq C_1\|\mu_h\|_{-1}^2 - C_2\|\mu_h\|_{-1}\|\mu_h\|_{-1,h} \tag{7.8}$$

iii)

$$\begin{aligned}
\mathscr{A}_h((\zeta_h, v_h, \mu_h); (\rho_h, q_h, 0)) &= \mathscr{A}((\zeta_h, v_h, \mu_h); (\rho_h, q_h, 0)) - \alpha\mathcal{S}_h((\zeta_h, v_h, \mu_h); (\rho_h, q_h, 0)) \\
&= \boldsymbol{a}((\zeta_h, v_h); (\rho_h, q_h)) - < \mu_h, q_h > \\
&- \alpha\mathcal{S}_h((\zeta_h, v_h, \mu_h); (\rho_h, q_h, 0)) \\
&= \int_0^\ell \zeta_h'\rho_h' \, dx + \varepsilon^{-2}\int_0^\ell (v_h' - \zeta_h)(q_h' - \rho_h) \, dx - < \mu_h, q_h > \\
&- \alpha\int_0^\ell \Big(h^2\varepsilon^4\zeta_h''\rho_h'' + h^2\varepsilon^2\left(\zeta_h''(q_h' - \rho_h) + \rho_h''(v_h' - \zeta_h)\right)\Big) \, dx \\
&- \alpha\int_0^\ell \Big(h^2(v_h' - \zeta_h)(q_h' - \rho_h) \\
&+ h^2(v_h'' - \zeta_h')(q_h'' - \rho_h')\Big) \, dx - \alpha\int_0^\ell h^2\varepsilon^2\mu_h(q_h'' - \mu_h') \, dx
\end{aligned}$$

Using (7.8) and Cauchy-Schwarz inequality

$$\begin{aligned}
\mathscr{A}_h((\zeta_h, v_h, \mu_h); (\rho_h, q_h, 0)) &\geq -\|\zeta_h'\|_0\|\rho_h'\|_0 - \varepsilon^{-2}\|v_h' - \zeta_h\|_0\|q_h' - \rho_h\|_0 \\
&+ C_1\|\mu_h\|_{-1}^2 - C_2\|\mu_h\|_{-1}\|\mu_h\|_{-1,h} - \alpha\varepsilon^4(h\|\zeta_h''\|_0)(h\|\rho_h''\|_0) \\
&- \alpha\varepsilon^2\big(h\|\zeta_h''\|_0\|h(q_h' - \rho_h)\|_0 + h\|\rho_h''\|_0\|h(v_h' - \zeta_h)\|_0\big) \\
&- \alpha\big((h\|v_h' - \zeta_h\|_0)(h\|q_h' - \rho_h\|_0) \\
&+ (h\|v_h'' - \zeta_h'\|_0)(h\|q_h'' - \rho_h'\|_0)\big) \\
&- \alpha\varepsilon^2(h\|\mu_h\|_0)(h\|q_h'' - \mu_h'\|_0)
\end{aligned}$$

Using (7.3), (7.7) and Young's inequalities

$$\begin{aligned}\mathscr{A}_h((\zeta_h,v_h,\mu_h);(\rho_h,q_h,0)) \geq & -(1+\alpha\varepsilon^4 C_I^{-1})\|\zeta_h'\|_0\|\rho_h'\|_0 + C_1\|\mu_h\|_{-1}^2\\ & - C_2\|\mu_h\|_{-1}\|\mu_h\|_{-1,h} - \alpha\varepsilon^2\sqrt{C_I^{-1}}\big(\|\zeta_h'\|_0\|h(q_h'-\rho_h)\|_0\\ & + \|\rho_h'\|_0\|h(v_h'-\zeta_h)\|_0\big) - \alpha\big((h\|v_h'-\zeta_h\|_0)(h\|q_h'-\rho_h\|_0)\big)\\ & - \alpha C_I^{-1}(\|v_h'-\zeta_h\|_0)(\|q_h'-\rho_h\|_0)\\ & - \alpha\varepsilon^2\sqrt{C_I^{-1}}(h\|\mu_h\|_0)(\|q_h'-\mu_h\|_0)\\ & -\varepsilon^{-2}\|v_h'-\zeta_h\|_0\|q_h'-\rho_h\|_0\\ & \geq (C_1-\frac{\epsilon}{2}C_2)\|\mu_h\|_{-1}^2 - \frac{\epsilon}{2}\left(\frac{\alpha\varepsilon^4}{C_I}+1\right)\|\rho_h'\|_0^2\\ & -\frac{1}{2\epsilon}\left(\frac{\alpha\varepsilon^4}{C_I}+1\right)\|\zeta_h'\|_0^2 - \frac{\alpha\varepsilon^2}{\sqrt{C_I}}\Big(\frac{\epsilon}{2}\big(\|\rho_h'\|_0^2+h^2\|q_h'-\rho_h\|_0^2\big)\\ & + \frac{1}{2\epsilon}\big(\|\zeta_h'\|_0^2+h^2\|v_h'-\zeta_h\|_0^2\big)\Big)\\ & - (\alpha h^2+\alpha C_I^{-1}+\varepsilon^{-2})\left(\frac{\epsilon}{2}\|q_h'-\rho_h\|_0^2+\frac{1}{2\epsilon}\|v_h'-\zeta_h\|_0^2\right)\\ & - \frac{\alpha\varepsilon^2}{\sqrt{C_I}}\left(\frac{\epsilon}{2}\|q_h'-\rho_h\|_0^2+\frac{1}{2\epsilon}h^2\|\mu_h\|_0^2\right) - \frac{C_2}{2\epsilon}\|\mu_h\|_{-1,h}^2\\ & \geq C_4\|\mu_h\|_{-1}^2 - C_5(\|\zeta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1,h}^2)\end{aligned}$$

We choose ϵ small enough so that $C_4>0$.

With the same steps we get

$$\mathscr{A}_h((\zeta_h,v_h,\mu_h);(\delta\rho_h,\delta q_h,0)) \geq C_4\|\mu_h\|_{-1}^2 - \delta C_5(\|\zeta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1,h}^2) \tag{7.9}$$

iv) From (7.6) , (7.9) we got

$$\begin{cases}\mathscr{A}_h((\zeta_h,v_h,\mu_h);(\zeta_h,v_h,-\mu_h)) \geq C_3(\|\zeta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1,h}^2)\\ \mathscr{A}_h((\zeta_h,v_h,\mu_h);(\delta\rho_h,\delta q_h,0)) \geq C_4\|\mu_h\|_{-1}^2 - \delta C_5(\|\zeta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1,h}^2)\end{cases}$$

So

$$\begin{aligned}\mathscr{A}_h((\zeta_h,v_h,\mu_h);(\zeta_h+\delta\rho_h,v_h+\delta q_h,-\mu_h)) \geq & \big(C_3-\delta C_5\big)(\|\zeta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1,h}^2)\\ & + C_4\|\mu_h\|_{-1}^2\end{aligned}$$

We choose $\delta>0$ such that $C_3-\delta C_5>0$ i.e., $0<\delta<\frac{C_3}{C_5}$

Hence there exist $u_h=\zeta_h+\delta\rho_h$, $p_h=v_h+\delta q_h$ such that

$$\mathscr{A}_h((\zeta_h,v_h,\mu_h);(u_h,p_h,-\mu_h)) \gtrsim \|\zeta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1}^2$$

v) Using (7.7), we obtain

$$\begin{aligned}\|u_h\|_1+\|p_h\|_1 &= \|\zeta_h+\delta\rho_h\|_1+\|v_h+\delta q_h\|_1\\ &\leq \|\zeta_h\|_1+\|v_h\|_1+\delta(\|\rho_h\|_1+\|q_h\|_1)\\ &\lesssim \|\zeta_h\|+\|v_h\|_1+\|\mu_h\|_{-1}\end{aligned}$$

□

Proposition 7.3. *For all* $(\zeta_h, v_h, \mu_h) \in V_h \times \Lambda_h$

$$\sup_{(u_h,p_h,\xi_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((\zeta_h, v_h, \mu_h), ((u_h, p_h, \xi_h))}{\|((u_h, p_h, \xi_h)\|_{\mathcal{H}}} \gtrsim \|(\zeta_h, v_h, \mu_h)\|_{\mathcal{H}} \tag{7.10}$$

Proof. We proved:

$$\mathscr{A}_h((\zeta_h, v_h, \mu_h); (u_h, p_h, -\mu_h)) \gtrsim \| (\zeta_h, v_h, \mu_h) \|^2_{\mathcal{H}}$$

i.e.,

$$\frac{\mathscr{A}_h ((\zeta_h, v_h, \mu_h); (u_h, p_h, -\mu_h))}{\| (\zeta_h, v_h, \mu_h) \|_{\mathcal{H}}} \gtrsim \| (\zeta_h, v_h, \mu_h) \|_{\mathcal{H}}$$

In addition, we have,

$$\| (u_h, p_h, -\mu_h) \|^2_{\mathcal{H}} = \|u_h\|^2_1 + \|p_h\|^2_1 + \|\mu_h\|^2_{-1} \lesssim \|\zeta_h\|^2_1 + \|v_h\|^2_1 + \|\mu_h\|^2_{-1} = \| (\zeta_h, v_h, \mu_h) \|^2_{\mathcal{H}}$$

i.e.,

$$\frac{1}{\| (u_h, p_h, -\mu_h) \|_{\mathcal{H}}} \gtrsim \frac{1}{\| (\zeta_h, v_h, \mu_h) \|_{\mathcal{H}}}$$

Then

$$\frac{\mathscr{A}_h ((\zeta_h, v_h, \mu_h); (u_h, p_h, -\mu_h))}{\| (u_h, p_h, -\mu_h) \|_{\mathcal{H}}} \gtrsim \| (\zeta_h, v_h, \mu_h) \|_{\mathcal{H}}$$

Thus

$$\sup_{(u_h,p_h,\xi_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((\zeta_h, v_h, \mu_h), (u_h, p_h, \xi_h))}{\|(u_h, p_h, \xi_h)\|_{\mathcal{H}}} \gtrsim \|(\zeta_h, v_h, \mu_h)\|_{\mathcal{H}}$$

□

7.2. **A posteriori error estimator for the stabilized formulation.** Following the approach used for the mixed formulation, the a posteriori error indicator for the stabilized formulation can be shown to be both reliable and optimal. By integrating the stabilization terms into the residual estimator, we prove these two key properties hold.

Theorem 7.4 (Reliability of the a posteriori error indicator)**.** *Let* (θ, w, λ) *be the solution of the continuous problem (4.1), and* $(\theta_h, w_h, \lambda_h)$ *be the solution of the discrete problem (7.1). Then, the following a posteriori reliability estimate holds:*

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \lesssim \eta + S_m \tag{7.11}$$

where η *and* S_m *are defined in the previous section.*

Proof. We prove Theorem 7.4 using Lemma 6.1 and Proposition 6.4, as well as the following results:

Lemma 7.5. Let z_h the Lagrange interpolation of z. Then

$$0 \leq \mathscr{L}_h((0, -z_h, 0)) - \mathscr{A}_h((\theta_h, w_h, \lambda_h); (0, -z_h, 0)). \tag{7.12}$$

Proposition 7.6. *Let* (θ, w, λ) *be the solution of the continuous problem (4.1), and* $(\theta_h, w_h, \lambda_h)$ *be the solution of the discrete problem (7.1). Then,*

$$\begin{aligned}
&\|\theta - \theta_h\|^2_1 + \|w - w_h\|^2_1 + \|\lambda - \lambda_h\|^2_{-1} \\
&\lesssim \mathscr{L}((\theta - \theta_h, z - z_h, \lambda_h - \lambda)) - \mathscr{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda)) \\
&+ \alpha h^2 \varepsilon^2 \int_0^\ell \Big(\big(- f - \varepsilon^{-2}(w_h'' - \theta_h') - \lambda_h \big)\big(z_h''\big) + \big(- \theta_h'' - \varepsilon^{-2}(w_h' - \theta_h)\big)\big(z_h'\big) \Big)\, dx
\end{aligned}$$

□

Theorem 7.7 (Optimality of the a posteriori error indicator ζ)**.** *Let (θ, w, λ) be the solution of the continuous problem (4.1), and $(\theta_h, w_h, \lambda_h)$ be the solution of the discrete problem (7.1). Then, the following a posteriori optimality estimate holds:*

$$\varepsilon^2\eta \lesssim \|\theta-\theta_h\|_1 + \|w-w_h\|_1 + \|\lambda-\lambda_h\|_{-1} + osc(f) \tag{7.13}$$

where η and $osc(f)$ are defined in the previous section.

Proof. This theorem is demonstrated using the same reasoning as applied in the proof of the indicator optimality for the mixed formulation. □

8. Numerical Tests

In this section, we conduct some numerical examples to illustrate and validate the theoretical results.

8.1. **Example with exact solution.** First of all, note that finding exact solutions for beam contact problems is not an obvious task even for very simple situations (see [4] for instance). We consider the following minimization problem, which consists of finding (θ, w) such that:

$$(\theta, w) := \underset{(\zeta,v)\in\mathcal{K}}{\arg\min} J(\zeta, v) \tag{8.1}$$

where

$$\begin{aligned} J(\zeta,v) :=& \frac{1}{2}\int_0^1 (\zeta')^2\, dx + \frac{\varepsilon^{-2}}{2}\int_0^1 (v'-\zeta)^2\, dx \\ \mathcal{K} :=& \{(\zeta,v)\in V\times W;\ \text{and } v\geq 0\} \\ V :=& \{\zeta\in H^1(0,1),\quad \zeta(0)=\alpha\} \\ W :=& \{v\in H^1(0,1),\quad v(0)=1\} \\ U :=& \{(\zeta,v)\in V\times W; v'(1)-\zeta(1)=0 \text{ and } \zeta'(1)=0\} \end{aligned}$$

we assume $\alpha < -3$.
Let us denote by $\gamma \in (0,1)$ the first point where the graph of w hits the x-axes and let (θ, w) be the minimizer of problem (8.1). Then the free boundary γ is the single real root of the following equation

$$\alpha\gamma^3 + 3\gamma^2 + 6\varepsilon^2 = 0, \quad \alpha < -3, \quad 0\leq t\leq 1 \tag{8.2}$$

Note also that on the set $\{x \mid w(x) > 0\}$,

$$w(x) = \frac{(\alpha\gamma+2)}{\gamma^3+6\varepsilon^2\gamma}x^3 + \left(\frac{6\varepsilon^2}{\gamma^2(\gamma^2+6\varepsilon^2)} - \left(\frac{3+2\alpha\gamma}{\gamma^2+6\varepsilon^2}\right)\right)x^2 + \left(\alpha - 6\varepsilon^2\frac{(\alpha\gamma+2)}{\gamma^3+6\varepsilon^2\gamma}\right)x + 1$$

The regularity of w is $C^{2,1}$, but it does not belong to $C^3[0,1]$.

8.1.1. *Numerical results for $\alpha = -6$ and $\varepsilon = 10^{-4}$ using penalty method.* For $\alpha = -6$ and $\varepsilon = 10^{-4}$ we have the exact solution

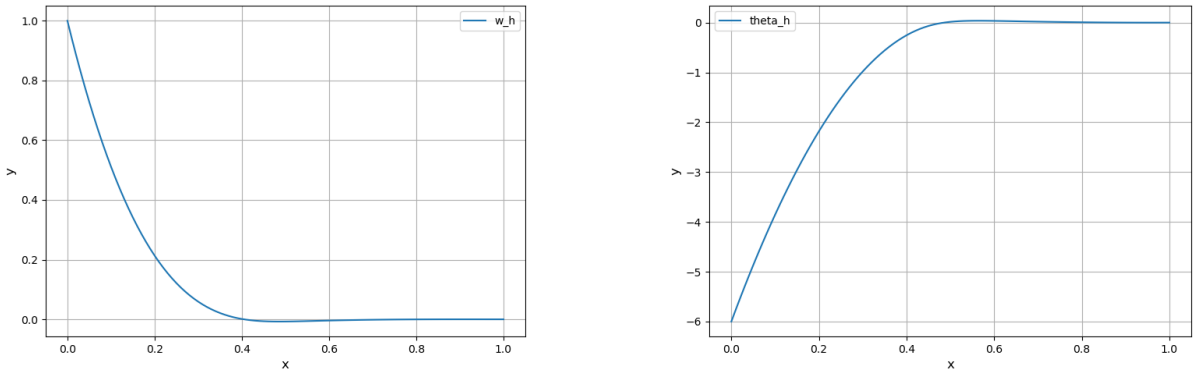
$$w(x) = \begin{cases} ax^3 + bx^2 + cx + 1 & \text{if } 0 < x < \gamma \\ 0 & \text{if } \gamma \leq x \leq 1 \end{cases} \tag{8.3}$$

where,

$$a = \frac{\alpha\gamma+2}{\gamma^3+6\varepsilon^2\gamma}, \quad b = \left(\frac{6\varepsilon^2}{\gamma^2(\gamma^2+6\varepsilon^2)} - \left(\frac{3+2\alpha\gamma}{\gamma^2+6\varepsilon^2}\right)\right), \quad c = \left(\alpha - 6\varepsilon^2\frac{(\alpha\gamma+2)}{\gamma^3+6\varepsilon^2\gamma}\right)$$

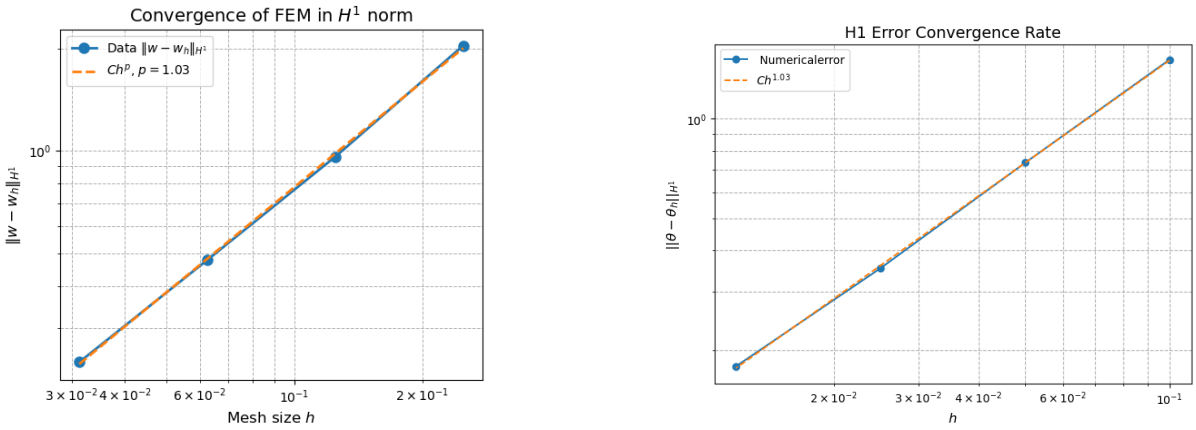
and θ si the solution of :

$$\begin{cases} -\varepsilon^2\theta'' + \theta = w' \text{ in } (0,\gamma) \\ \theta(0) = -6, \quad \theta(\gamma) = 0 \end{cases} \tag{8.4}$$



(A) w_h (B) θ_h

FIGURE 2. P1-P1 FEM with and without PSRI for $\varepsilon = 0.0001$



(A) w-rate of convergence (B) θ-rate of convergence

FIGURE 3. Rate of convergence for P1-P1 FEM with and with PSRI for $t = 0.0001$

In the first numerical experiment we implement a penalty method (see [25]) using $\mathbb{P}^1-\mathbb{P}^1$ Lagrange finite elements with PSRI[2] to compute approximate solution (θ_h, w_h) for very small value of ε. The figure 2.a and the figure 2.b show that the $\mathbb{P}^1 - \mathbb{P}^1$ FEM with PSRI is locking-free for both unknowns w and θ.

The mesh is refined uniformly and the penalty parameters were chosen in terms of the parameter ε and the mesh size h (see [25]), and the errors of the displacement w and the rotation θ in the H^1-norm are computed and tabulated. The resulting convergence curves are visualized in Fig. 3 as a function of the mesh parameter h. The parameter p stands for the rate of convergence $\mathcal{O}(h^p)$.

[2]Partially Selective Reduced Integration

8.1.2. *Numerical test using the Uzawa algorithm.* When the penalty method is used, the Lagrange multiplier λ can not be computed directly from the considered code. The simplest algorithm to do this in a systematic way is the Uzawa algorithm (see [20]). The Uzawa parameters were chosen in terms of the parameter ε and the mesh size h (see [20, Theorem 5.1]).
In this test, we consider a clamped Timoshenko beam $\Omega = (0, 1)$ over a rigid obstacle, defined by $\psi(x) = 4x^3 - 9x^2 + 6x - 1$, and a load $f = 0$. In this test we use $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^0$ Lagrange element the Uzawa parameter is chosen in terms of ε and h as $O(\frac{h}{\sqrt{\varepsilon}})$. Previously, we have shown that the solution to Timoshenko's obstacle problem converges to the corresponding Euler-Bernoulli obstacle problem as ε tends to 0. For this data, the exact solution of the corresponding Euler-Bernoulli obstacle problem is,

$$w(x) = \begin{cases} -4x^3 + 3x^2 & \text{if } x < \dfrac{1}{2} \\ 4x^3 - 9x^2 + 6x - 1 & \text{if } x \geq \dfrac{1}{2} \end{cases}$$

For this data, since the loading is $f = 0$, the Lagrange multiplier converges to zero when ε goes 0. In Table 1 we represent the L^∞ norm of $(w_h'' - \theta_h')$ or $\varepsilon^2\lambda$ as a function of ε.

ε	$\|\varepsilon^2\lambda\|_\infty$
0.01	5.63×10^{-2}
0.001	7.66×10^{-4}
0.0001	9.88×10^{-6}
0.00001	1.27×10^{-7}

TABLE 1. Values of $\|\varepsilon^2\lambda\|_{L^\infty}$ for different ε.

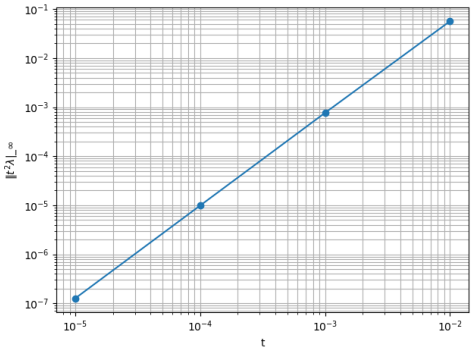
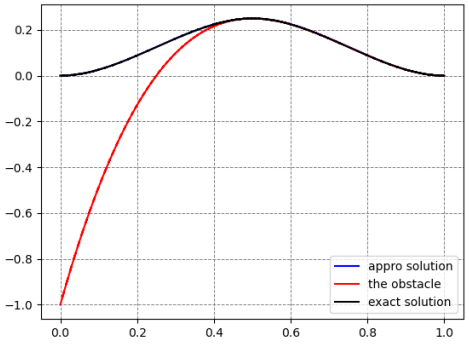


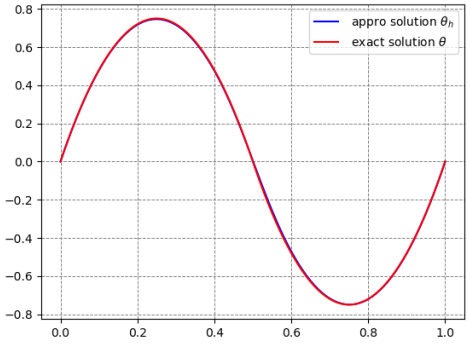
FIGURE 4. $\|w'' - \theta'\|_{L^\infty}$ as a function of ε.

8.2. **Numerical tests using the stabilized formulation.** We will prove numerically that the stabilized method leads to a stable system even with low-order finite elements using the Newton method. We replace the constrained minimization problem with an unconstrained minimization problem by adding a nonlinear term.

Since we use a Newton solver, the method converges with a good initial guess. We choose as initial guess $(\theta_0, w_0) = (w', w)$ where w is the exact solution of Euler-Bernoulli obstacle problem. We use the lowest order conforming finite element spaces, $V_h = V_h^1 \times V_h^1$ and $Q_h = V_h^0$. Recall that this choice leads to an unstable system for the mixed formulation.



(A) Plots for w, w_h and ψ.



(B) Plots for θ and θ_h.

FIGURE 5. Plots for $\mathbb{P}^2 - \mathbb{P}^1 - \mathbb{P}^0$ FEM-Uzawa algorithm.

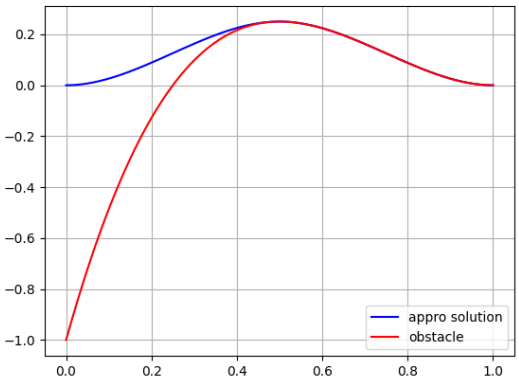


FIGURE 6. Plots for $\mathbb{P}^1 - \mathbb{P}^1 - \mathbb{P}^0$.

Figure 6 shows that, even for very small values of ε, the stabilized method is locking-free. Figure 7 shows excellent convergence properties for the considered test.

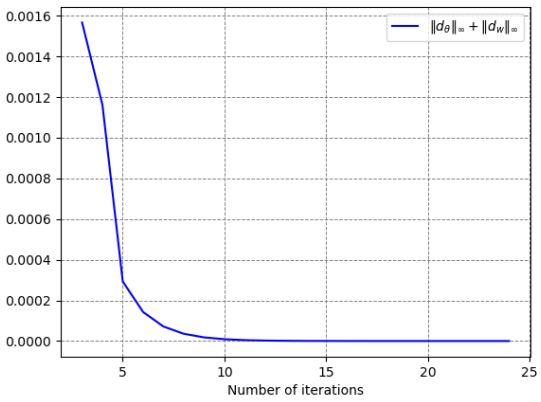


FIGURE 7. The L_∞-norm of d_w and d_θ for the stabilized formulation with $\mathbb{P}^1 - \mathbb{P}^1 - \mathbb{P}^0$

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