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COMPACTNESS OF POINTWISE CONTINUOUS OPERATORS IN GENERAL KÖTHE SEQUENCE SPACES

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Abstract. In this article, we present a comprehensive result characterizing the compactness of pointwise continuous operators acting on Köthe sequence spaces. Our results extend previous work on specific types of operators such as multiplication, difference, tridiagonal, Cesàro, Rhaly, composition and weighted composition operators and offer new insights into the general theory of compact operators on Köthe sequence spaces.

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1. Introduction

An operator $\mathcal{K} : X \to X$ is defined as compact if it is continuous and if the image of the closed unit ball $B_X = \{z \in X : \|z\|_X \leq 1\}$ under $\mathcal{K}$ is a relatively compact subset of X. In other words, $\overline{\mathcal{K}(B_X)}$ is a compact subset of X. The class of compact operators on a Banach space X exhibits many intriguing properties and has numerous applications. This includes finding the solution $z \in X$ for linear expressions of the type $(\lambda\mathcal{K} + I)z = y$. In such cases, $y \in X$ is predefined, λ is a scalar not equal to zero, and $\mathcal{K}$ acts as a compact operator on the space X.
In recent decades there has been substantial research on the compactness of operators defined by infinite matrices. Noteworthy contributions include Arora et al. [2], which investigates the properties of diagonal (multiplication) operators on Lorentz sequence spaces, and the work of Bala et al. [3], who explore multiplication operators on Orlicz-Lorentz sequence spaces. These results show that a multiplication operator $\mathcal{M}_u$, as defined in Section 3, is compact on Orlicz-Lorentz spaces if and only if the symbol $u \in c_0$, which means $u(n) \to 0$ as $n \to \infty$. This condition is likewise used to describe when $\mathcal{M}_u$ is compact on various Banach spaces of sequences, including Cesàro sequence spaces [12] as well as Cesàro–Orlicz spaces [19], among other examples; see [21] and [7] for further discussion and related references.

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Remarkable recent work by El-Shabrawya [11] provides characterizations of the compactness of generalized difference and Rhaly operators, which are defined by infinite lower triangular matrices (see Section 3 for definitions), on a broad class of Banach sequence spaces, including cs, h, and l^1, among others. More recently, Caicedo et al. [4] have characterized the continuity and compactness of tridiagonal operators on weighted l^2 spaces. These last operators are defined by infinite tridiagonal matrices, and they can be viewed as the limiting case of operators defined by finite tridiagonal matrices. This line of research about the properties of tridiagonal operators has been recently extended and generalized by Ramos-Fernández et al. [20] to the more general setting of weighted Orlicz sequences spaces including the cases when this operator apply a weighted l^∞ space into itself or into a weighted Orlicz sequence space and vice versa.
Other operators that have received considerable attention in recent decades include the well-known composition operator and its generalizations: weighted composition operators and differences of weighted composition operators. Recall that if $\varphi : \mathbb{N} \to \mathbb{N}$ is a function, then the composition operator with symbol φ is the transformation that maps a sequence $\mathbf{z} = \{z(j)\}$ to the sequence $\mathbf{y} = \mathcal{C}_\varphi(\mathbf{z}) = \mathbf{z}\circ\varphi$ defined by $y(j) = z(\varphi(j))$ for $j \in \mathbb{N}$. The weighted composition operator with symbols φ and a function $u : \mathbb{N} \to \mathbb{C}$ is defined by $\mathcal{W}_{u,\varphi}(\mathbf{z}) = \mathbf{u} \cdot (\mathbf{z} \circ \varphi)$, and an operator of the form $\mathcal{W}_{u,\varphi} - \mathcal{W}_{v,\psi}$ is known as a difference of weighted composition operators. The compactness of such operators acting on sequence Banach spaces has been widely studied by numerous authors. Notably, the work of Singh and Kumar [23] investigates the compactness of the composition operator on L^2 spaces of functions, while the influential study by Takagi [25] provides an in-depth analysis of the compactness of weighted composition operators acting between L_p function spaces. More recently, the study of the compactness of operators that generalize weighted composition when acting on sequence spaces has gained renewed interest. Particularly noteworthy is the work by Luan and Khoi [18], who characterized the functions for which the difference of two weighted composition operators is compact when acting on weighted l^2 spaces. This latter work inspired Carpintero et al. [6] to characterize the symbols that define compact weighted composition operators between two different weighted l^∞ spaces. Later on, Cardona-Gutiérrez and coauthors [5] broadened the work of Luan and Khoi [18] by considering a more general setting involving differences of two weighted composition operators between distinct weighted ℓ^p spaces.
In particular, in 2022 Carpintero and collaborators [6] established the following noteworthy condition characterizing the compactness of a continuous weighted composition operator acting between two distinct weighted ℓ^∞ spaces:

Theorem 1.1 (Theorem 3.1 in [6]). *Let ω_1 and ω_2 be two weight sequences. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function, $\mathbf{u}$ a complex sequence, and suppose that the operator $\mathcal{W}_{u,\varphi} : l^\infty(\omega_1) \to l^\infty(\omega_2)$ is continuous. The operator $\mathcal{W}_{u,\varphi} : l^\infty(\omega_1) \to l^\infty(\omega_2)$ is compact if and only if for all bounded-norm sequence $(\mathbf{z}_n) \subset l^\infty(\omega_1)$ such that $\mathbf{z}_n \to \mathbf{0}$ pointwise, we have*

$$\lim_{n\to\infty} \|\mathcal{W}_{u,\varphi}(\mathbf{z}_n)\|_{\infty,\omega_2} = 0,$$

where $\mathbf{z} \in l^\infty(\omega_1)$ *if and only if* $\|\mathbf{z}\|_{\infty,\omega_1} = \sup_{k\in\mathbb{N}} |z(k)|\,\omega_1(k) < \infty$.

Latter, in 2023 Caicedo et al. [4, Lemma 3.1] proved that a similar statement holds for continuous tridiagonal operators acting on weighted l^2 space and more recently, Cardona-Gutierrez et al. [5, Theorem 3.1] observed that the same phenomenon occurs for certain pointwise continuous operator acting between two weighted l^p spaces. This naturally leads to the question:

Is Theorem 1.1 valid for some other class of continuous operators K acting on Köthe sequence spaces?

We also note that analogous results hold for operators on Banach spaces of holomorphic functions, as shown in the Ph.D. thesis of M. Tjani [26].
In this article, we show that Theorem 1.1 also holds for certain pointwise continuous operators acting on Köthe sequence spaces with the Fatou property. To establish this result, Section 2 revisits the definition of Köthe sequence spaces and summarizes some of their key properties. Section 3 introduces the concept of pointwise continuous operators and provides examples illustrating that this concept is independent of the standard notion of continuity for operators on Banach spaces. We then state and prove the main result (Theorem 4.1) in Section 4, followed by several applications in Section 5. In particular, we demonstrate that many existing theorems concerning the compactness of multiplication, composition, weighted composition, Rhaly, generalized difference, and tridiagonal operators can be derived as consequences of our result.

2. Some remarks about Köthe sequence spaces

Throughout this work, we let $\mathbf{c} = \{c(k)\}$ represent a complex-valued sequence, whereas $(\mathbf{x}_n)$ refers to a sequence composed of sequences. Given these conventions, a Banach space of sequences $(X, \|\cdot\|_X)$ is categorized as a Köthe sequence space provided the following criteria are met:

(i) For any two sequences $\mathbf{c}$ and $\mathbf{d}$, if $\mathbf{d}$ belongs to X and $|c(k)| \le |d(k)|$ holds for every $k \in \mathbb{N}$, then $\mathbf{c}$ must also be an element of X such that $\|\mathbf{c}\|_X \le \|\mathbf{d}\|_X$.
(ii) The space X contains at least one sequence $\mathbf{y}$ where $y(m) > 0$ for all $m \in \mathbb{N}$.

Note that requirement (ii) guarantees that X incorporates the standard basis vectors $(\mathbf{e}_n)$, defined by $\mathbf{e}_n = \mathbf{1}_{\{n\}}$. Here, $\mathbf{e}_n$ is the sequence whose n-th entry is 1 and all other entries $e_n(k)$ are 0 for $k \neq n$. Furthermore, for any fixed $m \in \mathbb{N}$ and any $\mathbf{c} \in X$, the inequality $|c(m)e_m(k)| \le |c(k)|$ is satisfied for all $k \in \mathbb{N}$. Consequently, it follows that

$$|c(m)| \le \frac{1}{\|\mathbf{e}_m\|_X}\|\mathbf{c}\|_X$$

which demonstrates that the coordinate evaluation functional at m is a bounded operator on X. This last fact will be used several times in the proof of our main result. In addition, for any finite subset $B = \{n_1, n_2, \ldots, n_m\} \subset \mathbb{N}$, the characteristic sequence $\mathbf{1}_B$, given by

$$\mathbf{1}_B(k) = \begin{cases} 1, & k \in B, \\ 0, & k \notin B, \end{cases}$$

is contained in X. It should also be noted that the equality of two sequences $\mathbf{c}$ and $\mathbf{d}$ is defined by $c(k) = d(k)$ for every $k \in \mathbb{N}$. We denote the zero sequence as $\mathbf{0} = (0, 0, \dots)$. Prominent instances of Köthe sequence spaces include the l^p spaces for $1 \leq p \leq \infty$, the space c_0 of sequences that vanish at infinity, and the Lorentz sequence spaces $l_{(p,q)}$ where $1 < p \leq \infty$ and $1 \leq q \leq \infty$. The foundational development of Köthe sequence spaces can be traced back to the research of Köthe and Toeplitz [13], with subsequent contributions by Köthe [14, 15]. This framework was further extended through the work of Dieudonné [10], Cooper [9], and Lorentz and Wertheim [17].
We conclude this section by recalling that a Köthe sequence space X is said to satisfy the Fatou property whenever a norm-bounded sequence $(\mathbf{z}_n) \subset X$ fulfills $0 \leq \mathbf{z}_n \uparrow \mathbf{z}$ for some numerical sequence $\mathbf{z}$. In this case, $\mathbf{z}$ belongs to X and $\|\mathbf{z}_n\|_X$ converges to $\|\mathbf{z}\|_X$. It is well known that X enjoys the Fatou property if and only if, for every sequence $(\mathbf{z}_n) \subset X$ with $\liminf_{n\to\infty} \|\mathbf{z}_n\|_X < \infty$, one has $\liminf_{n\to\infty} |\mathbf{z}_n| \in X$ and satisfies

$$\left\| \liminf_{n\to\infty} |\mathbf{z}_n| \right\|_X \leq \liminf_{n\to\infty} \|\mathbf{z}_n\|_X .$$

We recommend the excellent book of Zaanen [27] for the study of Köthe spaces.

3. Pointwise continuous operators

In this section, we define the class of operators for which our results hold. We also provide examples that demonstrate the independence of the notions of continuous operators and pointwise continuous operators.

Definition 3.1. Let $(X, \|\cdot\|_X)$ be a Köthe sequence spaces. A linear operator $\mathcal{T}$ defined on X is said *pointwise continuous* if for all sequence $(\mathbf{z}_n) \subset X$ such that $\mathbf{z}_n \to \mathbf{0}$ pointwise, the sequence $(\mathbf{y}_n) = (\mathcal{T}\,\mathbf{z}_n)$ converges pointwise to zero.

We recall that a sequence $(\mathbf{z}_n) \subset X$ converges pointwise to zero if for each $m \in \mathbb{N}$ fixed the numerical sequence $\{z_n(m)\}$ goes to zero as $n \to \infty$. The above definition was inspired by certain operators defined by infinite matrices. For instance, for a fixed sequence $\alpha = \{\alpha(k)\}$ the Rhaly operator acting a Banach sequence space X is defined by

$$\mathcal{R}_\alpha\,\mathbf{z} = \begin{pmatrix} \alpha(1) & 0 & 0 & 0 & \cdots \\ \alpha(2) & \alpha(2) & 0 & 0 & \cdots \\ \alpha(3) & \alpha(3) & \alpha(3) & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \times \begin{pmatrix} z(1) \\ z(2) \\ z(3) \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha(1)z(1) \\ \alpha(2)z(1) + \alpha(2)z(2) \\ \vdots \end{pmatrix}. \tag{3.1}$$

This operator is a generalization of the well known Cesàro operator which is obtained when $\alpha(k) = \frac{1}{k}$ for all $k \in \mathbb{N}$ and this topic has attracted considerable attention from many authors; see, for instance, [16, 1, 11] and the references cited therein. Thus if $\mathbf{y} = \{y(k)\} = \mathcal{R}_\alpha \mathbf{z}$, then for any $k \in \mathbb{N}$ we have

$$y(k) = \alpha(k) \sum_{j=1}^{k} z(j)$$

and hence this operator is pointwise continuous independently if it is or not a continuous operator. Another important example of pointwise continuous operators is obtained by infinite tridiagonal matrices: given three sequences α, β and γ, the tridiagonal operator $\mathcal{T}_{\alpha,\beta,\gamma}$ defined over the Banach sequence space X is defined by

$$\begin{aligned} \mathcal{T}_{\alpha,\beta,\gamma}\, \mathbf{z} &= \begin{pmatrix} \alpha(1) & \beta(1) & 0 & 0 & \cdots \\ \gamma(1) & \alpha(2) & \beta(2) & 0 & \cdots \\ 0 & \gamma(2) & \alpha(3) & \beta(3) & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \times \begin{pmatrix} z(1) \\ z(2) \\ z(3) \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} \alpha(1)z(1) + \beta(1)z(2) \\ \gamma(1)z(1) + \alpha(2)z(2) + \beta(2)z(3) \\ \gamma(2)z(2) + \alpha(3)z(3) + \beta(3)z(4) \\ \vdots \end{pmatrix}. \end{aligned}$$

The continuity and compactness of tridiagonal operators acting on weighted l^2 spaces has been recently studied by Caicedo et al. in [4]. Clearly this operator is a generalization of the difference operator $\Delta_{\alpha,\beta}$ which is obtained when γ is the null sequence (see [11]) and the well known diagonal operator (or multiplication operator) which is obtained when $\beta = \gamma = \mathbf{0}$, the null sequence. This last operator has attracted significant attention in the literature and has been extensively studied by many authors; moreover, it continues to be regarded as a topic of considerable importance (see [8, 21, 22] and the references therein).

However, there are continuous operators defined by infinite matrices which are not pointwise continuous. For instance, the upper triangular matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

define a continuous operator from $c_S = \{\mathbf{z} : \mathcal{S}\mathbf{z} \in c\}$, into l^∞, where $\mathcal{S}\mathbf{z} = \mathbf{y} = \{y(j)\}$ is given by

$$y(j) = \sum_{k=1}^{j} x(k),$$

c is the space of all convergent sequences equipped with the norm of the supremum and l^∞ is the space of all bounded sequences (see [24]). The operator defined by the matrix A is not pointwise continuous since the sequence $(\mathbf{e}_n)$ converge pointwise to zero but $\mathbf{y}_n = A \times \mathbf{e}_n$ is the sequence

$$y_n(j) = \begin{cases} 1, & j \leq n, \\ 0, & j > n \end{cases}$$

and hence, for each $j \in \mathbb{N}$, we can see that $\lim_{n\to\infty} y_n(j) = 1$.

4. Compactness of bounded pointwise continuous operators

The aim of this section is to present a new criterion for the compactness of bounded and pointwise continuous operators acting on Köthe sequence spaces. More specifically, we will show that the argument used by Carpintero et al. (2022) in the proof of [6, Theorem 3.1] can be adapted to a more general setting. The result is stated as follows:

Theorem 4.1. *Let $(X, \|\cdot\|_X)$ be a Köthe space of sequences which have the Fatou property and suppose that $\mathcal{K} : X \to X$ is a bounded and pointwise continuous operator. $\mathcal{K}$ is a compact operator if and only if for all bounded sequences $(\mathbf{z}_n) \subset X$ such that $\mathbf{z}_n \to 0$ pointwise, we have*

$$\lim_{n\to\infty} \|\mathcal{K}\,\mathbf{z}_n\|_X = 0. \tag{4.1}$$

Proof. Suppose first that $\mathcal{K} : X \to X$ is a compact and pointwise continuous operator. Let $(\mathbf{z}_n)$ be any bounded sequence in X such that $\mathbf{z}_n \to 0$ pointwise. We are going to show that the relation (4.1) holds. By the way of contradiction, we shall assume that the relation (4.1) is false, then we can find an $\varepsilon_0 > 0$ and a subsequence $\left(\mathbf{z}_{n_j}\right)$ of $(\mathbf{z}_n)$ such that

$$\left\|\mathcal{K}\,\mathbf{z}_{n_j}\right\|_X \geq \varepsilon_0 \tag{4.2}$$

for all $j \in \mathbb{N}$. Next, observe that the sequence $\left(\mathbf{z}_{n_j}\right)$ remains bounded in X. Since the operator $\mathcal{K} : X \to X$ is compact, we may extract a further subsequence, if necessary, such that $\left(\mathcal{K}\mathbf{z}_{n_j}\right)$ converges. Consequently, there exists an element $\mathbf{y} \in X$ satisfying

$$\lim_{j\to\infty} \left\|\mathcal{K}\,\mathbf{z}_{n_j} - \mathbf{y}\right\|_X = 0.$$

We will show that $\mathbf{y}$ coincides with the zero sequence; namely, $y(k) = 0$ for every $k \in \mathbb{N}$. Indeed, for $k \in \mathbb{N}$ fixed, we have

$$\left|\mathcal{K}\,z_{n_j}(k) - y(k)\right| |e_k(j)| \leq \left|\mathcal{K}\,z_{n_j}(j) - y(j)\right|$$

for all $j \in \mathbb{N}$. Thus, by definition of Köthe spaces, then we find

$$\left|\mathcal{K}\,z_{n_j}(k) - y(k)\right| \|\mathbf{e}_k\|_X \leq \left\|\mathcal{K}\,\mathbf{z}_{n_j} - \mathbf{y}\right\|_X$$

and hence

$$|y(k)| = \lim_{j\to\infty} \left|\mathcal{K}\,z_{n_j}(k) - y(k)\right| \leq \lim_{j\to\infty} \frac{1}{\|\mathbf{e}_k\|_X} \left\|\mathcal{K}\,\mathbf{z}_{n_j} - \mathbf{y}\right\|_X = 0,$$

where we have used that $\mathcal{K} : X \to X$ is a pointwise continuous operator and $\mathbf{z}_n \to 0$ pointwise. Thus, we have arrived to the expression

$$\lim_{j\to\infty} \left\|\mathcal{K}\,\mathbf{z}_{n_j}\right\|_X = 0.$$

which contradicts the inequality (4.2). This proves the implication.
Conversely, we shall suppose that for all bounded sequences $(\mathbf{z}_n) \subset X$ such that $\mathbf{z}_n \to 0$ pointwise we have

$$\lim_{n\to\infty} \|\mathcal{K}\,\mathbf{z}_n\|_X = 0.$$

We are going to show that the operator $\mathcal{K} : X \to X$ is compact. With this end, let $(\mathbf{x}_j)$ be a sequence in X such that $\|\mathbf{x}_j\|_X \le 1$ for all $j \in \mathbb{N}$. The sequence of the first components $\{x_j(1)\}$ satisfies

$$|x_j(1)|\,|e_1(k)| \le |x_j(k)|$$

for all $k \in \mathbb{N}$ and hence

$$|x_j(1)| \le \frac{1}{\|\mathbf{e}_1\|_X}\,\|\mathbf{x}_j\|_X \le \frac{1}{\|\mathbf{e}_1\|_X}$$

which tell us that $\{x_j(1)\}$ is a bounded numerical sequence. By the Bolzano–Weierstrass theorem, the sequence $\{x_j(1)\}$ has a convergent subsequence $\{x_j^{(1)}(1)\}$; that is, there exists a complex number $x(1)$ such that

$$\lim_{j\to\infty} \left| x_j^{(1)}(1) - x(1) \right| = 0.$$

Next, we see the sequence of the second components of $\left(\mathbf{x}_j^{(1)}\right)$. As before, the sequence $\left\{x_j^{(1)}(2)\right\}$ has a convergent subsequence $\left\{x_j^{(2)}(2)\right\}$ and there exists $x(2) \in \mathbb{C}$ such that

$$\lim_{j\to\infty} \left| x_j^{(2)}(2) - x(2) \right| = 0.$$

Observe that also we have

$$\lim_{j\to\infty} \left| x_j^{(2)}(1) - x(1) \right| = 0$$

since $\left(\mathbf{x}_j^{(2)}\right)$ is a subsequence of $\left(\mathbf{x}_j^{(1)}\right)$. Repeating this process we can build a subsequence $(\mathbf{x}_{j_k})$ of $(\mathbf{x}_n)$ and a sequence $\mathbf{x} = (x(1), x(2), \cdots)$ of complex numbers such that $\mathbf{x}_{j_k} \to \mathbf{x}$ pointwise. Thus, since X has the Fatou property we obtain $\mathbf{x} \in X$, in fact,

$$\|\mathbf{x}\|_X = \left\| \liminf_{k\to\infty} |\mathbf{x}_{j_k}| \right\|_X \le \liminf_{k\to\infty} \|\mathbf{x}_{j_k}\|_X \le 1.$$

Finally, since the sequence $(\mathbf{x}_{j_k} - \mathbf{x})$ converges pointwise to zero, the hypothesis implies that

$$\lim_{k\to\infty} \|\mathcal{K}\,\mathbf{x}_{j_k} - \mathcal{K}\,\mathbf{x}\|_X = 0$$

and the operator $\mathcal{K} : X \to X$ is compact. $\square$

5. Some applications

Our Theorem 4.1 provides a genuine extension and generalization of Theorem 3.1 in [6], Lemma 3.1 in [4], and Theorem 2 in [5], among others. Below, we list some results that can be derived as special cases of our Theorem 4.1:

(1) Theorem 2.4 in [2]: Suppose that $\mathbf{u} \in l^\infty$.

The multiplication operator $\mathcal{M}_u : l_{(p,q)} \to l_{(p,q)}$ is compact if and only if $\mathbf{u} \in c_0$.

The aforementioned criterion further defines the compactness of the operator M_u when it operates within various other Köthe sequence spaces. This includes, for instance, Cesàro sequence spaces [12], Orlicz-Lorentz sequence spaces [3] and Cesàro-Orlicz sequence spaces [19], to name a few. Nevertheless, the definitive characterization of compactness for multiplication operators across the entire class of Köthe sequence spaces was established by Ramos-Fernández and Salas-Brown [21].

(2) Theorem 3.1 in [6]: See the formulation in Theorem 1.1.

(3) Theorem 3 in [5]: Let ω_1 and ω_2 be two weight sequences and let $p > 1$ be fixed. Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are complex sequences, $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ are functions and $\mathcal{W}_{u,\varphi}, \mathcal{W}_{v,\psi} : l^p(\omega_1) \to l^p(\omega_2)$ are continuous operators. The difference $\mathcal{W}_{u,\varphi} - \mathcal{W}_{v,\psi}$ from $l^p(\omega_1)$ into $l^p(\omega_2)$ is compact if and only if

$$\lim_{k\to\infty} \frac{\|(\mathcal{W}_{u,\varphi} - \mathcal{W}_{v,\psi})(\boldsymbol{e}_k)\|_{l^p(\omega_2)}}{\|\boldsymbol{e}_k\|_{l^p(\omega_1)}} = 0. \tag{5.1}$$

Here, a weight $\omega = \{\omega(k)\}$ is a real sequence such that $\omega(k) > 0$ for all $k \in \mathbb{N}$ and $\mathbf{z} \in l^p(\omega)$ if and only $\|\mathbf{z}\|^p_{l^p(\omega)} = \sum_{k=1}^{\infty} |z(k)|^p \omega^p(k) < \infty$.

(4) Theorem 3.2 in [4]: Let ω be a weight sequence and β, γ weights defined by $\beta(k) = \frac{\omega(k)}{\omega(k+1)}$ and $\gamma(k) = \frac{\omega(k+1)}{\omega(k)}$ with $k \in \mathbb{N}$. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be sequences such that $\mathcal{T}_{u,v,w} : l^2(\omega) \to l^2(\omega)$ is continuous.

The tridiagonal operator $\mathcal{T}_{u,v,w}$ is compact on $l^2(\omega)$ if and only if $\boldsymbol{u} \in c_0$, $\boldsymbol{v} \in c_0(\beta)$ and $\boldsymbol{w} \in c_0(\gamma)$.

We illustrate the application of Theorem 4.1 with two simple examples. The first provides a proof of a version of the first item, and the second demonstrates the compactness of the Rhaly operator acting on the l^1 space:

Example 5.1 (Compactness of multiplication operator)**.** Let $\mathbf{u} \in \ell^\infty$, and consider a Köthe sequence space $(X, \|\cdot\|_X)$ that satisfies the Fatou property. The multiplication operator $\mathcal{M}_u : X \to X$ is compact if and only if

$$\lim_{k\to\infty} |u(k)| = 0. \tag{5.2}$$

Proof. We have that for $\mathbf{u} \in l^\infty$, the multiplication operator $\mathcal{M}_u : X \to X$ is pointwise continuous. Thus, if we suppose that $\mathcal{M}_u : X \to X$ is a compact operator, then our Theorem 4.1 implies that

$$\lim_{n\to\infty} \|\mathcal{M}_u \mathbf{z}_n\|_X = 0,$$

for every sequence $(\mathbf{z}_n)$ in X that is norm-bounded and satisfies $\mathbf{z}_n \to 0$ pointwise. In particular, the sequence

$$\mathbf{z}_n = \frac{1}{\|\mathbf{e}_n\|_X}\mathbf{e}_n$$

remains bounded in X and converges pointwise to the zero sequence $\mathbf{0}$ and therefore

$$0 = \lim_{n\to\infty}\left\|\mathcal{M}_u\left(\frac{\mathbf{e}_n}{\|\mathbf{e}_n\|_X}\right)\right\|_X = \lim_{n\to\infty}\frac{1}{\|\mathbf{e}_n\|_X}\|\mathbf{u}\cdot\mathbf{e}_n\|_X = \lim_{n\to\infty}|u(n)|$$

and the relation (5.2) has been established.
Conversely, suppose that the relation (5.2) holds. We shall prove, using our Theorem 4.1, that the operator $\mathcal{M}_u : X \to X$ is compact. To proceed, let $(\boldsymbol{\xi}_n)$ be an arbitrary norm-bounded sequence in X that converges pointwise to zero. Our aim is to show that $\|\mathcal{M}_u\,\boldsymbol{\xi}_n\|_X \to 0$ as $n \to \infty$. First, note that there exists a constant $M > 0$ such that

$$\|\boldsymbol{\xi}_n\|_X \le M \quad \text{for all } n.$$

Fix $H \in \mathbb{N}$, and define the sequences

$$\begin{aligned}\boldsymbol{\eta}_n^H &= \big(u(1)\xi_n(1), u(2)\xi_n(2), \dots, u(H)\xi_n(H), 0, 0, \dots\big),\\ \boldsymbol{\zeta}_n^H &= \big(0, \dots, 0, u(H+1)\xi_n(H+1), u(H+2)\xi_n(H+2), \dots\big).\end{aligned}$$

Both $\boldsymbol{\eta}_n^H$ and $\boldsymbol{\zeta}_n^H$ belong to X, since for each $m \in \mathbb{N}$,

$$|\eta_n^H(m)|,\ |\zeta_n^H(m)| \le |u(m)\xi_n(m)|.$$

Also

$$\lim_{n\to\infty}\left\|\boldsymbol{\eta}_n^H\right\|_X \le \lim_{n\to\infty}\sum_{m=1}^{H}|u(m)|\,|\xi_n(m)|\,\|\mathbf{e}_m\|_X = 0 \tag{5.3}$$

since $\boldsymbol{\xi}_n \to 0$ pointwise. Also we have

$$\left|\zeta_n^H(m)\right| \le \sup_{k\ge H+1}\{|u(k)|\}\,|\xi_n(m)|$$

for all $m \in \mathbb{N}$. Hence, we can write

$$\left\|\boldsymbol{\zeta}_n^H\right\|_X \le \sup_{k\ge H+1}\{|u(k)|\}\,\|\boldsymbol{\xi}_n\|_X \le M\sup_{k\ge H+1}\{|u(k)|\}$$

and we obtain that

$$\limsup_{n\to\infty}\left\|\boldsymbol{\zeta}_n^H\right\|_X \le M\sup_{k\ge H+1}\{|u(k)|\}\,. \tag{5.4}$$

Finally from (5.3) and (5.4) we conclude that

$$\limsup_{n\to\infty}\|\mathbf{u}\cdot\boldsymbol{\xi}_n\|_X \le \limsup_{n\to\infty}\left(\left\|\boldsymbol{\eta}_n^H\right\|_X + \left\|\boldsymbol{\zeta}_n^H\right\|_X\right) \le M\sup_{k\ge H+1}\{|u(k)|\}$$

and the result follows from Theorem 4.1 since $\mathbf{u} \in c_0$. □

Example 5.2 (Theorem 3.3 in [11])**.** Let $\alpha = \{\alpha(k)\}$ be a fixed sequence. The Rhaly operator $\mathcal{R}_\alpha$ defined in (3.1) is compact on l^1 if and only if $\alpha \in l^1$.

Proof. It is enough to prove that if $\alpha \in l^1$ then $\mathcal{R}_\alpha : l^1 \to l^1$ is a compact operator. In order to use our Theorem 4.1, we consider an arbitrary sequence $(\mathbf{z}_n)$ in l^1 that is bounded with respect to the norm and converges pointwise to zero. We are going to show that $\|\mathcal{R}_\alpha \mathbf{z}_n\|_{l^1} \to 0$ as $n \to \infty$. Arguing as in the proof of Example 5.1, there exists $M > 0$ such that $\|\mathbf{z}_n\|_{l^1} \leq M$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we consider $\mathbf{y}_n = \mathcal{R}_\alpha \mathbf{z}_n$. Let $\varepsilon > 0$ be arbitrary. Since $\alpha \in l^1$, we can find $H \in \mathbb{N}$ such that $M \sum\limits_{k=H+1}^{\infty} |\alpha(k)| < \varepsilon$. Thus, we can write

$$\begin{aligned}
\|\mathbf{y}_n\|_{l^1} &= \sum_{k=1}^{H} \left| \alpha(k) \sum_{j=1}^{k} z_n(j) \right| + \sum_{k=H+1}^{\infty} \left| \alpha(k) \sum_{j=1}^{k} z_n(j) \right| \\
&\leq \sum_{k=1}^{H} \left| \alpha(k) \sum_{j=1}^{k} z_n(j) \right| + M \sum_{k=H+1}^{\infty} |\alpha(k)| < \sum_{k=1}^{H} \left| \alpha(k) \sum_{j=1}^{k} z_n(j) \right| + \varepsilon
\end{aligned}$$

and since $\mathbf{z}_n \to 0$ pointwise, we conclude that $\lim\limits_{n\to\infty} \|\mathbf{y}_n\|_{l^1} \leq \varepsilon$. This proves the result. □

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