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SUBCLASS OF MEROMORPHIC FUNCTIONS VIA KUMMER HURWITZ LERCH ZETA OPERATORS ON HILBERT SPACE

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Abstract. In this paper, we introduce the Kummer–Hurwitz–Lerch Zeta convolution operator to define a new subclass of meromorphic functions on Hilbert spaces. Using this operator, we construct functions that extend classical meromorphic function theory and derive sharp geometric bounds. We establish coefficient estimates, investigate extreme points, and examine closure properties under convex combinations. Additionally, the radii of meromorphic starlikeness, convexity, and close-to-convexity are determined. The Hadamard product of functions in this subclass is studied, and integral operators are explored to extend the geometric and analytic properties of the class.

Keywords. Meromorphic functions, Hilbert spaces, Coefficient bounds, Starlikeness, Convexity, Hadamard product, Integral operators
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1. Introduction

Let Σ denote the family of univalent meromorphic functions of the form

$$f(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} a_n \zeta^n, \tag{1.1}$$

where $\zeta \in \mathbb{U}^* = \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$ and $a_n \geq 0$.

Definition 1.1. For $f_j(\zeta) \in \Sigma$ $(j = 1, 2)$, defined by

$$f_j(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} a_{n,j} \zeta^n, \tag{1.2}$$

the Hadamard product is given as

$$(f_1 * f_2)(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} \zeta^n = (f_2 * f_1)(\zeta). \tag{1.3}$$

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Definition 1.2. The subclasses of meromorphic starlike and meromorphic convex functions, denoted respectively by Σ^* and Σ_c, are characterized by the conditions

$$-\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > 0, \qquad -\Re\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} > 0.$$

More generally, the families $\Sigma^*(\delta)$ and $\Sigma_c(\delta)$ $(0 \le \delta < 1)$ consist of functions satisfying

$$-\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > \delta, \tag{1.4}$$

and

$$-\Re\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} > \delta. \tag{1.5}$$

In this work, we define the operator $\mathcal{L}_x^{\delta}(\gamma, \beth, \varrho)$ to generate a new subclass of meromorphic functions and obtain associated geometric bounds. Inspired by Lagad et al. [17], we consider the function

$$\overline{K}(\gamma; \beth, \zeta) = \frac{1}{\zeta} + \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\beth)_{n+1}} \frac{\zeta^n}{(n+1)!}, \qquad (\gamma \in \mathbb{C},\ \beth \in \mathbb{C} \setminus \{0, -1, \dots\},\ \zeta \in E^*). \tag{1.6}$$

Combining (1.6) with the Hadamard product (1.3) and using convolution, we define a linear convolution operator for $f(\zeta) \in \Sigma$ as

$$\mathcal{L}_x^{\delta}(\gamma, \beth, \varrho)\, f(\zeta) = \overline{K}(\gamma; \beth, \zeta) * A_{x,\delta}(\zeta) * f(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \Phi_n(\delta, \varrho, \gamma, x, \beth)\, a_n \zeta^n, \tag{1.7}$$

with

$$\Phi_n(\delta, \varrho, \gamma, x, \beth) = \frac{(\gamma)_{n+1}}{(\beth)_{n+1}} \frac{(\varrho)_{n+1}}{(n+1)!\,(n+1)!} \left(\frac{\delta+1}{\delta+n+1}\right)^x. \tag{1.8}$$

For further results on meromorphic functions linked with Hurwitz–Lerch zeta functions, see [1, 2].

Recent studies have introduced new subclasses of bi-univalent functions using various operators and polynomials, including the Wright function with the Jung-Kim-Srivastav operator [11], the symmetric q-derivative [12], and Fekete–Szegö inequalities with Cauchy and Bernoulli polynomials [4].

Let H be a complex Hilbert space and $L(H)$ the algebra of bounded linear operators on H. For a complex-valued function f analytic in a domain E containing the spectrum $\sigma(\mathfrak{X})$ of a bounded operator $\mathfrak{X}$, the operator $f(\mathfrak{X})$ is defined via the Riesz-Dunford integral [7]:

$$f(\mathfrak{X}) = \frac{1}{2\pi i} \int_C (zI - \mathfrak{X})^{-1} f(z)\, dz,$$

where I is the identity on H and C is a positively oriented simple closed contour enclosing $\sigma(\mathfrak{X})$ [7]. Equivalently, $f(\mathfrak{X})$ can be expressed as the series

$$f(\mathfrak{X}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, \mathfrak{X}^n,$$

convergent in the operator norm.
In this study, we focus on a subclass of Σ_p connected with the integral operator $\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\zeta)$.

Definition 1.3. Let $0 \leq \varsigma < 1$ and $0 \leq \eta < 1$. A function $f \in \Sigma_p$, given by (1.1), belongs to the class $\mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$ if

$$\frac{\mathfrak{X}\,(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))'}{(\varsigma-1)\,\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}) + \varsigma\, A\,(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))'} > \eta, \tag{1.9}$$

where the operator $\mathcal{L}_x^\delta(\gamma, \beth, \varrho)$ is defined in (1.7).

Remark 1.4. Kavitha et al. [16] introduced a generalized subclass $M_p(\delta, \alpha)$: For $0 \leq \delta < 1$ and $0 \leq \alpha < 1$, a function $f \in \Sigma_p$ from (1.1) belongs to $M_p(\delta, \alpha)$ if and only if

$$\frac{\zeta f'(\zeta)}{(\alpha-1) f(\zeta) + \alpha \zeta f'(\zeta)} > \delta. \tag{1.10}$$

Lemma 1.5. *Let $w = u + iv$. Then $\Re(w) > \delta$ if and only if $|w-1| < |w+1-2\delta|$. Using this lemma, an equivalent form of Definition (1.9) can be obtained.*

Definition 1.6. Let $0 \leq \varsigma < 1$ and $0 \leq \eta < 1$. A function $f \in \Sigma_p$, given by (1.1), belongs to the class $\mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$ if

$$\begin{aligned} &\Big|\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))' - \big[(\varsigma-1)\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}) + \varsigma\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))'\big]\Big| \\ < \; &\Big|\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))' + (1-2\eta)\big[(\varsigma-1)\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}) + \varsigma\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))'\big]\Big| \end{aligned} \tag{1.11}$$

for all operators $\mathfrak{X}$ satisfying $|\mathfrak{X}| < 1$ and $\mathfrak{X} \neq \Pi$, where Π is the zero operator on H.

In this work, we derive bounds on the coefficients and determine the radii of starlikeness and convexity for functions belonging to the class $\mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$.

2. Coefficient Bounds

Theorem 2.1. *Let $f \in \Sigma_p$ be given by (1.1). Then f belongs to the class $\mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$ for all proper contractions $\mathfrak{X} \neq \Pi$ if and only if*

$$\sum_{n=1}^{\infty} \big[n + \eta - \eta\varsigma(n+1)\big]\, \Phi_n(\delta, \varrho, \gamma, x, \beth) a_n \leq 1 - \eta. \tag{2.1}$$

The bound is sharp for

$$f(\zeta) = \frac{1}{\zeta} + \frac{1-\eta}{[n + \eta - \eta\varsigma(n+1)]\Phi_n(\delta, \varrho, \gamma, x, \beth)}\zeta^n, \quad n \geq 1. \tag{2.2}$$

Proof. Suppose that (1.9) holds. Then

$$\begin{aligned} &\Big|\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))' - \big[(\varsigma-1)\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}) + \varsigma\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))'\big]\Big| \\ -&\Big|\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))' + (1-2\eta)\big[(\varsigma-1)\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}) + \varsigma\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \beth, \varrho)\, f(\mathfrak{X}))'\big]\Big| \end{aligned}$$

$$= \left| \sum_{n=1}^{\infty} (n+1)(1-\varsigma)\, \Phi_n(\delta, \varrho, \gamma, x, \daleth)\, a_n\, \mathfrak{X}^n \right|$$
$$- \left| 2(1-\eta)\, \mathfrak{X}^{-1} - \sum_{n=1}^{\infty} [\, n + (1-2\eta)(\varsigma - 1 + \varsigma n)\,]\, \Phi_n(\delta, \varrho, \gamma, x, \daleth)\, a_n\, \mathfrak{X}^n \right|$$

$$\le \sum_{n=1}^{\infty} (n+1)(1-\varsigma)\Phi_n(\delta, \varrho, \gamma, x, \daleth) a_n |\mathfrak{X}|^n - 2(1-\eta)|\mathfrak{X}^{-1}|$$
$$+ \sum_{n=1}^{\infty} [n + (1-2\eta)(\varsigma - 1 + \varsigma n)]\Phi_n(\delta, \varrho, \gamma, x, \daleth) a_n |\mathfrak{X}|^n$$

$$= 2 \sum_{n=1}^{\infty} [n + \eta - \eta\varsigma(n+1)]\Phi_n(\delta, \varrho, \gamma, x, \daleth) a_n |\mathfrak{X}|^n - 2(1-\eta)|\mathfrak{X}^{-1}|$$

Choosing $\mathfrak{X} = rI$, $0 < r < 1$, and letting $r \to 1^-$, we obtain (2.1). Conversely, if (2.1) holds, a similar computation shows that $f \in \mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$. □

Remark 2.2. For $\varsigma = 0$, the inequality reduces to

$$\frac{-\mathfrak{X}(\mathcal{L}_x^\delta(\gamma, \daleth, \varrho)\, f(\mathfrak{X}))'}{\mathcal{L}_x^\delta(\gamma, \daleth, \varrho)\, f(\mathfrak{X})} > \eta, \tag{2.3}$$

which implies that $\mathcal{L}_x^\delta(\gamma, \daleth, \varrho)\, f(\mathfrak{X})$ belongs to $\Sigma_p^*(\eta)$ if and only if

$$\sum_{n=1}^{\infty} (n+\eta)\, \Phi_n(\delta, \varrho, \gamma, x, \daleth)\, a_n \le 1 - \eta. \tag{2.4}$$

Corollary 2.3. *If $f \in \Sigma_p$ from (1.1) belongs to $\mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$, then*

$$a_n \le \frac{1-\eta}{[n + \eta - \eta\varsigma(n+1)]\Phi_n(\delta, \varrho, \gamma, x, \daleth)}, \quad n \ge 1.$$

This bound is sharp for the function given in (2.2).

Remark 2.4. If $\mathcal{L}_x^\delta(\gamma, \daleth, \varrho)\, f(\mathfrak{X}) \in \Sigma_p^*(\eta)$, then

$$a_n \le \frac{1-\eta}{(n+\eta)\Phi_n(\delta, \varrho, \gamma, x, \daleth)}, \quad n \ge 1.$$

Moreover, taking $\varsigma = 0$ in (2.1) recovers the generalized corollary of Kavitha et al. [16] for functions $f \in \Sigma_p$ on $\mathbb{U}^*$.

Theorem 2.5. *The set $\mathcal{K}_p(\varsigma, \eta, \mathfrak{X})$ is invariant under convex combinations.*

Proof. Let

$$f(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} a_n \zeta^n, \quad g(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} b_n \zeta^n$$

be two functions in $\mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. By Theorem 2.1,

$$\sum_{n=1}^{\infty}[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)a_n\leq 1-\eta$$

and

$$\sum_{n=1}^{\infty}[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)b_n\leq 1-\eta.$$

For $0\leq\tau\leq 1$, define

$$h(\zeta)=\tau f(\zeta)+(1-\tau)g(\zeta)=\frac{1}{\zeta}+\sum_{n=1}^{\infty}[\tau a_n+(1-\tau)b_n]\zeta^n.$$

Then,

$$\begin{aligned}\sum_{n=1}^{\infty}&[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)[\tau a_n+(1-\tau)b_n]\\ &=\tau\sum_{n=1}^{\infty}[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)a_n\\ &\quad+(1-\tau)\sum_{n=1}^{\infty}[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)b_n\\ &\leq\tau(1-\eta)+(1-\tau)(1-\eta)=1-\eta.\end{aligned}$$

Hence, $h\in\mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. □

3. Extreme Points

Theorem 3.1. *Let*

$$f_0(\zeta)=\frac{1}{\zeta}$$

and for $n\geq 1$

$$f_n(\zeta)=\frac{1}{\zeta}+\frac{1-\eta}{[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}\zeta^n. \tag{3.1}$$

Then a function $f\in\mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$ if and only if it can be written as

$$f(\zeta)=\sum_{n=0}^{\infty}\varrho_n f_n(\zeta),\qquad \varrho_n\geq 0,\quad \sum_{n=0}^{\infty}\varrho_n=1.$$

Proof. Suppose

$$f(\zeta)=\sum_{n=0}^{\infty}\varrho_n f_n(\zeta),\qquad \varrho_n\geq 0,\quad \sum_{n=0}^{\infty}\varrho_n=1.$$

Then

$$f(\zeta)=\varrho_0 f_0(\zeta)+\sum_{n=1}^{\infty}\varrho_n f_n(\zeta)=\frac{1}{\zeta}+\sum_{n=1}^{\infty}\varrho_n\frac{1-\eta}{[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}\zeta^n.$$

Therefore,

$$\sum_{n=1}^{\infty}[\, n+\eta-\eta\varsigma(n+1)\,]\, \Phi_n(\delta,\varrho,\gamma,x,\daleth)\frac{\varrho_n(1-\eta)}{[\, n+\eta-\eta\varsigma(n+1)\,]\, \Phi_n(\delta,\varrho,\gamma,x,\daleth)}$$
$$= (1-\eta)\sum_{n=1}^{\infty}\varrho_n = (1-\eta)(1-\varrho_0) \leq 1-\eta.$$

Hence, $f \in \mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. ∎

Therefore, from Theorem (2.1), we have $f \in \mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. Conversely, assume $f \in \mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. By Corollary (2.3),

$$a_n \leq \frac{1-\eta}{[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}, \qquad n \geq 1.$$

Define

$$\varrho_n = \frac{[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\eta}a_n, \qquad n \geq 1,$$

and

$$\varrho_0 = 1-\sum_{n=1}^{\infty}\varrho_n.$$

Then

$$f(\zeta) = \varrho_0 f_0(\zeta) + \sum_{n=1}^{\infty}\varrho_n f_n(\zeta).$$

∎

4. Radii of Starlikeness and Convexity

In this section, we determine the radii within which functions in the class $\mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$ are meromorphically close-to-convex, starlike, and convex.

Theorem 4.1. *Let $f \in \mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. Then f is meromorphically close-to-convex of order ϱ $(0 \leq \varrho < 1)$ in the disk $|\zeta| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\varrho)\,[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{n(1-\eta)}\right]^{\frac{1}{n+1}}, \qquad n \geq 1.$$

The result is sharp for the extremal function f defined by (1.1).

Proof. It is enough to show

$$\left|\frac{f'(\mathfrak{X})}{\mathfrak{X}^{-2}}+1\right| < 1-\varrho. \tag{4.1}$$

By Theorem (2.1), we have

$$\sum_{n=1}^{\infty}\frac{[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\eta}a_n \leq 1.$$

Thus,

$$\left|\frac{f'(\mathfrak{X})}{\mathfrak{X}^{-2}}+1\right| = \left|\sum_{n=1}^{\infty}na_n\mathfrak{X}^{n+1}\right| \leq \sum_{n=1}^{\infty}na_n|\mathfrak{X}|^{n+1} < 1-\varrho$$

holds provided that

$$\frac{n|\mathfrak{X}|^{n+1}}{1-\varrho} \leq \frac{[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\eta}.$$

Therefore, inequality (4.1) is satisfied if

$$|\mathfrak{X}|^{n+1} \leq \frac{(1-\varrho)\,[n+\eta-\eta\varsigma(n+1)]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{n(1-\eta)}, \qquad n \geq 1,$$

which establishes the close-to-convexity radius and concludes the proof. $\square$

Theorem 4.2. *Let $f \in \mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. Then f is meromorphically starlike of order ϱ $(0 \leq \varrho < 1)$ in the disk $|\zeta| < r_2$, where*

$$r_2 = \inf_n \left[\left(\frac{1-\varrho}{n+2-\varrho}\right)\frac{[\,n+\eta-\eta\varsigma(n+1)\,]\,\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\eta}\right]^{\frac{1}{n+1}}, \qquad n \geq 1.$$

The result is sharp for the extremal function f given by (3.1).

Proof. Using the approach similar to Theorem (4.1), we obtain

$$\left|\frac{\mathfrak{X}f'(\mathfrak{X})}{f(\mathfrak{X})}+1\right| < 1-\varrho, \qquad |\zeta| < r_2.$$

$\square$

Theorem 4.3. *Let $f \in \mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. Then f is meromorphically convex of order ϱ $(0 \leq \varrho < 1)$ in the disk $|\zeta| < r_3$, where*

$$r_3 = \inf_n \left[\left(\frac{1-\varrho}{n+2-\varrho}\right)\frac{[\,n+\eta-\eta\varsigma(n+1)\,]\,\Phi_n(\delta,\varrho,\gamma,x,\beth)}{n(1-\eta)}\right]^{\frac{1}{n+1}}, \qquad n \geq 1.$$

The result is sharp for the extremal function

$$f_n(\zeta) = \frac{1}{\zeta} + \frac{n(1-\eta)}{[\,n+\eta-\eta\varsigma(n+1)\,]\,\Phi_n(\delta,\varrho,\gamma,x,\beth)}\,\zeta^n, \qquad n \geq 1.$$

Proof. By an argument analogous to Theorem (4.1), we have

$$\left|\frac{\mathfrak{X}f''(\mathfrak{X})}{f'(\mathfrak{X})}+2\right| < 1-\varrho, \qquad |\zeta| < r_3,$$

which establishes the stated radius of convexity. $\square$

5. Hadamard Product

Theorem 5.1. *Let $f, g \in \Sigma_p$ defined by (1.1) and belonging to $\mathcal{K}_p(\varsigma,\eta,\mathfrak{X})$. Then the Hadamard product $f * g \in M_p(\daleth,\varsigma,\mathfrak{X})$, where*

$$\daleth \leq 1 - \frac{(1-\eta)^2(n+1)(1-\varsigma)}{(1-\eta)^2(1-\varsigma(n+1)) + [\,n+\eta-\eta\varsigma(n+1)\,]^2\Phi_n(\delta,\varrho,\gamma,x,\beth)}.$$

Proof. From Theorem (2.1), we have

$$\sum_{n=1}^{\infty} \frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\eta} a_n \leq 1 \tag{5.1}$$

and

$$\sum_{n=1}^{\infty} \frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\eta} b_n \leq 1. \tag{5.2}$$

We want the largest $\gimel$ such that

$$\sum_{n=1}^{\infty} \frac{[\,n+\gimel-\gimel\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\gimel} a_n b_n \leq 1.$$

By the Cauchy–Schwarz inequality and (5.1), (5.2), we obtain

$$\sum_{n=1}^{\infty} \frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\eta} \sqrt{a_n b_n} \leq 1. \tag{5.3}$$

It suffices to satisfy

$$\frac{[\,n+\gimel-\gimel\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\gimel} a_n b_n \leq \frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}{1-\eta} \sqrt{a_n b_n},$$

i.e.,

$$\sqrt{a_n b_n} \leq \frac{(1-\gimel)[\,n+\eta-\eta\varsigma(n+1)\,]}{(1-\eta)[\,n+\gimel-\gimel\varsigma(n+1)\,]}. \tag{5.4}$$

Also, from (5.3),

$$\sqrt{a_n b_n} \leq \frac{1-\eta}{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)}. \tag{5.5}$$

Combining (5.4) and (5.5), it is enough that

$$\frac{1-\eta}{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\daleth)} \leq \frac{(1-\gimel)[\,n+\eta-\eta\varsigma(n+1)\,]}{(1-\eta)[\,n+\gimel-\gimel\varsigma(n+1)\,]},$$

which simplifies to

$$\gimel \leq \frac{[\,n+\eta-\eta\varsigma(n+1)\,]^2\Phi_n(\delta,\varrho,\gamma,x,\daleth) - n(1-\eta)^2}{[\,n+\eta-\eta\varsigma(n+1)\,]^2\Phi_n(\delta,\varrho,\gamma,x,\daleth) + (1-\eta)^2[1-\varsigma(n+1)]} = \varphi(n).$$

Since $\varphi(n+1)-\varphi(n) > 0$, $\varphi(n)$ is increasing and $\varphi(n) \geq \varphi(1)$. Hence the stated bound for $\gimel$ holds. $\square$

6. Hadamard Squares

Theorem 6.1. *Let $f, g \in \Sigma_p$ as in (1.1) and assume $f, g \in \mathcal{K}_p(\varsigma,\eta,\zeta)$. Define*

$$k(\zeta) = \frac{1}{\zeta} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\zeta^n.$$

Then $k \in \mathcal{K}_p(\varsigma,\eta,\zeta)$ and

$$\gimel \leq 1 - \frac{2(1-\eta)^2\,\Phi_n(\delta,\varrho,\gamma,x,\daleth)\,[\,1-\varsigma(n+1)+n\,]}{[\,n+\eta-\eta\varsigma(n+1)\,]^2(\Phi_n(\delta,\varrho,\gamma,x,\daleth))^2 + 2(1-\eta)^2\,\Phi_n(\delta,\varrho,\gamma,x,\daleth)\,[\,1-\varsigma(n+1)\,]}.$$

Proof. Since $f, g \in \mathcal{K}_p(\varsigma, \eta, \zeta)$, we have

$$\sum_{n=1}^{\infty}\left(\frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\eta}a_n\right)^2 \leq 1 \tag{6.1}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\eta}b_n\right)^2 \leq 1. \tag{6.2}$$

Adding (6.1) and (6.2) yields

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\eta}\right)^2(a_n^2+b_n^2) \leq 1.$$

To find the maximal $\daleth$ such that

$$\sum_{n=1}^{\infty}\frac{[\,n+\daleth-\daleth\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\daleth}(a_n^2+b_n^2) \leq 1, \tag{6.3}$$

it suffices that

$$\frac{[\,n+\daleth-\daleth\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\daleth} \leq \frac{1}{2}\left(\frac{[\,n+\eta-\eta\varsigma(n+1)\,]\Phi_n(\delta,\varrho,\gamma,x,\beth)}{1-\eta}\right)^2.$$

This leads to

$$\daleth \leq \frac{[\,n+\eta-\eta\varsigma(n+1)\,]^2\Phi_n(\delta,\varrho,\gamma,x,\beth)-2n(1-\eta)^2\Phi_n(\delta,\varrho,\gamma,x,\beth)}{[\,n+\eta-\eta\varsigma(n+1)\,]^2\Phi_n(\delta,\varrho,\gamma,x,\beth)+2(1-\eta)^2\Phi_n(\delta,\varrho,\gamma,x,\beth)[1-\varsigma(n+1)]}$$

$$=1-\frac{2(1-\eta)^2\Phi_n(\delta,\varrho,\gamma,x,\beth)[1-\varsigma(n+1)+n]}{[\,n+\eta-\eta\varsigma(n+1)\,]^2(\Phi_n(\delta,\varrho,\gamma,x,\beth))^2+2(1-\eta)^2\Phi_n(\delta,\varrho,\gamma,x,\beth)[1-\varsigma(n+1)]}=\varphi(n).$$

Since $\varphi(n+1) > \varphi(n)$ for all n, we conclude that $\varphi(n)$ is increasing and $\varphi(n) \geq \varphi(1)$, giving the stated bound for $\daleth$. □

7. Integral Transforms

In this section, we examine integral transforms of functions in $\mathcal{K}_p(\varsigma, \eta, \zeta)$ similar to those considered by Goel and Sohi [8].

Theorem 7.1. *Let $f \in \Sigma_p$ as in (1.1) belong to $\mathcal{K}_p(\varsigma, \eta, \zeta)$. Define*

$$F(\zeta) = \frac{c}{\zeta^c}\int_0^{\zeta} t^{c-1}f(t)\,dt, \qquad 0 < c < \infty.$$

Then $F \in M_p(\daleth, \varsigma, \zeta)$, with

$$\daleth = \frac{1-(1-\eta)(1+2\varsigma)+c}{(1+\eta-2\eta\varsigma)(c+2)+(1-\eta)(1-2\varsigma)}.$$

The bound is sharp for

$$f(\zeta) = \frac{1}{\zeta}+\frac{(1-\eta)(\delta+2)}{(1+\eta-2\eta\varsigma)\delta}\,\zeta.$$

Proof. Let $f \in \Sigma_p$ as in (1.1) belong to $\mathcal{K}_p(\varsigma, \eta, \zeta)$. Then

$$F(\zeta) = \frac{c}{\zeta^c}\int_0^{\zeta} t^{c-1} f(t)\, dt = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n \zeta^n.$$

We need

$$\sum_{n=1}^{\infty} \frac{c[\, n + \daleth - \daleth\varsigma(n+1)\,]\Phi_n(\delta, \varrho, \gamma, x, \beth)}{(1-\daleth)(c+n+1)}\, a_n \leq 1. \tag{7.1}$$

Since $f \in \mathcal{K}_p(\varsigma, \eta, \zeta)$,

$$\sum_{n=1}^{\infty} \frac{[\, n + \eta - \eta\varsigma(n+1)\,]\Phi_n(\delta, \varrho, \gamma, x, \beth)}{1-\eta}\, a_n \leq 1.$$

Inequality (7.1) holds if

$$\frac{c[n + \daleth - \daleth\varsigma(n+1)]}{(1-\daleth)(c+n+1)} \leq \frac{[\, n + \eta - \eta\varsigma(n+1)\,]}{1-\eta}.$$

This gives

$$\begin{aligned}
\daleth &\leq \frac{[\, n + \eta - \eta\varsigma(n+1)\,](n+c+1) - (1-\eta)cn}{[\, n + \eta - \eta\varsigma(n+1)\,](n+c+1) + c(1-\eta)[1-\varsigma(n+1)]} \\
&= \frac{1 - (1-\eta)[1+\varsigma(n+1)] + cn}{[\, n + \eta - \eta\varsigma(n+1)\,](n+c+1) + (1-\eta)[1-\varsigma(n+1)]}.
\end{aligned}$$

Denoting the right-hand side by $\phi(n)$, a direct computation shows $\phi(n)$ increases with n for $n \geq 1$, so $\phi(n) \geq \phi(1)$. This yields the stated bound for $\daleth$. $\square$

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