Gulf Journal of Mathematics ISSN: 2309-4966
Vol 22, Issue 2 (2026) 1-18 https://doi.org/10.56947/gjom.v22i2.3984

LINEAR APPROXIMATION OF NONLINEAR POLYNOMIAL CATENARY SYSTEMS WITH n COMPARTMENTS

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Abstract. Nonlinear polynomial compartmental systems arise in population dynamics and mass transfer modeling. This paper addresses the inverse problem of identifying exchange coefficients from partial measurements of $n-1$ compartments after injecting a known quantity into the first. We approximate the nonlinear system by a linear one via a Taylor expansion around a reference point, allowing explicit computation of approximate coefficients using the Jacobian. This approach provides a practical framework for parameter identification in nonlinear compartmental systems while leveraging classical linear system techniques.

Keywords. Nonlinear polynomial, Catenary systems
2020 Mathematics Subject Classification. Primary 34A30; Secondary 34A34

1. Introduction

Compartmental systems constitute a fundamental modeling tool in applied mathematics and biomathematics, providing a structured framework for describing the distribution and transfer of quantities such as populations, chemical substances, or energy among interconnected compartments. These systems are widely used in diverse applications, including population dynamics, pharmacokinetics, epidemiology, and environmental science, where understanding the interactions and flows between compartments is essential for both theoretical analysis and practical decision-making [5, 6, 9].
Linear compartmental systems, characterized by constant exchange coefficients, have been extensively studied, and their solutions are well understood, often admitting uniqueness and stability under standard conditions. However, many real-world systems exhibit nonlinear behavior, where the transfer of material between compartments depends on the current states of the source and target compartments. Polynomial nonlinearities, in particular, arise naturally in models of population interactions, enzyme kinetics, and other processes governed by mass-action laws. These nonlinear systems present significant analytical challenges, motivating the development of approximation and identification methods

Date: Received: Dec 20, 2025; Revised: Jan 15, 2026; Accepted: Jan 31, 2026.
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to enable practical analysis.
A major difficulty in studying nonlinear compartmental systems lies in parameter identification. In many experimental or observational settings, it is not feasible to measure all compartments, and data are often available only from a subset of compartments. This leads to inverse problems, where the goal is to estimate unknown exchange coefficients from partial observations of the system's state. Addressing this challenge is critical for both accurate modeling and prediction, as well as for designing interventions in biological, ecological, or chemical systems.
In this work, we focus on nonlinear polynomial compartmental systems and propose a methodology to determine the exchange coefficients by measuring the amount of substance in a minimal number of compartments, specifically from compartment 1 to $n-1$. To achieve this, we linearize the nonlinear system using a Taylor series expansion around a reference point, which enables explicit computation of approximate exchange coefficients via the Jacobian matrix of the nonlinear vector field. This approach provides a practical and computationally tractable framework for parameter identification, while retaining the essential dynamics of the original nonlinear system. The proposed method offers valuable insights into the behavior of complex compartmental systems and opens the door for further applications in biomathematics and applied mathematics.

2. Preliminary Definitions and Notations

Definition 2.1. We say that a matrix $M = (m_{ij})$ of order n is partially strictly diagonally dominant if:

$$|m_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |m_{ij}| \quad and \quad \exists i_0 \in \{1, 2, \cdots, n\} \,/ |m_{i_0 i_0}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |m_{i_0 j}|$$

Definition 2.2. Let $M = (a_{ij})$ be a matrix of order n, and $P_1; P_2; \cdots; P_n$, n distinct points of the Euclidean plane.
The directed graph of the matrix M associated with the points $P_1; P_2; \cdots; P_n$ is called the set of vectors $\overrightarrow{P_iP_j}$ (or arcs) such that $a_{ij} \neq 0$ and $i \neq j$.
The directed graph of M is denoted $G(M)$

$$G(M) = \left\{ \overrightarrow{P_iP_j} / a_{ij} \neq 0 \quad and \quad i \neq j \right\}$$

Definition 2.3. The directed graph of the matrix M is said to be strongly vectorially connected if for all points $P_i; P_j$, there exists a path of $G(M)$ going from P_i to P_j, that is to say that:

$$\forall (i, j) \in (\{1, 2, \cdots, n\})^2, i \neq j \quad \exists \{i_1, i_2, \cdots, i_q\} \subset \{1, 2, \cdots, n\} \quad with:$$
$$\left\{ \overrightarrow{P_iP_{i_1}}, \overrightarrow{P_{i_1}P_{i_2}}, \cdots, \overrightarrow{P_{i_q}P_j} \right\} \subset G(M)$$

we can remark:

$$\overrightarrow{P_iP_j} = \overrightarrow{P_iP_{i_1}} + \overrightarrow{P_{i_1}P_{i_2}} + \cdots + \overrightarrow{P_{i_q}P_j}$$

Proposition 2.4. [11] *A square matrix is irreducible if, and only if, its directed graph is strongly vectorially connected.*

Proposition 2.5. *Any compartmental matrix associated with an open compartmental system verifying the following hypothesis* ($\mathbb{H}_{Comp}$)*:*

$$\begin{cases} \forall j \in \{1,2,\cdots,n\}, \exists i_0 \in \{1,2,\cdots,n\} / k_{i_0 j} \neq 0 \\ \forall i \in \{1,2,\cdots,n\}, \exists j_0 \in \{1,2,\cdots,n\} / k_{i j_0} \neq 0 \end{cases}$$

the compartmental matrix associated with this system is:

$$A = \begin{pmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{pmatrix}$$

with:

$$k_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} - k_{ie}$$

is invertible.

Proof. It suffices to show that A^t is partially strict and irreducible diagonally dominant.
Indeed, for all $i \in \{1,2,\cdots,n\}$

$$|k_{ii}| = \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} + k_{ie} \Rightarrow |k_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij}$$

as the system is open there exists i_0 such that $k_{i_0} > 0$ therefore:

$$|k_{i_0 i_0}| > \sum_{\substack{j=1 \\ j \neq i_0}}^{n} k_{i_0 j}$$

the matrix A^t is then partially strictly diagonally dominant.
Moreover, to show that A^t is irreducible, it suffices to show that $G(A^t)$ is strongly vectorially connected.
To do this, we will reason by recurrence on the rank of the matrix A^t.
If $n = 2$

$$A^t = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

$k_{12} \neq 0$ and $k_{21} \neq 0$ by hypothesis ($\mathbb{H}_{Comp}$), so:

$$G(A^t) = \left\{ \overrightarrow{P_1 P_2}, \overrightarrow{P_2 P_1} \right\}$$

is strongly vectorially connected.
Suppose that for an n-compartmental system (system composed of n compartments) verifying the hypothesis ($\mathbb{H}_{Comp}$) with compartmental matrix $A_{[n]}$, $G(A^t_{[n]})$ is strongly vectorially connected and consider an $(n+1)$-compartmental system verifying the hypothesis ($\mathbb{H}_{Comp}$): $G(A^t_{[n]})$ is strongly vectorially connected and is equivalent to:

$$\forall (i,j) \in (\{1,2,\cdots,n\})^2, i \neq j, \exists \{i_1, i_2, \cdots, i_q\} \subset \{1,2,\cdots,n\} \quad \textit{with}:$$

$$\left\{\overrightarrow{P_iP_{i_1}}, \overrightarrow{P_{i_1}P_{i_2}}, \cdots, \overrightarrow{P_{i_q}P_j}\right\} \subset G(A^t_{[n]})$$

Let the matrix be $A^t_{[n+1]}$ and P_i, P_j $\quad i \leq n+1$, and $j \leq n+1$.
We will distinguish three cases:

(1) $i < n+1$ and $j < n+1$
There exists a path from P_i to Pj because $G(A^t_[n])$ is strongly vectorially connected by induction hypothesis; like $G(A_{[n]}) \subset G(A_{[n+1]})$ this path from P_i to P_j is in $(A^t_{[n+1]})$

(2) $i = n+1$ and $j < n+1$
hypothesis $(\mathbb{H}_{Comp})$ implies that:

$$\exists j_0 \in \{1, 2, \cdots, n\} \, / k_{(n+1)j_0} \neq 0$$

so

$$\overrightarrow{P_{n+1}P_{j_0}} \subset G(A^t_{[n+1]}).$$

Or $j_0 \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, n\}$, recurrence failure allows you to confirm that there is a chemistry between P_{j_0} and P_j, which means:

$$\exists \{i_1, i_2, \cdots, i_s\} \subset \{1, 2, \cdots, n\} \, / \left\{\overrightarrow{P_{j_0}P_{i_1}}, \overrightarrow{P_{i_1}P_{i_2}}, \cdots, \overrightarrow{P_{i_s}P_j} \subset G(A^t_{[n+1]})\right\}.$$

Finally, $\overrightarrow{P_{n+1}P_{j_0}} \subset G(A^t_{[n+1]})$ implies that:

$$\left\{\overrightarrow{P_{n+1}P_{j_0}}, \overrightarrow{P_{j_0}P_{i_1}}, \cdots, \overrightarrow{P_{i_s}P_j}\right\} \subset G(A^t_{[n+1]}).$$

(3) $i = n+1$ and $j = n+1$
The demonstration is analogous to case (2)
Therefore the directed graph of A^t is strongly vectorially connected and consequently A^t is invertible.

□

We examine the nonlinear polynomial compartmental catenary system, denoted by (S_{N-Lin}), as illustrated in Figure 1.

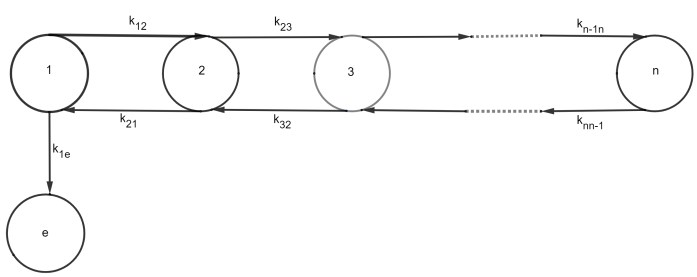

FIGURE 1. (S_{N-Lin}) Nonlinear polynomial compartimental catenary system

These systems are governed by the following hypothesis: the quantity transferred from compartment i to compartment j is given by

$$k_{ij} x_i^{\alpha} x_j^{\beta},$$

where $\beta = 0$ if compartment j represents the external environment (see [5], [6], [9]). Here, $x_i(t)$ denotes the mass quantity in compartment i at time t, k_{ij} is the exchange coefficient, and α and β are constants characterizing the compartmental system. This assumption is known as the polynomial exchange hypothesis of order $(\alpha + \beta)$.
According to the mass balance principle, each compartment satisfies a set of nonlinear differential equations (see [4]). The identification process is carried out by subjecting the system to an instantaneous injection of a quantity of substance a a into the first compartment. Consequently, the compartmental catenary system is governed by the following set of differential equations, together with the corresponding initial condition:

$$\begin{cases} x_1'(t) = k_{21} x_2^{\alpha}(t) x_1^{\beta}(t) - k_{12} x_1^{\alpha}(t) x_2^{\beta}(t) - k_{1e} x_1^{\alpha}(t) \\ x_i'(t) = k_{i-1i} x_{i-1}^{\alpha}(t) x_i^{\beta}(t) + k_{i+1i} x_{i+1}^{\alpha}(t) x_i^{\beta}(t) \\ \qquad - \left(k_{ii-1} x_{i-1}^{\beta}(t) + k_{ii+1} x_{i+1}^{\beta}(t)\right) x_i^{\alpha}(t) \quad \forall i \in \{2, 3, ..., n-1\} \\ x_n'(t) = k_{n-1n} x_{n-1}^{\alpha}(t) x_n^{\beta}(t) - k_{nn-1} x_n^{\alpha}(t) x_{n-1}^{\beta}(t) \\ x_1(0) = a \\ x_i(0) = 0 \quad \forall i \in \{2, 3, ..., n\} \end{cases} \tag{2.1}$$

we denote by:

$$\begin{array}{rcl} X : [0, +\infty[& \longrightarrow & \mathbb{R}^n \\ t & \longrightarrow & X^T(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \end{array}$$

The state function of the catenary compartmental system (S_{N-Lin}) is given by:

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$
$$(x_1, x_2, \ldots, x_n) \longmapsto F(x_1, \ldots, x_n) = \big(f_1(x_1, \ldots, x_n),\ f_2(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)\big)$$

such that:

$$\begin{cases} f_1(x_1, x_2, \cdots, x_n) = k_{21} x_2^{\alpha} x_1^{\beta} - k_{12} x_1^{\alpha} x_2^{\beta} - k_{1e} x_1^{\alpha} \\ f_i(x_1, x_2, \cdots, x_n) = k_{i-1i} x_{i-1}^{\alpha} x_i^{\beta} + k_{i+1i} x_{i+1}^{\alpha} x_i^{\beta} - \left(k_{ii-1} x_{i-1}^{\beta} + k_{ii+1} x_{i+1}^{\beta}\right) x_i^{\alpha} \quad \forall i \in \{2, 3, ..., n-1\} \\ f_n(x_1, x_2, \cdots, x_n) = k_{n-1n} x_{n-1}^{\alpha} x_n^{\beta} - k_{nn-1} x_n^{\alpha} x_{n-1}^{\beta} \end{cases}$$

Using these notations, the differential system (2.1) can be written in vector form as:

$$\begin{cases} X'(t) = F^T(X^T(t)) \\ \\ X(0) = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{cases} \tag{2.2}$$

3. Initial study

For the linearization of the system (2.2) our aim is to apply the Taylor formula in the neighbourhood of the initial condition $(a_0, 0, 0)$.
The partial derivatives of the function F being:

$$\begin{cases} \dfrac{\partial f_1}{\partial x_1}(x_1, x_2, \cdots, x_n) = \beta k_{21} x_2^{\alpha} x_1^{\beta-1} - \alpha k_{12} x_1^{\alpha-1} x_2^{\beta} - \alpha k_{1e} x_1^{\alpha-1} \\ \\ \dfrac{\partial f_1}{\partial x_2}(x_1, x_2, \cdots, x_n) = \alpha k_{21} x_2^{\alpha-1} x_1^{\beta} - \beta k_{12} x_1^{\alpha} x_2^{\beta-1} \\ \\ \dfrac{\partial f_1}{\partial x_i}(x_1, x_2, \cdots, x_n) = 0 \quad \forall i \in \{3, \cdots, n\} \end{cases}$$

and for all $i \in \{2, 3, ..., n-1\}$, we have:

$$\begin{cases} \dfrac{\partial f_i}{\partial x_{i-1}}(x_1, x_2, \cdots, x_n) = \alpha k_{i-1i} x_{i-1}^{\alpha-1} x_i^{\beta} - \beta k_{ii-1} x_{i-1}^{\beta-1} x_i^{\alpha} \\ \\ \dfrac{\partial f_i}{\partial x_i}(x_1, x_2, \cdots, x_n) = \beta k_{i-1i} x_{i-1}^{\alpha} x_i^{\beta-1} + \beta k_{i+1i} x_{i+1}^{\alpha} x_i^{\beta-1} - \alpha \left(k_{ii-1} x_{i-1}^{\beta} + k_{ii+1} x_{i+1}^{\beta} \right) x_i^{\alpha-1} \\ \\ \dfrac{\partial f_i}{\partial x_{i+1}}(x_1, x_2, \cdots, x_n) = \alpha k_{i+1i} x_{i+1}^{\alpha-1} x_i^{\beta} - \beta k_{ii+1} x_{i+1}^{\beta-1} x_i^{\alpha} \end{cases}$$

and:

$$\begin{cases} \dfrac{\partial f_n}{\partial x_i}(x_1, x_2, \cdots, x_n) = 0 \quad \forall i \in \{1, 2, 3, ..., n-2\} \\ \\ \dfrac{\partial f_n}{\partial x_{n-1}}(x_1, x_2, \cdots, x_n) = \alpha k_{n-1n} x_{n-1}^{\alpha-1} x_n^{\beta} - \beta k_{nn-1} x_n^{\alpha} x_{n-1}^{\beta-1} \\ \\ \dfrac{\partial f_n}{\partial x_n}(x_1, x_2, \cdots, x_n) = \beta k_{n-1n} x_{n-1}^{\alpha} x_n^{\beta-1} - \alpha k_{nn-1} x_n^{\alpha-1} x_{n-1}^{\beta} \end{cases}$$

The function F is differentiable in all point $(x_1, x_2, \cdots, x_n)$ such that $x_i \neq 0$ $\forall i \in \{1, 2, \cdots, n\}$, for all $\alpha > 0$ and all $\beta > 0$, and the Jacobain matrix is given by:

$$(DF_{(x_1, x_2, x_3)}) = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Where matrix components are given by the form:

$$\begin{cases} a_{11} = -\alpha k_{12} x_1^{\alpha-1} x_2^{\beta} + \beta k_{21} x_2^{\alpha} x_1^{\beta-1} - \alpha k_{1e} x_1^{\alpha-1}, \\ \forall i \in \{2,3,\cdots,n-1\} : \begin{cases} a_{ii-1} = \alpha k_{i-1i} x_{i-1}^{\alpha-1} x_i^{\beta} - \beta k_{ii-1} x_{i-1}^{\beta-1} x_i^{\alpha}, \\ a_{ii} = \beta k_{i-1i} x_{i-1}^{\alpha} x_i^{\beta-1} + \beta k_{i+1i} x_{i+1}^{\alpha} x_i^{\beta-1} - \alpha \left(k_{ii-1} x_{i-1}^{\beta} + k_{ii+1} x_{i+1}^{\beta} \right) x_i^{\alpha-1} \\ a_{ii+1} = \alpha k_{i+1i} x_{i+1}^{\alpha-1} x_i^{\beta} - \beta k_{ii+1} x_{i+1}^{\beta-1} x_i^{\alpha} \end{cases}, \\ a_{ij} = 0 \quad \forall i,j \in \{1,2,\cdots,n\} \ \ and \ \ | i-j | \geq 2, \\ a_{nn} = \beta k_{n-1n} x_{n-1}^{\alpha} x_n^{\beta-1} - \alpha k_{nn-1} x_n^{\alpha-1} x_{n-1}^{\beta}. \end{cases}$$

Remark 3.1. 1)If $\alpha < 1$ or $\beta < 1$ F is not differentiable in $(a, 0, \cdots, 0)$.
2) If $\alpha \geq 1$ and $\beta \geq 1$ F is differentiable in $(a, 0, \cdots, 0)$.

The Taylor formula applied in neighbourhood of $(a, 0, \cdots, 0)$ leads to:

$$\begin{aligned} F^T(x_1, x_2, \cdots, x_n) = F^T(a, 0, \cdots, 0) + (DF)_{(a,0,\cdots,0)} (x_1, x_2, \cdots, x_n)^T \\ + \left(D^2 F\right)_{(x_{\theta 1}, x_{\theta 2}, \cdots, x_{\theta n})} (x_1, x_2, \cdots, x_n)^2 \end{aligned}$$

with: $(x_{\theta 1}, x_{\theta 2}, \cdots, x_{\theta n}) = (x_1 + \theta (x_1 - a), x_2 + \theta . x_2, \cdots, x_n + \theta . x_n)$ with $|\theta| < 1$.
For t sufficiently small, Our aim is to approach the differential system (2.1) on $[0, t_0]$ by the following linear differential system:

$$Z'(t) = F^T(a, 0, \cdots, 0) + (DF)_{(a,0,\cdots,0)} (z_1 - a, z_2, \cdots, z_n)^T \tag{3.1}$$

with:

$$(DF)_{(a,0,\cdots,0)} = \begin{pmatrix} \alpha k_{1e} a^{\alpha-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We note that this matrix is not well suited for approximating the exchange coefficients, so we introduce a temporization procedure. First, we choose a time $t_0 > 0$ close to zero and consider the differential system which governs the catenary compartmental system (shown in Figure 1) at time t_0.

4. Direct problem study

We consider the following differential system:

$$\begin{cases} X'(t) = F(X(t)^T) \\ X(0) = \begin{pmatrix} a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{cases} \tag{4.1}$$

where

$$X(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\\vdots\\x_i(t)\\\vdots\\x_n(t)\end{pmatrix},\quad F(X^T)=\begin{pmatrix}k_{21}x_2^\alpha x_1^\beta-k_{12}x_1^\alpha x_2^\beta-k_{1e}x_1^\alpha\\k_{12}x_1^\alpha x_2^\beta+k_{32}x_3^\alpha x_2^\beta-(k_{21}x_1^\beta+k_{23}x_3^\beta)x_2^\alpha\\\vdots\\k_{i-1i}x_{i-1}^\alpha x_i^\beta+k_{i+1i}x_{i+1}^\alpha x_i^\beta-\left(k_{ii-1}x_{i-1}^\beta+k_{ii+1}x_{i+1}^\beta\right)x_i^\alpha\\\vdots\\k_{n-1n}x_{n-1}^\alpha x_n^\beta-k_{nn-1}x_n^\alpha x_{n-1}^\beta\end{pmatrix}.$$

Remark 4.1. In this work, we limit our analysis to the situation where $\alpha\geq 1$ and $\beta\geq 1$.

According to the Cauchy–Lipschitz theorem, a local solution exists and is unique because:

- The function F is continuous in a neighborhood of the initial point,
- F satisfies a local Lipschitz condition with respect to X.

We suppose that there exists $t_j^*>0$ such that $\forall j\in\{2,\cdots,n\}\quad x_j(t)=0$ for all t in $[0,t_j^*]$.
Then:
Based on the system (4.1), for all t in $[0,t_j^*]$ we have:

$$\begin{cases}x_j'(t)=0\\x_j(t)=0.\end{cases}$$

We have $x_j'(t)=0\ \ \forall t\in[0,t_j^*]$ then $x_j(t)=cte$ on $[0,t_j^*]$ and we have $x_j(0)=0$ then $x_j(t)=0\ \ \forall t\in[0,t_j^*]$.
Therefore, the problem is a $j-1$-compartment problem. Which contradicts the hypothesis that the system has n compartments.
This is a contradiction. We conclude that $x_j(t)\neq 0$ for all t in $[0,t_j^*]$.

5. Linearization of the system (2.2)

$$\begin{cases}x_1'(t)=k_{21}x_2^\alpha(t)x_1^\beta(t)-k_{12}x_1^\alpha(t)x_2^\beta(t)-k_{1e}x_1^\alpha(t)\\[2ex] x_i'(t)=k_{i-1i}x_{i-1}^\alpha(t)x_i^\beta(t)+k_{i+1i}x_{i+1}^\alpha(t)x_i^\beta(t)\\[2ex] \qquad\quad-\left(k_{ii-1}x_{i-1}^\beta(t)+k_{ii+1}x_{i+1}^\beta(t)\right)x_i^\alpha(t)\qquad\forall i\in\{2,3,\cdots,n-1\}\\[2ex] x_n'(t)=k_{n-1n}x_{n-1}^\alpha(t)x_i^\beta(t)-k_{nn-1}x_n^\alpha(t)x_{n-1}^\beta(t)\\[2ex] x_i(t_0)=a_i\qquad\forall i\in\{1,2,\cdots,n\}\end{cases}\tag{5.1}$$

That can be written in the following equivalent vector form:

$$\begin{cases}X'(t)=F^T\left(X^T(t)\right)\\[2ex] X^T(t_0)=(a_1,a_2,\cdots,a_n)\end{cases}\tag{5.2}$$

The Taylor formula on the interval $[t_0,T]$ is given by:

$$F^T(x_1,x_2,\cdots,x_n)=F^T\left(a_1,a_2,\cdots,a_n\right)+(DF)_{(a_1,a_2,\cdots,a_n)}\left(x_1-a_1,x_2-a_2,\cdots,x_n-a_n\right)^T$$

$$+\left(D^2F\right)_{(x_{\theta 1},x_{\theta 2},\cdots,x_{\theta n})}(x_1-a_1,x_2-a_2,\cdots,x_n-a_n)^2$$

with:

$(x_{\theta 1},x_{\theta 2},\cdots,x_{\theta n})=(x_1+\theta\,(x_1-a_1)\,,x_2+\theta(x_2-a_2),\cdots,x_n+\theta(x_n-a_n))$ with $|\theta|<1$.

Where the components of matrix $\left(DF_{(a_1,a_2,\cdots,a_n)}\right)$ are written as:

$$\begin{cases} a_{11}=-\alpha k_{12}a_1^{\alpha-1}a_2^{\beta}+\beta k_{21}a_2^{\alpha}a_1^{\beta-1}-\alpha k_{1e}a_1^{\alpha-1} \\ \forall i\in\{2,3,\cdots,n-1\}\begin{cases} a_{ii-1}=\alpha k_{i-1i}a_{i-1}^{\alpha-1}a_i^{\beta}-\beta k_{ii-1}a_{i-1}^{\beta-1}a_i^{\alpha} \\ a_{ii}=\beta k_{i-1i}a_{i-1}^{\alpha}a_i^{\beta-1}+\beta k_{i+1i}a_{i+1}^{\alpha}a_i^{\beta-1}-\alpha\left(k_{ii-1}a_{i-1}^{\beta}+k_{ii+1}a_{i+1}^{\beta}\right)a_i^{\alpha-1} \\ a_{ii+1}=\alpha k_{i+1i}a_{i+1}^{\alpha-1}a_i^{\beta}-\beta k_{ii+1}a_{i+1}^{\beta-1}a_i^{\alpha}\end{cases} \\ a_{ij}=0 \qquad \forall(i,j)\in(\{1,2,\cdots,n\})^2 \quad with \quad \mid i-j\mid\geq 2 \\ a_{nn}=\beta k_{n-1n}a_n^{\alpha}a_n^{\beta-1}-\alpha k_{nn-1}a_n^{\alpha-1}a_{n-1}^{\beta}\end{cases},$$

We pose:

$$\left(DF_{(a_1,a_2,\cdots,a_n)}\right)=\begin{pmatrix} b_{11} & b_{21} & 0 & \cdots & 0 \\ b_{12} & b_{22} & b_{32} & \ddots & \vdots \\ 0 & b_{23} & b_{33} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{nn-1} \\ 0 & \cdots & 0 & b_{n-1n} & b_{nn}\end{pmatrix}$$

and

$$F^T(a_1,a_2,\cdots,a_n)=(f_1,f_2,\cdots,f_n)$$

with:

$$\begin{cases} f_1=k_{21}b^{\alpha}a_1^{\beta}-k_{12}a_1^{\alpha}b^{\beta}-k_{1e}a_1^{\alpha} \\ f_i=k_{i-1i}a_{i-1}^{\alpha}a_i^{\beta}+k_{i+1i}a_{i+1}^{\alpha}a_i^{\beta}-\left(k_{ii-1}a_1^{\beta}+k_{ii+1}a_{i+1}^{\beta}\right)a_i^{\alpha} \quad \forall i\in\{2,3,\cdots,n-1\} \\ f_n=k_{n-1n}a_{n-1}^{\alpha}a_n^{\beta}-k_{nn-1}a_n^{\alpha}a_{n-1}^{\beta}\end{cases}$$

For t sufficiently small, we propose to approach the differential system (5.1) on $[t_0,T]$ by the following linear differential system:

$$Z'(t)=F^T(a_1,a_2,\cdots,a_n)+(DF)_{(a_1,a_2,\cdots,a_n)}(z_1(t)-a_1,z_2(t)-a_2,\cdots,z_n(t)-a_n)^T \tag{5.3}$$

Lemma 5.1. *We can prove that there exists γ,δ and ω such that:*

$$(DF_{(a_1,a_2,\cdots,a_n)}).\begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{pmatrix}=F^T(a_1,a_2,\cdots,a_n)$$

indeed

$$\begin{pmatrix} b_{11} & b_{21} & 0 & \cdots & 0 \\ b_{12} & b_{22} & b_{32} & \ddots & \vdots \\ 0 & b_{23} & b_{33} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{nn-1} \\ 0 & \cdots & 0 & b_{n-1n} & b_{nn} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \vdots \\ \gamma_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_n \end{pmatrix} \tag{5.4}$$

Thanks to Proposition (2.5), the solutions of system (5.4) can be easily found by applying Cramer's method.
Therefore we consider the linear differential system:

$$Z'(t) = (DF_{(a_1,a_2,\cdots,a_n)}). \begin{pmatrix} z_1(t) - a_1 + \gamma_1 \\ z_2(t) - a_2 + \gamma_2 \\ \vdots \\ z_n(t) - a_n + \gamma_n \end{pmatrix}$$

The change of the state function of the compartmental catenary system:

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} z_1(t) - a_1 + \gamma_1 \\ z_2(t) - a_2 + \gamma_2 \\ \vdots \\ z_n(t) - a_n + \gamma_n \end{pmatrix} \tag{5.5}$$

permits to reduce the system (5.2) to the canonical form:

$$\begin{cases} Y'(t) = (DF)_{(a_1,a_2,\cdots,a_n)}).Y(t) \\ \\ Y^T(t_0) = (\gamma_1, \gamma_2, \cdots, \gamma_n) \end{cases} \tag{5.6}$$

The matrix $(DF)\,(a_1, a_2, \cdots, a_n)$ has the general structure of a compartmental matrix. Therefore, we can fictitiously associate with it the linear compartmental system, denoted by (S_{Lin}), represented by the following figure 2:

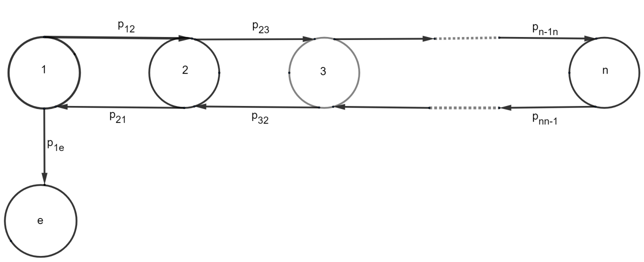

FIGURE 2. (S_{Lin}) Linear model approximation

With for all $i \in \{1, 2, \cdots, n\}$, we have:

$$\begin{cases} p_{1e} = \alpha k_{1e} a_1^{\alpha-1} \\ \\ p_{ii+1} = \alpha k_{ii+1} a_i^{\alpha-1} a_{i+1}^{\beta} - \beta k_{i+1i} a_{i+1}^{\alpha} a_i^{\beta-1} \\ \\ p_{i+1i} = \alpha k_{i+1i} a_{i+1}^{\alpha-1} a_i^{\beta} - \beta k_{ii+1} a_i^{\alpha} a_{i+1}^{\beta-1} \\ \\ p_{n1} = p_{1n} = 0 \end{cases}$$

6. Determination of the initial condition and the associated problem

The parameters k_{ii+1} *and* k_{i+1i} for all $i \in \{1, 2, \cdots, n-1\}$ and the constants α, β characterize the polynomial compartmental catenary system. But the initial condition a_i for all $i \in \{1, 2, \cdots, n\}$ depend on the choice of the time t_0. This involves that the signs of p_{ii+1} *and* p_{i+1i} for all $i \in \{1, 2, \cdots, n-1\}$ are related to t_0 and not know. to be sur that the linear model (S_{VL}) corresponds to measurable physical reality p_{ii+1} *and* p_{i+1i} for all $i \in \{1, 2, \cdots, n-1\}$ must be positive:

$$\begin{cases} p_{ii+1} > 0 \\ \\ p_{i+1i} > 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha k_{ii+1} a_i^{\alpha} - \beta k_{i+1i} a_{i+1}^{\alpha-\beta} a_i^{\beta} > 0 \\ \\ \alpha k_{i+1i} a_{i+1}^{\alpha-\beta} a_i^{\beta} - \beta k_{ii+1} a_i^{\alpha} > 0 \end{cases}$$

This raises the questions:

(1) Do there exist values of a_i for all $i \in \{1, 2, \cdots, n\}$ such that $p_{ii+1} > 0$ and $p_{i+1i} > 0$ for all $i \in \{1, 2, \cdots, n-1\}$
(2) Moreover,a_i for all $i \in \{1, 2, \cdots, n\}$ being tied. does exist a value t_0 making this condition verified ?

Proposition 6.1. *: The real numbers a_i for all $i \in \{1, 2, \cdots, n\}$, such that:*

$$p_{ii+1} > 0 \quad \text{and} \quad p_{i+1i} > 0 \qquad \forall i \in \{1, 2, \cdots, n-1\}$$

exist if and only if $\alpha > \beta$.

Proof:
Knowing that:

$$\begin{cases} p_{ii+1} > 0 \\ \\ p_{i+1i} > 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha k_{ii+1} a_i^{\alpha} - \beta k_{i+1i} a_{i+1}^{\alpha-\beta} a_i^{\beta} > 0 \\ \\ \alpha k_{i+1i} a_{i+1}^{\alpha-\beta} a_i^{\beta} - \beta k_{ii+1} a_i^{\alpha} > 0 \end{cases}$$

and

Let $\Delta_1 = k_{ii+1} a_i^{\alpha}$ and $\Delta_2 = k_{i+1i} a_{i+1}^{\alpha-\beta}$

$$\begin{cases} p_{ii+1} > 0 \\ \\ p_{i+1i} > 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha\Delta_1 - \beta\Delta_2 > 0 \\ \\ \alpha\Delta_2 - \beta\Delta_1 > 0 \end{cases}$$

If $\alpha \leq \beta$ the solutions set:

$$\begin{cases} \alpha\Delta_1 - \beta\Delta_2 > 0 \\ \\ \alpha\Delta_2 - \beta\Delta_1 > 0 \end{cases}$$

is empty .
If $\alpha > \beta$ the solutions set :

$$\begin{cases} \alpha\Delta_1 - \beta\Delta_2 > 0 \\ \\ \alpha\Delta_2 - \beta\Delta_1 > 0 \end{cases}$$

is not empty.

7. Determination of the exchange coefficients of the system (S_{Lin}) and the system (S_{N-Lin})

Next we should check if the values found above are suitable. We are going to show that the choice of t_0 is compatible with one real system as soon as the eigenvalues λ_i for all $i \in \{1, 2, \cdots, n\}$ obtained by the minimization of an error functional (defined below) are negative.

The amount of material in the compartment 1 to compartment n-1 of the system (S_{N-Lin}) is measured at the moments t_k, $1 \leq k \leq m$.
These measurements will be associated to the linear model (S_{VL}) which leads to a new linear model (S_{Lin}) governed by the differential system:

$$\begin{cases} \overline{Y}'(t) = A.\overline{Y}(t) \qquad \forall t \geq t_0 \\ \\ \overline{Y}(t_0) = (\gamma_1, \gamma_2, \cdots, \gamma_n)^T \end{cases} \tag{7.1}$$

where:

$$A = \begin{pmatrix} \overline{p}_{11} & \overline{p}_{21} & 0 & \cdots & 0 \\ \overline{p}_{12} & \overline{p}_{22} & \overline{p}_{32} & \ddots & \vdots \\ 0 & \overline{p}_{23} & \overline{p}_{33} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \overline{p}_{nn-1} \\ 0 & \cdots & 0 & \overline{p}_{n-1n} & \overline{p}_{nn} \end{pmatrix}$$

The solution of this system has the form:

$$\forall j \in \{1, 2, \cdots, n\}, \qquad \overline{y}_j(t) = \sum_{i=1}^{n} \beta_i^j e^{\lambda_i t} \qquad \forall t \geq t_0$$

.
Let us consider the functional:

$$J^*\left(\beta_i^j, \lambda_j, 1 \leq j \leq n-1, 1 \leq i \leq n\right) = \sum_{k=1}^{p} \left(\sum_{j=1}^{n-1} \left(x_j(t_{l_k}) - \sum_{i=1}^{n} \beta_i^j e^{\lambda_i t_{l_k}} \right)^2 \right) \tag{7.2}$$

with $1 \leq l_1 \leq m-1$ and $2 \leq l_p \leq m$. Indices $l_1, l_2, ..., l_p$ are chosen such that:

$$minJ^*\left(\beta_i^j, \lambda_i, 1 \leq j \leq n-1, 1 \leq i \leq n\right)$$

is realized for $\lambda_i < 0 \quad \forall i \in \{1, 2, \cdots, n\}$. As matter of fact let us prove the following theorem:

Theorem 7.1. *Let $\beta_i^{j*}, \lambda_i^*, 1 \leq j \leq n-1, 1 \leq i \leq n$ be the values such that:*

$$\min J^* \left(\beta_i^j, \lambda_i, 1 \leq j \leq n-1, 1 \leq i \leq n\right) = J^* \left(\beta_i^{j*}, \lambda_j^*, 1 \leq j \leq n-1, 1 \leq i \leq n\right)$$

if it exists $l_1, l_2, ..., l_p$ such that:

(1) $\lambda_i^* < 0 \quad \forall i \in \{1, 2, \cdots, n\}$

(2) $\sum_{i=1}^{i=n} \lambda_i^* \beta_i^{j-1*} \beta_i^{j*} > 0$ *with* $\beta_i^{j*} \neq 0$ *for all* $2 \leq j \leq n-1, 1 \leq i \leq n$

Then for all $j \in \{1, 2, \cdots, n-1\} \qquad \overline{p}_{jj+1} > 0 \quad$ *and* $\quad \overline{p}_{j+1j} > 0$

Proof. Let A be the matrix of the linear model (S_{Lin}):

$$A = \begin{pmatrix} \overline{p}_{11} & \overline{p}_{21} & 0 & \cdots & 0 \\ \overline{p}_{12} & \overline{p}_{22} & \overline{p}_{32} & \ddots & \vdots \\ 0 & \overline{p}_{23} & \overline{p}_{33} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \overline{p}_{nn-1} \\ 0 & \cdots & 0 & \overline{p}_{n-1n} & \overline{p}_{nn} \end{pmatrix}$$

The functions $\varphi_i, \ i \in \{1, 2, \cdots, n\}$ defined by:

$$\varphi_i(t) = \exp(\lambda_i^* t) \ , \forall t \in \mathbb{R}, \ \forall i \in \{1, 2, \cdots, n\}$$

being linearly independent we can conclude that for every integer i in $\{1, 2, \cdots, n\}$ and from the relationship (7.1), we have:

$$\lambda_i^* \beta_i^{1*} = -\overline{p}_{12} \beta_i^{1*} - \overline{p}_{1e} \beta_i^{1*} + \overline{p}_{21} \beta_i^{2*} \tag{7.3}$$

$$\lambda_i^* \beta_i^{j*} = \overline{p}_{j-1j} \beta_i^{j-1*} - \overline{p}_{jj-1} \beta_i^{j*} - \overline{p}_{jj+1} \beta_i^{j*} + \overline{p}_{j+1j} \beta_i^{j+1} \qquad \forall j \in \{2, 3, \cdots, n-1\} \tag{7.4}$$

$$\lambda_i^* \beta_i^{n} = \overline{p}_{n-1n} \beta_i^{n-1*} - \overline{p}_{nn-1} \beta_i^{n} \tag{7.5}$$

First step:

Multiplying both sides of the equation (7.5) by β_i^n and summing over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n} \lambda_i^* (\beta_i^n)^2 = \overline{p}_{n-1n} \sum_{i=1}^{i=n} \beta_i^{n-1*} \beta_i^n - \overline{p}_{nn-1} \sum_{i=1}^{i=n} (\beta_i^n)^2$$

Orthogonality of the vector system $\left\{ (\beta_1^n, \beta_2^n, \cdots, \beta_n^n)^T \quad \textit{and} \quad \left(\beta_1^{j*}, \beta_2^{j*}, \cdots, \beta_n^{j*}\right)^T, \quad j \in \{1, 2, \cdots, n\} \right\}$ (see[7][9]), allows us to conclude that:

$$\sum_{i=1}^{i=n} \lambda_i^* (\beta_i^n)^2 = -\overline{p}_{nn-1} \sum_{i=1}^{i=n} (\beta_i^n)^2 = -\overline{p}_{nn-1}$$

so:

$$\overline{p}_{nn-1} = -\sum_{i=1}^{i=n} \lambda_i^* (\beta_i^n)^2 > 0 \tag{7.6}$$

Multiplying both sides of the equation (7.4) for $j = n-1$ by β_i^n and summing over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{n-1*}\beta_i^n=\overline{p}_{n-2n-1}\sum_{i=1}^{i=n}\beta_i^{n-2*}\beta_i^n-\overline{p}_{n-1n-2}\sum_{i=1}^{i=n}\beta_i^{n-1*}\beta_i^n-\overline{p}_{n-1n}\sum_{i=1}^{i=n}\beta_i^{n-1*}\beta_i^n+\overline{p}_{nn-1}\sum_{i=1}^{i=n}\beta_i^n\beta_i^n$$

Orthogonality of the vector system $\left\{(\beta_1^n,\beta_2^n,\cdots,\beta_n^n)^T \quad and \quad \left(\beta_1^{j*},\beta_2^{j*},\cdots,\beta_n^{j*}\right)^T, \quad j\in\{1,2,\cdots,n\}\right\}$, allows us to conclude that:

$$\overline{p}_{nn-1}=\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{n-1*}\beta_i^n$$

Based on relationship (7.6), we conclude that

$$\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{n-1*}\beta_i^n>0$$

Multiplying both sides of the equation (7.5) by β_i^{n-1*} and summing over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n}\lambda_i^*\beta_i^n\beta_i^{n-1*}=\overline{p}_{n-1n}\sum_{i=1}^{i=n}(\beta_i^{n-1*})^2-\overline{p}_{nn-1}\sum_{i=1}^{i=n}\beta_i^n\beta_i^{n-1*}$$

Orthogonality of the vector system $\left\{(\beta_1^n,\beta_2^n,\cdots,\beta_n^n)^T \quad and \quad \left(\beta_1^{j*},\beta_2^{j*},\cdots,\beta_n^{j*}\right)^T, \quad j\in\{1,2,\cdots,n\}\right\}$, allows us to conclude that:

$$\overline{p}_{n-1n}=\sum_{i=1}^{i=n}\lambda_i^*\beta_i^n\beta_i^{n-1*}>0$$

Second step:

Multiplying both sides of the equation (7.4) for all $j\in\{2,3,\cdots,n-1\}$ by β_i^{j-1*} and summing over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{j*}\beta_i^{j-1*}=\overline{p}_{j-1j}\sum_{i=1}^{i=n}(\beta_i^{j-1*})^2-\overline{p}_{jj-1}\sum_{i=1}^{i=n}\beta_i^{j*}\beta_i^{j-1*}-\overline{p}_{jj+1}\sum_{i=1}^{i=n}\beta_i^{j*}\beta_i^{j-1*}+\overline{p}_{j+1j}\sum_{i=1}^{i=n}\beta_i^{j+1}\beta_i^{j-1*}$$

Orthogonality of the vector system $\left\{(\beta_1^n,\beta_2^n,\cdots,\beta_n^n)^T \quad and \quad \left(\beta_1^{j*},\beta_2^{j*},\cdots,\beta_n^{j*}\right)^T, \quad j\in\{1,2,\cdots,n\}\right\}$, allows us to conclude that:

$$\overline{p}_{j-1j}=\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{j*}\beta_i^{j-1*}>0$$

Also multiplying both sides of the equation (7.4) for all $j\in\{2,3,\cdots,n-2\}$ by β_i^{j+1*} and summing over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{j*}\beta_i^{j+1*}=\overline{p}_{j-1j}\sum_{i=1}^{i=n}\beta_i^{j-1*}\beta_i^{j+1*}-\overline{p}_{jj-1}\sum_{i=1}^{i=n}\beta_i^{j*}\beta_i^{j+1*}-\overline{p}_{jj+1}\sum_{i=1}^{i=n}\beta_i^{j*}\beta_i^{j+1*}+\overline{p}_{j+1j}\sum_{i=1}^{i=n}(\beta_i^{j+1})^2$$

Orthogonality of the vector system $\left\{(\beta_1^n,\beta_2^n,\cdots,\beta_n^n)^T \quad and \quad \left(\beta_1^{j*},\beta_2^{j*},\cdots,\beta_n^{j*}\right)^T, \quad j\in\{1,2,\cdots,n\}\right\}$, allows us to conclude that:

$$\overline{p}_{j+1j}=\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{j*}\beta_i^{j+1*}>0$$

Third step:

Multiplying both sides of the equation (7.5) by β_i^{2*} and summing over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{1*}\beta_i^{2*}=-\overline{p}_{12}\sum_{i=1}^{i=n}\beta_i^{1*}\beta_i^{2*}-\overline{p}_{1e}\sum_{i=1}^{i=n}\beta_i^{1*}\beta_i^{2*}+\overline{p}_{21}\sum_{i=1}^{i=n}(\beta_i^{2*})^2$$

Orthogonality of the vector system $\left\{(\beta_1^n,\beta_2^n,\cdots,\beta_n^n)^T \quad and \quad \left(\beta_1^{j*},\beta_2^{j*},\cdots,\beta_n^{j*}\right)^T, \quad j\in\{1,2,\cdots,n\}\right\}$, allows us to conclude that:

$$\overline{p}_{21}=\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{1*}\beta_i^{2*}>0$$

Conclusion:

For all $i\in\{1,2,\cdots,n-1\} \qquad \overline{p}_{ii+1}>0 \quad and \quad \overline{p}_{i+1i}>0$ ☐

Corollary 7.2. *We can choose $t_0=t_{i1}$.*

Remark 7.3. We can remark that the matrix A is compartmental matrix.

7.1. **Identification of the system** (S_{Lin})**.**

Proposition 7.4. ***Completeness of the partial measures matrix*** [7]***:***
Let us introduce the matrix noted B_i^* defined by:

$$B_i^*=\begin{pmatrix}\beta_1^{1*} & \beta_1^{2*} & \cdots & \beta_1^{n-1*} & \beta_1^n\\ \beta_2^{1*} & \beta_2^{2*} & \cdots & \beta_2^{n-1*} & \beta_2^n\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \beta_n^{1*} & \beta_n^{2*} & \cdots & \beta_n^{n-1*} & \beta_n^n\end{pmatrix}$$

then we have:

$$\beta_k^3=\beta_k^{3*}=-\sum_{j=1}^{j=n-1}\beta_k^{j*}-\overline{p}_{1e}\frac{\beta_k^{1*}}{\lambda_k^*} \qquad \forall k\in\{1,2,\cdots,n\}. \tag{7.7}$$

Theorem 7.5. *Under the same assumptions as those of the theorem (7.1) the system (S_{Lin}) can be identified.*

Proof. Thanks to the theorem (7.1), we have:

$$\overline{p}_{j+1j}=\overline{p}_{jj+1}=\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{j*}\beta_i^{j+1*} \qquad \forall j\in\{1,2,\cdots,n-1\} \tag{7.8}$$

Using the results of the proposition (7.4) we can easily calculate the exchange coefficients of the system (S_{Lin}) except for the coefficient $\overline{p}_{1e}$.
Moreover, to identify $\overline{p}_{1e}$ we multiply both sides of the equation (7.3) by β_i^{1*} and sum over indices i from 1 to n, we get:

$$\sum_{i=1}^{i=n} \lambda_i^* (\beta_i^{1*})^2 = -\overline{p}_{12} \sum_{i=1}^{i=n} (\beta_i^{1*})^2 - \overline{p}_{1e} \sum_{i=1}^{i=n} (\beta_i^{1*})^2 + \overline{p}_{21} \sum_{i=1}^{i=n} \beta_i^{1*} \beta_i^{2*}$$

Orthogonality of the vector system $\left\{ \left(\beta_1^{j*}, \beta_2^{j*}, \cdots, \beta_n^{j*} \right)^T, \quad j \in \{1, 2, \cdots, n\} \right\}$, allows us to conclude that:

$$\sum_{i=1}^{i=n} \lambda_i^* (\beta_i^{1*})^2 = -\overline{p}_{12} \sum_{i=1}^{i=n} (\beta_i^{1*})^2 - \overline{p}_{1e} \sum_{i=1}^{i=n} (\beta_i^{1*})^2 = -\overline{p}_{12} - \overline{p}_{1e}$$

then:

$$\overline{p}_{1e} = -\overline{p}_{12} - \sum_{i=1}^{i=n} \lambda_i^* (\beta_i^{1*})^2.$$

So the excretion coefficient $\overline{p}_{1e}$ and the exchange coefficients

$$\overline{p}_{jj+1}, \quad \textit{and} \quad \overline{p}_{j+1j} \quad \forall j \in \{1, 2, \cdots, n-1\}$$

□

Remark 7.6. The quantities a_i for all $i \in \{1, 2, \cdots, n-1\}$ are known as we have measured compartments $1, 2, \cdots, n-1$.
The quantity in compartment n alone remains unknown.

Proposition 7.7. *The quantity of substance a_n is given by:*

$$a_n = \frac{1}{\overline{p}_{nn-1}} \left[(\alpha + \beta) \left(\sum_{i=1}^{i=n} \lambda_i^* \beta_i^{n*} e^{\lambda_i^* t_0} \right) + a_{n-1} \overline{p}_{n-1n} \right]$$

Proof. We have

$$x_n'(t_0) = g_n(a_1, a_2, \cdots, a_n) = y_n'(t_0) = z_n'(t_0) = \overline{z}_n'(t_0)$$

so:

$$k_{nn-1} a_{n-1}^{\beta} a_n^{\alpha} - k_{n-1n} a_n^{\beta} a_{n-1}^{\alpha} = \sum_{i=1}^{i=n} \lambda_i^* \beta_i^{n*} e^{\lambda_i^* t_0}$$

and we have

$$\overline{p}_{n-1n} = \alpha k_{n-1n} a_n^{\beta} a_{n-1}^{\alpha-1} - \beta k_{nn-1} a_{n-1}^{\beta-1} a_n^{\alpha}$$

$$\overline{p}_{nn-1} = \alpha k_{nn-1} a_{n-1}^{\beta} a_n^{\alpha-1} - \beta k_{n-1n} a_n^{\beta-1} a_{n-1}^{\alpha}$$

therefore:

$$a_n \overline{p}_{nn-1} - a_{n-1} \overline{p}_{n-1n} = (\alpha + \beta) \left(k_{nn-1} a_{n-1}^{\beta} a_n^{\alpha} - k_{n-1n} a_n^{\beta} a_{n-1}^{\alpha} \right)$$

consequently:

$$a_n = \frac{1}{\overline{p}_{nn-1}} \left[(\alpha + \beta) \left(k_{nn-1} a_{n-1}^{\beta} a_n^{\alpha} - k_{n-1n} a_n^{\beta} a_{n-1}^{\alpha} \right) + a_{n-1} \overline{p}_{n-1n} \right]$$

Finally, we have:

$$a_n = \frac{1}{\overline{p}_{nn-1}}\left[(\alpha+\beta)\left(\sum_{i=1}^{i=n}\lambda_i^*\beta_i^{n*}e^{\lambda_i^* t_0}\right)+a_{n-1}\overline{p}_{n-1n}\right]$$

□

7.2. Identification of the system (S_{N-Lin}).

Proposition 7.8. *Under the same assumptions as those of the theorem (7.1) the system (S_{N-Lin}) can be approached. More precisely:*
For all $j \in \{1, 2, \cdots, n-1\}$, we have:

$$\begin{cases} k_{1e} = \dfrac{\overline{p}_{1e}}{\alpha a_1^{\alpha-1}} \\[2ex] k_{jj+1} = \dfrac{a_j\alpha\overline{p}_{jj+1}+a_{j+1}\beta\overline{p}_{j+1j}}{(\alpha^2-\beta^2)a_{j+1}^{\beta}a_j^{\alpha}} \\[2ex] k_{j+1j} = \dfrac{a_j\beta\overline{p}_{jj+1}+a_{j+1}\alpha\overline{p}_{j+1j}}{(\alpha^2-\beta^2)a_{j+1}^{\alpha}a_j^{\beta}} \end{cases}$$

Proof. We recall that:

$$\overline{p}_{1e} = \alpha k_{1e}a_1^{\alpha-1}$$

then

$$k_{1e} = \frac{\overline{p}_{1e}}{\alpha a_1^{\alpha-1}}$$

and we recall that:

$$\begin{cases} \alpha k_{jj+1}a_{j+1}^{\beta}a_j^{\alpha-1} - \beta k_{j+1j}a_j^{\beta-1}a_{j+1}^{\alpha} = \overline{p}_{jj+1} \\[2ex] \alpha k_{j+1j}a_j^{\beta}a_{j+1}^{\alpha-1} - \beta k_{jj+1}a_{j+1}^{\beta-1}a_j^{\alpha} = \overline{p}_{j+1j} \end{cases}$$

we note that k_{jj+1} and k_{j+1j} for all $j \in \{1, 2, \cdots, n-1\}$ are the solutions of a linear algebraic system whose determinant is not equal to zero:

$$\begin{cases} k_{jj+1} = \dfrac{a_j\alpha\overline{p}_{jj+1}+a_{j+1}\beta\overline{p}_{j+1j}}{(\alpha^2-\beta^2)a_{j+1}^{\beta}a_j^{\alpha}} \\[2ex] k_{j+1j} = \dfrac{a_j\beta\overline{p}_{jj+1}+a_{j+1}\alpha\overline{p}_{j+1j}}{(\alpha^2-\beta^2)a_{j+1}^{\alpha}a_j^{\beta}} \end{cases}$$

□

8. Conclusion

In this work, we studied nonlinear compartmental systems of polynomial type, which frequently arise in population dynamics and other applied mathematical models. Our focus was on determining the exchange coefficients between compartments by measuring the amount of substance in a minimal number of compartments, rather than requiring full observation of all compartments.

We modeled the system using nonlinear differential equations derived from the law of mass conservation and proposed a method to approximate the system through a Taylor series expansion. This linear approximation allows for a computational approach to estimate the exchange coefficients efficiently.

The approach presented here demonstrates that, by injecting a known quantity of substance into the first compartment and observing the system over time, it is possible to obtain meaningful estimates of the interactions between compartments using limited measurements. This method provides a practical framework for analyzing complex compartmental systems in applied mathematics and biomathematics, can be determined from the measurements obtained from compartments 1 through $n-1$.

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