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ANALYSIS OF SIR EPIDEMIC MODEL WITH RELAPSE INCORPORATING ORNSTEIN–UHLENBECK PROCESS

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Abstract. In this work, we analyze a stochastic SIR epidemic model that incorporates the Ornstein–Uhlenbeck process, relapse, and generalized nonlinear incidence rate. First, we demonstrate that, under all favorable initial conditions, there is only one possible global solution. The new stochastic model threshold values, $\mathcal{D}_\beta$ and $\mathcal{D}_\alpha$, are created. We create the necessary conditions for the disease to persist and become extinct. Lastly, we present a few computer simulations that demonstrate the theoretical understanding we have acquired. Our findings highlight the importance of recognizing how stochasticity affects epidemic transmission, especially when complicated incidence mechanisms and stochastic environmental variables are involved.

Keywords. Stochastic SIR epidemic model, Relapse, Ornstein–Uhlenbeck process, Extinction, Persistence in mean
2020 Mathematics Subject Classification. Primary 46L55; Secondary 44B20

1. Introduction

1.1. **Background.** Among the numerous causes of death, infectious diseases rank among the most common. An infection has the potential to spread globally and become endemic in a short period of time. For this reason, infectious diseases are of great interest and concern to governments and lawmakers. Programs and research teams of scientists are attempting to regulate and eradicate these diseases. This research team includes mathematicians, who are continually proposing and discussing a variety of mathematical models. To model disease transmission in humans or animals, compartmental models divide the population into small groups of susceptible, infected, and recovered individuals ($\mathcal{S}$, $\mathcal{I}$, and $\mathcal{R}$, respectively). If a susceptible individual in the population falls ill, they join the infectious group and then transition to the recovered group once the infection period is finished. Reactivating latent infections in certain disorders, such as herpes and tuberculosis, can cause healed patients to relapse and become contagious again [5]. SIRI models can be used to model these kinds of diseases. One of the earliest epidemic models with relapse in a steady population with a

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bilinear incidence rate was created by Tuder in [24]. A nonlinear incidence rate was incorporated into the model by Moreira and Wang [19]. In order to explain the dynamics of drinking habits resulting from interactions between individuals in shared drinking environments, Sanchez et al. [21] introduced a SIRI epidemic model with nonlinear relapse. Based on the cited publications and the fact that relapse is caused by contact with the infected, it is more fair to propose a bilinear relapse rate $\alpha\mathcal{R}(t)\mathcal{I}(t)$. We examine the following SIR compartmental model in a population with a bilinear relapse rate.

$$\left\{\begin{array}{lcl} d\mathcal{S} &=& \big[b - b\mathcal{S} - \beta\mathcal{S}\times\mathcal{I}\big]\mathrm{d}t,\\ d\mathcal{I} &=& \big[-(b+\delta)\mathcal{I} + \beta\mathcal{S}\times\mathcal{I} + \alpha\mathcal{R}\times\mathcal{I}\big]\mathrm{d}t,\\ d\mathcal{R} &=& \big[-b\mathcal{R} + \delta\mathcal{I} - \alpha\mathcal{R}\times\mathcal{I}\big]\mathrm{d}t,\end{array}\right. \tag{1.1}$$

where b, β, δ, and α are positive constants that, in turn, stand for the birth and death rates, the infection coefficient, the rate at which infected individuals recover, and the rate at which immunity is lost (the relapse rate).

An essential part of mathematical epidemiology is calculating the incidence rate, which is defined as the number of new cases per unit of time for a given disease. According to [1, 7, 25], the bilinear incidence rate $\beta\mathcal{S}\mathcal{I}$ has been widely utilized in prior research. However, there are certain advantages to using a non-linear incidence rate, such as saturation and near-bilinearity. In fact, the incidence rate is most likely not linear across the entire infected and vulnerable compartment. This encourages us to select a generalized incidence rate of the kind $\beta\mathcal{S}g(\mathcal{I})$, which is more suitable for many scenarios than other incidence rates. After taking into account all the prior notations, we express the model (1.1) in the following way:

$$\left\{\begin{array}{lcl} d\mathcal{S} &=& \big[b - b\mathcal{S} - \beta\mathcal{S}\times g(\mathcal{I})\big]\mathrm{d}t,\\ d\mathcal{I} &=& \big[-(b+\delta)\mathcal{I} + \beta\mathcal{S}\times g(\mathcal{I}) + \alpha\mathcal{R}\times\mathcal{I}\big]\mathrm{d}t,\\ d\mathcal{R} &=& \big[-b\mathcal{R} + \delta\mathcal{I} - \alpha\mathcal{R}\times\mathcal{I}\big]\mathrm{d}t,\end{array}\right. \tag{1.2}$$

where the function $g(.)$ is a non-negative twice continuously differentiable function. In this work, we suppose that $g(.)$ satisfies

$$\left\{\begin{array}{l} g(0) = 0, \quad g'(0) > 0,\\ \\ -\varsigma \leq \left(\dfrac{g(x)}{x}\right)' \leq 0, \quad \forall x > 0,\end{array}\right.$$

with ς is a positive constant.

1.2. **Stochastic Model.** Stochastic differential equations are being used more and more to model population dynamics, including the spread of epidemics. This is because they take into account the randomness of birth, death, and transmission rates, which makes predictions more accurate and helps develop effective control strategies (see, for example, [2, 6, 9, 11, 13, 22, 32]). Stochastic SIR epidemic models with relapse are used to investigate infectious illness spread. These models take into account the randomness of illness transmission and the risk of relapse

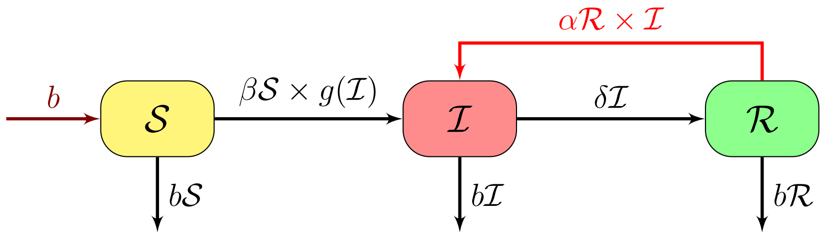

FIGURE 1. The compartmental diagram illustrates the SIR model with relapse and global incidence rate.

in recovered people. According to [3, 8, 10, 12], relapse has a major impact on dynamics, especially persistence and extinction.

A more accurate description of the disease dynamics is achieved by incorporating Ornstein–Uhlenbeck (OU) process perturbations into infectious disease models, which improves the modeling of the inherent randomness and uncertainty frequently encountered in real disease systems [4, 16, 23, 31]. In model (1.2), we assume β and α follow the OU process to determine the impact of ambient noise on transmission rate [17, 18, 28, 29]:

$$\begin{cases} d\beta(t) &= \theta_1 \times \big(\widetilde{\beta} - \beta(t)\big)\mathrm{d}t + \rho_1 d\mathcal{B}_1(t), \\ d\alpha(t) &= \theta_2 \times \big(\widetilde{\alpha} - \alpha(t)\big)\mathrm{d}t + \rho_2 d\mathcal{B}_2(t), \end{cases} \tag{1.3}$$

where θ_1, θ_2 are the rates of reversion and $\widetilde{\beta}, \widetilde{\alpha}$ are the long-run mean values of the infection rates β, α. $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent standard Brownian motion parameters defined on a complete probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$, where $\rho_i (i = 1, 2)$ denotes the intensity of $\mathcal{B}_i$. Typically, all parameters are thought to be nonnegative.

Using the stochastic integral format, we can derive the explicit form solution for the arithmetic OU process:

$$\begin{cases} \beta(t) &= \widetilde{\beta} + \big(\beta_0 - \widetilde{\beta}\big)e^{-\theta_1 t} + \rho_1 \displaystyle\int_0^t e^{-\theta_1(t-\tau)} d\mathcal{B}_1(\tau), \\ \alpha(t) &= \widetilde{\alpha} + \big(\alpha_0 - \widetilde{\alpha}\big)e^{-\theta_2 t} + \rho_2 \displaystyle\int_0^t e^{-\theta_2(t-\tau)} d\mathcal{B}_2(\tau), \end{cases} \tag{1.4}$$

with initial values $\beta_0 = \beta(0)$ and $\alpha_0 = \alpha(0)$. The term $\rho_i \int_0^t e^{-\theta_i(t-\tau)} d\mathcal{B}_i(\tau)$ is easily determined to follow the normal distribution $\mathcal{N}\left(0, \frac{\rho_i^2 \times (1-e^{-2\theta_i t})}{2\theta_i}\right)$. Thus, it is equivalent to

$$\rho_i \int_0^t e^{-\theta_i(t-\tau)} d\mathcal{B}_i(\tau) = \frac{\rho_i}{\sqrt{2\theta_i}} \times \sqrt{1 - e^{-2\theta_i t}} \frac{d\mathcal{B}_i(t)}{\mathrm{d}t} \quad a.s.$$

Consequently, (1.4) can be written as follows:

$$\begin{cases} \beta(t) &= \widetilde{\beta} + \big(\beta_0 - \widetilde{\beta}\big)e^{-\theta_1 t} + \sigma_1(t) \times \dfrac{d\mathcal{B}_1(t)}{\mathrm{d}t}, \\ \alpha(t) &= \widetilde{\alpha} + \big(\alpha_0 - \widetilde{\alpha}\big)e^{-\theta_2 t} + \sigma_2(t) \times \dfrac{d\mathcal{B}_2(t)}{\mathrm{d}t}, \end{cases} \tag{1.5}$$

where

$$\sigma_i(t) = \frac{\rho_i \times \sqrt{1 - e^{-2\theta_i t}}}{\sqrt{2\theta_i}}, \ \ i = 1, 2.$$

By substituting (1.5) into model (1.2), the stochastic SIR model with relapse and global incidence rate incorporating OU process may be obtained as follows:

$$\begin{cases} d\mathcal{S} &= \Big[b - b\mathcal{S} - \big(\widetilde{\beta} + (\beta_0 - \widetilde{\beta})e^{-\theta_1 t}\big)\mathcal{S}g(\mathcal{I})\Big]\mathrm{d}t - \sigma_1(t)\mathcal{S}g(\mathcal{I})d\mathcal{B}_1(t), \\ d\mathcal{I} &= \Big[-(b+\delta)\mathcal{I} + \big(\widetilde{\beta} + (\beta_0 - \widetilde{\beta})e^{-\theta_1 t}\big)\mathcal{S}g(\mathcal{I}) + \big(\widetilde{\alpha} + (\alpha_0 - \widetilde{\alpha})e^{-\theta_2 t}\big)\mathcal{R}\mathcal{I}\Big]\mathrm{d}t \\ & \quad +\sigma_1(t)\mathcal{S}g(\mathcal{I})d\mathcal{B}_1(t) + \sigma_2(t)\mathcal{R}\mathcal{I}d\mathcal{B}_2(t), \\ d\mathcal{R} &= \Big[-b\mathcal{R} + \delta\mathcal{I} - \big(\widetilde{\alpha} + (\alpha_0 - \widetilde{\alpha})e^{-\theta_2 t}\big)\mathcal{R}\mathcal{I}\Big]\mathrm{d}t - \sigma_2(t)\mathcal{R}\mathcal{I}d\mathcal{B}_2(t). \end{cases} \tag{1.6}$$

2. Preliminaries

In the present work, we define $\mathbb{R}^d_+ = \big\{(v_1, ..., v_d) \mid v_1 > 0, ..., \ v_d > 0\big\}$. This is one way to express the system (1.6):

$$d\mathcal{X} = f(\mathcal{X})\,\mathrm{d}t + \ell_1(\mathcal{X})\,d\mathcal{B}_1 + \ell_2(\mathcal{X})\,d\mathcal{B}_2, \tag{2.1}$$

where $\mathcal{X} = \big(\mathcal{S}, \mathcal{I}, \mathcal{R}\big)$. The diffusion matrix is defined as follows:

$$M(\mathcal{X}) = \big(m_{kj}(\mathcal{X})\big), \ \ m_{kq}(\mathcal{X}) = \ell_{1,k}(\mathcal{X})\ell_{1,q}(\mathcal{X}) + \ell_{2,k}(\mathcal{X})\ell_{2,q}(\mathcal{X}).$$

Furthermore, we have included a generator $\mathcal{L}$ that is linked to (2.1) so that it can be used in later parts of this research. For any φ that is twice continuously differentiable,

$$\mathcal{L}\varphi(\mathcal{X}) = \sum_{k=1}^{3} f_k(\mathcal{X})\frac{\partial\varphi(\mathcal{X})}{\partial\mathcal{X}_k} + \frac{1}{2}\sum_{k,q=1}^{3} m_{kq}(\mathcal{X})\frac{\partial^2\varphi(\mathcal{X})}{\partial\mathcal{X}_k\partial\mathcal{X}_q}.$$

The existence and uniqueness of a global solution for model (1.6) will be covered in this section. The following subset must be specified before we can start investigating the model:

$$\mathcal{O} = \Big\{\mathcal{X} = \big(v_1, v_2, v_3\big) \in \mathbb{R}^3_+; \ \ v_1 + v_2 + v_3 = 1\Big\}.$$

Theorem 2.1. *Assume that $\mathcal{X}(0) \in \mathcal{O}$. Then $\mathbb{P}\big(\mathcal{X}(t) \in \mathcal{O}\big) = 1$ for all $t \geq 0$. This evidence supports the claim that the set $\mathcal{O}$ is almost surely positively invariant by the system (1.6).*

Proof. Let $\mathcal{X}(0) = (\mathcal{S}(0), \mathcal{I}(0), \mathcal{R}(0)) \in \mathcal{O}$ and $t \geq 0$. In system (1.6), we can readily demonstrate that the total population $N = \mathcal{S} + \mathcal{R} + \mathcal{I}$ verifies on $[0, t]$,

$$dN(v) = b\,(1 - N(v))\,\mathrm{d}v \ \ a.s.,$$

which implies by integration

$$N(v) - 1 = (N(0) - 1)e^{-bv} \qquad \text{for all} \quad v \in [0, t].$$

Hence for $\mathcal{X}(0) = \big(\mathcal{S}(0), \mathcal{I}(0), \mathcal{R}(0)\big) \in \mathcal{O}$ we have $N(v) - 1 = 0$ and then, if

$$\mathcal{X}(v) = (\mathcal{S}(v), \mathcal{I}(v), \mathcal{R}(v)) \in \mathbb{R}^3_+ \ \ \text{for all} \quad v \in [0, t] \ \ a.s.,$$

we get

$$\mathcal{S}(v)+\mathcal{R}(v)+\mathcal{I}(v)=1, \text{ and } \mathcal{S}(v),\ \mathcal{R}(v),\ \mathcal{I}(v)\in(0,1) \text{ for all } \ v\in[0,t]\ a.s. \tag{2.2}$$

Since the coefficients of (1.6) are locally Lipschitz-continuous, for any given initial value $\mathcal{X}(0)=(\mathcal{S}(0),\mathcal{I}(0),\mathcal{R}(0))\in\mathcal{O}$, there is a unique maximal local solution $\mathcal{X}(t)=(\mathcal{S}(t),\ \mathcal{I}(t),\ \mathcal{R}(t))$ for $t\in[0,\nu_e)$, where ν_e is the explosion time. Let $q_0\in\mathbb{N}$ such that $\mathcal{S}(0),\mathcal{R}(0),\mathcal{I}(0)\in[\frac{1}{q_0},q_0]$. For every integer $q>q_0$ considering the stopping times

$$\nu_q=\inf\Big\{t\in[0,\nu_e):\min\big\{\mathcal{R}(t),\mathcal{I}(t),\mathcal{S}(t)\big\}\le\frac{1}{q} \text{ or } \max\big\{\mathcal{R}(t),\mathcal{I}(t),\mathcal{S}(t)\big\}\ge q\Big\}.$$

Let us define the functions for $\mathcal{X}=(\mathcal{S},\mathcal{I},\mathcal{R})\in\mathbb{R}^3_+$:

$$\varphi\big(\mathcal{X}\big)=(-\log\mathcal{S}+\mathcal{S}-1)+(-\log\mathcal{I}+\mathcal{I}-1)+(-\log\mathcal{R}+\mathcal{R}-1).$$

φ is nonnegative function. Itô's formula gives us for any $v\in[0,t\wedge\nu_\varepsilon]$:

$$d\varphi(\mathcal{X})=\mathcal{L}\varphi(\mathcal{X})\mathrm{d}v+\sigma_1(v)\left(\frac{1}{\mathcal{S}}-\frac{1}{\mathcal{I}}\right)\mathcal{S}g(\mathcal{I})d\mathcal{B}_1(v)+\sigma_2(v)\left(\frac{1}{\mathcal{R}}-\frac{1}{\mathcal{I}}\right)\mathcal{R}\mathcal{I}d\mathcal{B}_2(v), \tag{2.3}$$

where

$$\begin{aligned}\mathcal{L}\varphi(\mathcal{X}) \;&=\; \left(1-\frac{1}{\mathcal{S}}\right)\Big(b-b\mathcal{S}-\big(\widetilde{\beta}+(\beta_0-\widetilde{\beta})e^{-\theta_1 v}\big)\mathcal{S}g(\mathcal{I})\Big)+\frac{1}{2}\sigma_1^2(v)(g(\mathcal{I}))^2\\ &\quad\left(1-\frac{1}{\mathcal{I}}\right)\Big(-(b+\delta)\mathcal{I}+\big(\widetilde{\beta}+(\beta_0-\widetilde{\beta})e^{-\theta_1 v}\big)\mathcal{S}g(\mathcal{I})\\ &\quad+\big(\widetilde{\alpha}+(\alpha_0-\widetilde{\alpha})e^{-\theta_2 v}\big)\mathcal{R}\mathcal{I}\Big)+\frac{1}{2}\sigma_1^2(v)\left(\mathcal{S}\frac{g(\mathcal{I})}{\mathcal{I}}\right)^2+\frac{1}{2}\sigma_2^2(v)\mathcal{R}^2\\ &\quad\left(1-\frac{1}{\mathcal{R}}\right)\Big(-b\mathcal{R}+\delta\mathcal{I}-\big(\widetilde{\alpha}+(\alpha_0-\widetilde{\alpha})e^{-\theta_2 v}\big)\mathcal{R}\mathcal{I}\Big)+\frac{1}{2}\sigma_2^2(v)\mathcal{I}^2.\end{aligned}$$

By (2.2), we assert that $\mathcal{S}(v),\mathcal{R}(v),\mathcal{I}(v)\in(0,1)$ for all $v\in[0,t\wedge\nu_\varepsilon]$ a.s. Using the inequalities $\big(\widetilde{\beta}+(\beta_0-\widetilde{\beta})e^{-\theta_1 v}\big)\le\beta_0\vee\widetilde{\beta}$, $\big(\widetilde{\alpha}+(\alpha_0-\widetilde{\alpha})e^{-\theta_2 v}\big)\le\alpha_0\vee\widetilde{\alpha}$, $\sigma_1^2(v)\le\frac{\rho_1^2}{2\theta_1}$ and $\sigma_2^2(v)\le\frac{\rho_2^2}{2\theta_2}$, we easily obtain

$$\mathcal{L}\varphi(\mathcal{X}) \;\le\; 4b+2\delta+(\beta_0\vee\widetilde{\beta})g(\mathcal{I})+\frac{\rho_1^2}{4\theta_1}(g(\mathcal{I}))^2+\frac{\rho_1^2}{4\theta_1}\left(\frac{g(\mathcal{I})}{\mathcal{I}}\right)^2+\alpha_0\vee\widetilde{\alpha}+\frac{\rho_2^2}{2\theta_2}$$

Since

$$\left(\frac{g(\mathcal{I})}{\mathcal{I}}\right)'\le0,\ \ \forall\mathcal{I}>0.$$

Thus $\frac{g(\mathcal{I})}{\mathcal{I}}\le g'(0)$, which implies $g(\mathcal{I})\le g'(0)\times\mathcal{I}$, thus one gets

$$\mathcal{L}\varphi(\mathcal{X}) \;\le\; 4b+2\delta+(\beta_0\vee\widetilde{\beta})\times g'(0)+\frac{\rho_1^2}{2\theta_1}(g'(0))^2+\alpha_0\vee\widetilde{\alpha}+\frac{\rho_2^2}{2\theta_2}. \tag{2.4}$$

From (2.3) and (2.4), we have

$$d\varphi(\mathcal{X}) \;\le\; K\mathrm{d}v+\sigma_1(v)\times\left(g(\mathcal{I})-\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\right)d\mathcal{B}_1(v)+\sigma_2(v)\times(\mathcal{I}-\mathcal{R})\,d\mathcal{B}_2(v),$$

where $K \triangleq 4b + 2\delta + (\beta_0 \vee \widetilde{\beta}) \times g'(0) + \frac{\rho_1^2}{2\theta_1}(g'(0))^2 + \alpha_0 \vee \widetilde{\alpha} + \frac{\rho_2^2}{2\theta_2}$. Thus, integration leads to the following:

$$\varphi(\mathcal{X}(v)) \;\; \leq \;\; \varphi(\mathcal{X}(0)) + Kv + \Upsilon_1(v) + \Upsilon_2(v),$$

where

$$\Upsilon_1(v) = \int_0^v \sigma_1(s) \left(g(\mathcal{I}) - \frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}} \right) d\mathcal{B}_1(s),$$

and

$$\Upsilon_2(v) = \int_0^v \sigma_2(s) \left(\mathcal{I}(s) - \mathcal{R}(s) \right) d\mathcal{B}_2(s).$$

Hence, $\Upsilon_1(v)$ and $\Upsilon_2(v)$ are real-valued local martingales with

$$\mathbb{E}(\Upsilon_1(v)) = \mathbb{E}(\Upsilon_2(v)) = 0.$$

Then

$$\mathbb{E}\big(\varphi(\mathcal{X}(t \wedge \nu_q))\big) \leq \varphi(\mathcal{X}(0)) + K(t \wedge \nu_q). \tag{2.5}$$

This, together with (2.2), gives that

$$\mathbb{E}\Big(\varphi(\mathcal{X}(\nu_q))\mathbb{1}_{(\nu_q \leq t)}\Big) \leq \mathbb{E}\Big(\varphi(\mathcal{X}(t \wedge \nu_q))\Big) \leq \varphi(\mathcal{X}(0)) + Kt. \tag{2.6}$$

Suppose that $\nu_e < \infty$, then there is $t > 0$ such that $\mathbb{P}\left(\nu_e < t\right) > 0$. Hence, for any $q > q_0$ we have $\mathbb{P}\left(\nu_q < t\right) > 0$. By the definition of ν_q, we have

$$\left\{ (q - 1 - \log q) \wedge \left(\frac{1}{q} - 1 - \log \frac{1}{q} \right) \right\} \leq \varphi(\mathcal{X}(0)) + Kt.$$

Letting $q \longrightarrow \infty$, yields $\infty = \varphi(\mathcal{X}(0)) + Kt < \infty$. Then, $\mathbb{P}\left(\nu_e \leq t\right) = 0$ for all $t \geq 0$. Hence $\mathbb{P}\left(\nu_e = \infty\right) = 1$, consequently the solution to system (1.6) is positive and global. The proof of the theorem is thus completed. □

3. Disease extinction

In this section, we will look at the criteria necessary for the disease to become extinct. Initially, we shall consider the two positive numbers:

$$\mathcal{D}_\beta = \frac{\widetilde{\beta} \times g'(0)}{\delta + b + \frac{(\rho_1 g'(0))^2}{4\theta_1}}, \qquad \mathcal{D}_\alpha = \frac{\widetilde{\alpha}}{\delta + b + \frac{\rho_2^2}{4\theta_2}}.$$

Theorem 3.1. *Let $\mathcal{X} = (\mathcal{S}, \mathcal{I}, \mathcal{R})$ be the solution of the system (1.6), starting with $\mathcal{X}(0) \in \mathcal{O}$. If even one of the following assumptions are true*

$$(\mathcal{C}_1) \qquad \mathcal{D}_\beta < 1, \;\; \widetilde{\alpha} \geq \frac{\rho_2^2}{2\theta_2} \;\; \textit{and} \;\; \widetilde{\beta} - \widetilde{\alpha} \geq \frac{\rho_1^2}{2\theta_1} g'(0),$$

$$(\mathcal{C}_2) \qquad \mathcal{D}_\alpha < 1, \;\; \widetilde{\beta} \geq \frac{\rho_1^2}{2\theta_1} g'(0) \;\; \textit{and} \;\; \widetilde{\alpha} - \widetilde{\beta} g'(0) \geq \frac{\rho_2^2}{2\theta_2},$$

then

$$\mathbb{P}\left(\limsup_{t \longrightarrow \infty} \frac{\log\left(\mathcal{I}(t)\right)}{t} < 0 \right) = 1.$$

Another way of putting it is that the disease should fade away at an exponential rate.

Proof. Using the Itô formula on $\mathcal{I} \mapsto \log(\mathcal{I})$, We are able to obtain:

$$\begin{aligned}
d\log(\mathcal{I}) &= \Bigg[-(\delta+b) + \big(\widetilde{\beta} + (\beta_0 - \widetilde{\beta})e^{-\theta_1 t}\big)\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big) - \frac{1}{2}\sigma_1^2(t)\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2 \\
&\quad + \big(\widetilde{\alpha} + (\alpha_0 - \widetilde{\alpha})e^{-\theta_2 t}\big)\mathcal{R} - \frac{1}{2}\sigma_2^2(t)\mathcal{R}^2 \Bigg] \mathrm{d}t \\
&\quad + \sigma_1(t)\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big) d\mathcal{B}_1(t) + \sigma_2(t)\mathcal{R} d\mathcal{B}_2(t) \\
&= \Bigg[-(\delta+b) + \widetilde{\beta}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big) - \frac{\rho_1^2}{4\theta_1}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2 + \widetilde{\alpha}\mathcal{R} - \frac{\rho_2^2}{4\theta_2}\mathcal{R}^2 \\
&\quad + (\beta_0 - \widetilde{\beta})e^{-\theta_1 t}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big) + \frac{\rho_1^2}{4\theta_1}e^{-2\theta_1 t}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2 + (\alpha_0 - \widetilde{\alpha})e^{-\theta_2 t}\mathcal{R} \\
&\quad + \frac{\rho_2^2}{4\theta_2}e^{-2\theta_2 t}\mathcal{R}^2 \Bigg] \mathrm{d}t + \sigma_1(t)\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big) d\mathcal{B}_1(t) + \sigma_2(t)\mathcal{R} d\mathcal{B}_2(t) \\
&= \Bigg[\Psi\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}, \mathcal{R}\Big) + G\Big(t, \frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}, \mathcal{R}\Big) \Bigg] \mathrm{d}t \\
&\quad + \sigma_1(t)\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big) d\mathcal{B}_1(t) + \sigma_2(t)\mathcal{R} d\mathcal{B}_2(t), \qquad (3.1)
\end{aligned}$$

where we set

$$\begin{cases}
\Psi(v_1, v_2) = -(b+\delta) + \widetilde{\beta}v_1 - \dfrac{\rho_1^2}{4\theta_1}v_1^2 + \widetilde{\alpha}v_2 - \dfrac{\rho_2^2}{4\theta_2}v_2^2, \\
G(t, v_1, v_2) = (\beta_0 - \widetilde{\beta})e^{-\theta_1 t}v_1 + \dfrac{\rho_1^2}{4\theta_1}e^{-2\theta_1 t}v_1^2 + (\alpha_0 - \widetilde{\alpha})e^{-\theta_2 t}v_2 + \dfrac{\rho_2^2}{4\theta_2}e^{-2\theta_2 t}v_2^2.
\end{cases} \qquad (3.2)$$

Division by t after integrating from 0 to t yields

$$\begin{aligned}
\frac{\log\big(\mathcal{I}(t)\big)}{t} &= \frac{1}{t}\int_0^t \Psi\Big(\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}, \mathcal{R}(\tau)\Big)\mathrm{d}\tau + \frac{\log\big(\mathcal{I}(0)\big)}{t} \\
&\quad + \frac{1}{t}\int_0^t G\Big(\tau, \frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}, \mathcal{R}(\tau)\Big)\mathrm{d}\tau + \frac{\Upsilon_3(t)}{t} + \frac{\Upsilon_4(t)}{t}, \quad (3.3)
\end{aligned}$$

where

$$\Upsilon_3(t) = \int_0^t \sigma_1(\tau) \times \tfrac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)} d\mathcal{B}_1(\tau) \text{ and } \Upsilon_4(t) = \int_0^t \sigma_2(\tau) \times \mathcal{R}(\tau) d\mathcal{B}_2(\tau). \quad (3.4)$$

Assuming $(\mathcal{C}_1)$ holds, since $\widetilde{\alpha} \geq \frac{\rho_2^2}{2\theta_2}$, the function $v_2 \mapsto \Psi(., v_2)$ is increasing on $(0,1)$. Consequently, from $\mathcal{R} \leq 1 - \mathcal{S} \leq 1 - \dfrac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}g'(0)}$, we obtain:

$$\begin{aligned}
\Psi\Big(\frac{\mathcal{S}\times g(\mathcal{I})}{\mathcal{I}}, \mathcal{R}\Big) &\leq \Psi\Big(\frac{\mathcal{S}\times g(\mathcal{I})}{\mathcal{I}}, 1 - \frac{\mathcal{S}g(\mathcal{I})}{g'(0)\times \mathcal{I}}\Big)\\
&= -\frac{\rho_2^2}{4\theta_2} + \widetilde{\alpha} - (b+\delta) + \left(\frac{\rho_2^2}{2\theta_2 g'(0)} - \frac{\widetilde{\alpha}}{g'(0)} + \widetilde{\beta}\right)\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\\
&\quad - \left(\frac{\rho_1^2}{4\theta_1} + \frac{\rho_2^2}{4\theta_2 (g'(0))^2}\right) \times \Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2\\
&\triangleq H\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big). \qquad (3.5)
\end{aligned}$$

Since $\widetilde{\beta} - \widetilde{\alpha} \geq \frac{\rho_1^2}{2\theta_1} g'(0)$, the function H is increasing on $(0, g'(0))$, which gives $H\big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\big) \leq H(g'(0))$. By substituting this inequality in (3.5), we are able to derive

$$\begin{aligned}
\Psi\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}, \mathcal{R}\Big) &\leq -\frac{\rho_1^2}{4\theta_1}(g'(0))^2 + \widetilde{\beta}g'(0) - (b+\delta)\\
&= \left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2 + b + \delta\right)\big(\mathcal{D}_\beta - 1\big). \qquad (3.6)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\int_0^t G\Big(\tau, \frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}, \mathcal{R}(\tau)\Big)\mathrm{d}\tau\right|\\
\leq\ & |\beta_0 - \widetilde{\beta}|g'(0)\int_0^t e^{-\theta_1\tau}\mathrm{d}\tau + \frac{\rho_1^2}{4\theta_1}(g'(0))^2\int_0^t e^{-2\theta_1\tau}\mathrm{d}\tau\\
&+|\alpha_0 - \widetilde{\alpha}|\int_0^t e^{-\theta_2\tau}\mathrm{d}\tau + \frac{\rho_2^2}{4\theta_2}\int_0^t e^{-2\theta_2\tau}\mathrm{d}\tau\\
\leq\ & \frac{g'(0)}{\theta_1}|\beta_0 - \widetilde{\beta}| + \frac{\rho_1^2}{8\theta_1^2}(g'(0))^2 + \frac{1}{\theta_2}|\alpha_0 - \widetilde{\alpha}| + \frac{\rho_2^2}{8\theta_2^2},
\end{aligned}$$

which implies that

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t G\Big(\tau, \frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}, \mathcal{R}(\tau)\Big)\mathrm{d}\tau = 0. \qquad (3.7)$$

Since $\Upsilon_3(t)$ and $\Upsilon_4(t)$ are real-valued local martingales having the following quadratic variation

$$< \Upsilon_3(t), \Upsilon_3(t) > = \int_0^t \sigma_1^2(\tau)\Big(\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}\Big)^2\mathrm{d}\tau \ \leq\ \frac{\rho_1^2}{2\theta_1}(g'(0))^2 t \ <\ \infty, \qquad (3.8)$$

and

$$< \Upsilon_4(t), \Upsilon_4(t) > = \int_0^t \sigma_2^2(\tau)\mathcal{R}^2(\tau)\mathrm{d}\tau \ \leq\ \frac{\rho_2^2}{2\theta_2}t \ <\ \infty. \qquad (3.9)$$

We have by using the large number theorem for martingales [15] along with (3.8) and (3.9) that we can say the following:

$$\lim_{t\to\infty}\frac{\Upsilon_3(t)}{t}=0 \text{ and } \lim_{t\to\infty}\frac{\Upsilon_4(t)}{t}=0. \tag{3.10}$$

Then, from (3.3), (3.6), (3.7), (3.10), and the condition $\mathcal{D}_\beta<1$, we obtain

$$\limsup_{t\to\infty}\frac{\log\big(\mathcal{I}(t)\big)}{t}\leq\left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+b+\delta\right)\big(\mathcal{D}_\beta-1\big)<0.$$

Now, if $(\mathcal{C}_2)$ holds, in light of $\widetilde{\beta}\geq\frac{\rho_1^2}{2\theta_1}g'(0)$, the function $v_1\mapsto\Psi(v_1,.)$ is increasing on $(0,g'(0))$. Consequently, from $\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\leq\mathcal{S}g'(0)\leq(1-\mathcal{R})g'(0)$, we derive that

$$\begin{aligned}\Psi\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}},\mathcal{R}\Big) &\leq \Psi\Big((1-\mathcal{R})g'(0),\mathcal{R}\Big)\\ &= -\left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+\frac{\rho_2^2}{4\theta_2}\right)\mathcal{R}^2+\left(\frac{\rho_1^2}{2\theta_1}(g'(0))^2-\widetilde{\beta}g'(0)+\widetilde{\alpha}\right)\mathcal{R}\\ &\quad -\frac{\rho_1^2}{4\theta_1}(g'(0))^2+\widetilde{\beta}g'(0)-(b+\delta)\\ &\triangleq Q(\mathcal{R}).\end{aligned} \tag{3.11}$$

In light of the fact that $\widetilde{\alpha}-\widetilde{\beta}g'(0)\geq\frac{\rho_2^2}{2\theta_2}$, the function Q is increasing on $(0,1)$, so $Q(\mathcal{R})\leq Q(1)$. We put this inequality into (3.11) to get

$$\Psi\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}},\mathcal{R}\Big) \leq \left(-\frac{\rho_2^2}{4\theta_2}+\widetilde{\alpha}-(b+\delta)\right)=\left(\frac{\rho_2^2}{4\theta_2}+b+\delta\right)\big(\mathcal{D}_\alpha-1\big). \tag{3.12}$$

Using (3.3), (3.7), (3.10), (3.12), and the condition $\mathcal{D}_\alpha<1$, we obtain

$$\limsup_{t\longrightarrow\infty}\frac{\log\big(\mathcal{I}(t)\big)}{t}\leq\left(\frac{\rho_2^2}{4\theta_2}+\delta+b\right)\big(\mathcal{D}_\alpha-1\big)<0.$$

The proof is so completed. □

4. Persistence in mean

In epidemic models, extinction and persistence of diseases are the most fascinating topics to research. We discussed the first one in Section 3. Following this section, we shall show that the disease is persistent for a long time. According to reference [26], the following lemma is required.

Lemma 4.1. *Assume that $Y\in\mathcal{C}(\mathbb{R}_+\times\Omega;\mathbb{R}_+)$ and $\mathfrak{M}\in\mathcal{C}(\mathbb{R}_+\times\Omega;\mathbb{R})$ such that $\lim_{t\longrightarrow\infty}\frac{\mathfrak{M}(t)}{t}=0$. If for all $t\geq0$, $\log\big(Y(t)\big)\geq\pi_1t-\pi_2\int_0^tY(\tau)d\tau+\mathfrak{M}(t)$. Then,*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^tY(\tau)d\tau\geq\frac{\pi_1}{\pi_2}\ a.s,$$

π_1 and π_2 are both positive constants.

Theorem 4.2. *If $\mathcal{D}_\beta > 1$ and $\widetilde{\alpha} \geq \frac{\rho_2^2}{2\theta_2}$, so that, starting with any given value $\mathcal{X}(0) = (\mathcal{S}(0), \mathcal{I}(0), \mathcal{R}(0)) \in \mathcal{O}$, the solution for system (1.6) satisfies the property that*

$$\begin{aligned}
(i)\ \liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t \mathcal{S}(\tau)d\tau &\geq \frac{b}{(\beta_0 \vee \widetilde{\beta})g'(0)+b},\\
(ii)\ \liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t \mathcal{I}(\tau)d\tau &\geq \zeta,\\
(iii)\ \liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t \mathcal{R}(\tau)d\tau &\geq \frac{\delta\times\zeta}{\big(b+\alpha_0\vee\widetilde{\alpha}\big)},
\end{aligned}$$

for some positive constant ζ.

Proof. (i) Based on $\widetilde{\beta} + (\beta_0 - \widetilde{\beta})e^{-\theta_1 t} \leq \beta_0 \vee \widetilde{\beta}$ and the first equation in (1.6), we may deduce that

$$d\mathcal{S} + \big[bS + (\beta_0 \vee \widetilde{\beta})\mathcal{S}g(\mathcal{I})\big]\mathrm{dt} \ \geq\ b\mathrm{dt} - \sigma_1\mathcal{S}g(\mathcal{I})d\mathcal{B}_1.$$

Using $\mathcal{I} \in (0,1)$ and $\frac{g(\mathcal{I})}{\mathcal{I}} \leq g'(0)$, integrating the above inequality and dividing both sides by t, we get

$$\begin{aligned}
\frac{b+(\beta_0\vee\widetilde{\beta})g'(0)}{t}\int_0^t \mathcal{S}(\tau)\mathrm{d}\tau &\geq b - \frac{\mathcal{S}(t)-\mathcal{S}(0)}{t} - \frac{1}{t}\int_0^t \sigma_1(\tau)\mathcal{S}(\tau)g(\mathcal{I}(\tau))d\mathcal{B}_1(\tau)\\
&= b - \frac{\mathcal{S}(t)-\mathcal{S}(0)}{t} - \frac{\Upsilon_5(t)}{t}, \qquad (4.1)
\end{aligned}$$

where

$$\Upsilon_5(t) \triangleq \int_0^t \sigma_1(\tau)\mathcal{S}(\tau)g(\mathcal{I}(\tau))d\mathcal{B}_1(\tau). \qquad (4.2)$$

$\Upsilon_5(t)$ is real-valued local martingales having the following quadratic variation

$$<\Upsilon_5(t),\Upsilon_5(t)> \ =\ \int_0^t \sigma_1^2(\tau)\mathcal{S}^2(\tau)(g(\mathcal{I}(\tau)))^2\mathrm{d}\tau \ \leq\ \frac{\rho_1^2}{2\theta_1}(g'(0))^2 t \ <\ \infty.$$

From the strong law of large numbers for local martingales and the fact that $\mathcal{S} \in (0,1)$, we obtain

$$\lim_{t\longrightarrow\infty}\left(-\frac{\mathcal{S}(t)-\mathcal{S}(0)}{t} - \frac{\Upsilon_5(t)}{t}\right) = 0, \qquad (4.3)$$

which, when combined with (4.1), produces

$$\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t \mathcal{S}(\tau)\mathrm{d}\tau \ \geq\ \frac{b}{b+(\beta_0\vee\widetilde{\beta})g'(0)}.$$

(ii) From the formula (3.3), we can obtain

$$\begin{aligned}
\log\big(\mathcal{I}(t)\big) &= \log\big(\mathcal{I}(0)\big) + \int_0^t \Psi\Big(\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}, \mathcal{R}(\tau)\Big)\mathrm{d}\tau\\
&\quad + \int_0^t G\Big(\tau, \frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}, \mathcal{R}(\tau)\Big)\mathrm{d}\tau + \Upsilon_3(t) + \Upsilon_4(t). \qquad (4.4)
\end{aligned}$$

Considering that $\mathcal{S},\mathcal{R}\in(0,1)$ and $\frac{g(\mathcal{I})}{\mathcal{I}}\leq g'(0)$, one can conclude

$$\begin{aligned}
\Psi\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}},\mathcal{R}\Big) &= -(b+\delta)+\widetilde{\beta}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)+\widetilde{\alpha}\mathcal{R}-\frac{1}{2}\left(\frac{\rho_1^2}{2\theta_1}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2+\frac{\rho_2^2}{2\theta_2}\mathcal{R}^2\right)\\
&\geq -(b+\delta)+\widetilde{\beta}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)-\frac{\rho_1^2}{4\theta_1}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2+\left(\widetilde{\alpha}-\frac{\rho_2^2}{4\theta_2}\right)\mathcal{R}^2\\
&= -(b+\delta)+\widetilde{\beta}g'(0)-\frac{\rho_1^2}{4\theta_1}(g'(0))^2+\left(\widetilde{\alpha}-\frac{\rho_2^2}{4\theta_2}\right)\mathcal{R}^2\\
&\quad +\widetilde{\beta}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)-\widetilde{\beta}g'(0)-\frac{\rho_1^2}{4\theta_1}\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)^2+\frac{\rho_1^2}{4\theta_1}(g'(0))^2\\
&\geq -(b+\delta)+\widetilde{\beta}g'(0)-\frac{\rho_1^2}{4\theta_1}(g'(0))^2+\left(\widetilde{\alpha}-\frac{\rho_2^2}{4\theta_2}\right)\mathcal{R}^2\\
&\quad -\Big(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\Big)\Big(g'(0)-\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)\\
&= \left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+b+\delta\right)\big(\mathcal{D}_\beta-1\big)+\left(\widetilde{\alpha}-\frac{\rho_2^2}{4\theta_2}\right)\mathcal{R}^2\\
&\quad -\left(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\right)\left(g'(0)-\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\right).
\end{aligned}$$

From $\widetilde{\alpha}\geq\frac{\rho_2^2}{2\theta_2}$, we have

$$\begin{aligned}
\Psi\Big(\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}},\mathcal{R}\Big) &\geq \left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+\delta+b\right)\big(\mathcal{D}_\beta-1\big)\\
&\quad -\left(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\right)\left(g'(0)-\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\right).
\end{aligned}$$

If we insert another inequality into (4.4); we get

$$\begin{aligned}
\log\big(\mathcal{I}(t)\big) &\geq t\left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+\delta+b\right)\big(\mathcal{D}_\beta-1\big)+\log\big(\mathcal{I}(0)\big)\\
&\quad -\left(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\right)\int_0^t\Big(g'(0)-\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}\Big)\mathrm{d}\tau \qquad (4.5)\\
&\quad +\int_0^t G\Big(\tau,\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)},\mathcal{R}(\tau)\Big)\mathrm{d}\tau+\Upsilon_3(t)+\Upsilon_4(t).
\end{aligned}$$

From the first equation in the system (1.6); we get

$$d\mathcal{S}\geq\left[\frac{b}{g'(0)}\Big(g'(0)-\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)-b\mathcal{S}\Big(1-\frac{g(\mathcal{I})}{\mathcal{I}g'(0)}\Big)-(\beta_0\vee\widetilde{\beta})g'(0)\mathcal{I}\right]\mathrm{d}t-\sigma_1\mathcal{S}g(\mathcal{I})d\mathcal{B}_1(t).$$

Take note of that

$$-\varsigma\leq\left(\frac{g(v)}{v}\right)'=\frac{\dfrac{g(v)}{v}-\lim\limits_{v\to0}\dfrac{g(v)}{v}}{v}=\frac{\dfrac{g(v)}{v}-g'(0)}{v}\leq 0.$$

We acquire that

$$g'(0)-\frac{g(\mathcal{I})}{\mathcal{I}}\leq \varsigma\mathcal{I}.$$

As a result, the inequality that follows

$$b\mathcal{S}\Big(1-\frac{g(\mathcal{I})}{\mathcal{I}g'(0)}\Big)\leq \frac{b\varsigma}{g'(0)}\mathcal{I}.$$

Which implies

$$d\mathcal{S} \;\geq\; \left[\frac{b}{g'(0)}\Big(g'(0)-\frac{\mathcal{S}g(\mathcal{I})}{\mathcal{I}}\Big)-\Big(\frac{b\varsigma}{g'(0)}+(\beta_0\vee\widetilde{\beta})g'(0)\Big)\mathcal{I}\right]\mathrm{d}t-\sigma_1\mathcal{S}g(\mathcal{I})d\mathcal{B}_1(t),$$

then

$$\begin{aligned}-\int_0^t\Big(g'(0)-\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)}\Big)\mathrm{d}\tau \;&\geq\; \frac{g'(0)}{b}\Bigg\{-\Big(\frac{b\varsigma}{g'(0)}+(\beta_0\vee\widetilde{\beta})g'(0)\Big)\int_0^t\mathcal{I}(\tau)\mathrm{d}\tau\\ &\qquad \mathcal{S}(0)-\mathcal{S}(t)-\Upsilon_5(t)\Bigg\}. \qquad (4.6)\end{aligned}$$

We get by substituting (4.6) into (4.5) and rearranging

$$\begin{aligned}\log\big(\mathcal{I}(t)\big) \;&\geq\; \left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+b+\delta\right)\times\big(\mathcal{D}_\beta-1\big)t\\ &\quad -\frac{g'(0)}{b}\Big(\frac{b\varsigma}{g'(0)}+(\beta_0\vee\widetilde{\beta})g'(0)\Big)\left(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\right)\int_0^t\mathcal{I}(\tau)\mathrm{d}\tau+\mathfrak{M}(t),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{M}(t) \;&\triangleq\; -\frac{g'(0)}{b}\left(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\right)\Big(\mathcal{S}(t)-\mathcal{S}(0)+\Upsilon_5(t)\Big)+\log\big(\mathcal{I}(0)\big)\\ &\quad +\int_0^t G\Big(\tau,\frac{\mathcal{S}(\tau)g(\mathcal{I}(\tau))}{\mathcal{I}(\tau)},\mathcal{R}(\tau)\Big)\mathrm{d}\tau+\Upsilon_3(t)+\Upsilon_4(t).\end{aligned}$$

Moreover, from (3.7), (3.10) and (4.3), we obtain $\lim\limits_{t\longrightarrow\infty}\frac{\mathfrak{M}(t)}{t}=0$ a.s. When paired with Lemma 4.1, this leads to

$$\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t\mathcal{I}(\tau)\mathrm{d}\tau \;\geq\; \zeta,$$

where

$$\zeta \;=\; \frac{\left(\frac{\rho_1^2}{4\theta_1}(g'(0))^2+b+\delta\right)\big(\mathcal{D}_\beta-1\big)}{\frac{g'(0)}{b}\Big(\frac{b\varsigma}{g'(0)}+(\beta_0\vee\widetilde{\beta})g'(0)\Big)\left(\widetilde{\beta}-\frac{\rho_1^2}{4\theta_1}g'(0)\right)} \;>\; 0.$$

(iii) Integrating the third equation of the system (1.6) yields

$$\begin{aligned}\mathcal{R}(t)-\mathcal{R}(0) &= -\int_0^t \big(\widetilde{\alpha}+(\alpha_0-\widetilde{\alpha})e^{-\theta_2 t}\big)\mathcal{R}(\tau)\mathcal{I}(\tau)\mathrm{d}\tau - b\int_0^t \mathcal{R}(\tau)\mathrm{d}\tau \\ &\quad +\delta\int_0^t \mathcal{I}(\tau)\mathrm{d}\tau - \int_0^t \sigma_2(\tau)\mathcal{I}(\tau)\mathcal{R}(\tau)\mathrm{d}\mathcal{B}_2(\tau) \\ &\geq -\big((\alpha_0\vee\widetilde{\alpha})+b\big)\int_0^t \mathcal{R}(\tau)\mathrm{d}\tau + \delta\int_0^t \mathcal{I}(\tau)\mathrm{d}\tau - \Upsilon_6(t),\end{aligned}$$

where

$$\Upsilon_6(t) \triangleq \int_0^t \sigma_2(\tau)\mathcal{I}(\tau)\mathcal{R}(\tau)\mathrm{d}\mathcal{B}_2(\tau).$$

Then,

$$\frac{1}{t}\int_0^t \mathcal{R}(\tau)\mathrm{d}\tau \;\geq\; \frac{1}{t\big((\alpha_0\vee\widetilde{\alpha})+b\big)}\Big(\delta\int_0^t \mathcal{I}(\tau)\mathrm{d}\tau - (\mathcal{R}(t)-\mathcal{R}(0)) - \Upsilon_6(t)\Big).$$

Based on the large-number martingale theorem and the knowledge that $\mathcal{R}\in(0,1)$,

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\times\big(-\mathcal{R}(t)+\mathcal{R}(0)-\Upsilon_6(t)\big)=0.$$

Next, we deduce (iii) from (ii). Which concludes the proof. □

5. Numerical examples

In this section, we present a number of numerical simulations of our theoretical findings using the saturation incidence rate as an example [20], i.e., let $g(\mathcal{I}) = \frac{\mathcal{I}}{1+0.001\times\mathcal{I}}$. To solve the system (1.6) with a given initial value, we employ the classical numerical method, Milstein's Higher Order Method [14]. The values of $b=0.09$, $\delta=0.28$, $\beta_0=0.2$, $\alpha_0=0.18$, and $(\mathcal{S}(0),\mathcal{I}(0),\mathcal{R}(0))=(0.7,0.2,0.1)$ are selected to be the same in every example.

Example 5.1 (Case $\mathcal{D}_\beta<1$)**.** For the extinction of the disease, we choose $\widetilde{\beta}=0.33, \widetilde{\alpha}=0.27$, $\theta_1=0.41$, $\theta_2=0.40$, $\rho_1=0.23$ and $\rho_2=0.31$. Since $\widetilde{\alpha}\geq\frac{\rho_2^2}{2\theta_2}$, $\widetilde{\beta}-\widetilde{\alpha}\geq\frac{\rho_1^2}{2\theta_1}g'(0)$ and $\mathcal{D}_\beta=0.8187<1$. The stochastic model (1.6) predicts that the disease will eventually disappear from the population in accordance with Theorem 3.1. See Figure 2 for more illustrations of this example.

Example 5.2 (Case $\mathcal{D}_\alpha<1$)**.** Consider the parameters $\widetilde{\beta}=0.28$, $\widetilde{\alpha}=0.42$, $\theta_1=0.41$, $\theta_2=0.40$, $\rho_1=0.25$ and $\rho_2=0.3$. Since $\widetilde{\beta}\geq\frac{\rho_1^2}{2\theta_1}g'(0)$, $\widetilde{\alpha}-\widetilde{\beta}g'(0)\geq\frac{\rho_2^2}{2\theta_2}$ and $\mathcal{D}_\alpha=0.9384<1$. The stochastic model (1.6) predicts that the disease will eventually disappear from the population in accordance with Theorem 3.1. Figure 3 provides further instances of this.

Example 5.3 (Case $\mathcal{D}_\beta>1$)**.** For illustrating the persistence condition of Theorem 4.2, we set $\widetilde{\beta}=0.45$, $\widetilde{\alpha}=0.45$, $\theta_1=0.41$, $\theta_2=0.3$, $\rho_1=0.2$ and $\rho_2=0.3$. It is easy to calculate that $\widetilde{\alpha}\geq\frac{\rho_2^2}{2\theta_2}$ and $\mathcal{D}_\beta=1.141>1$, then the condition of

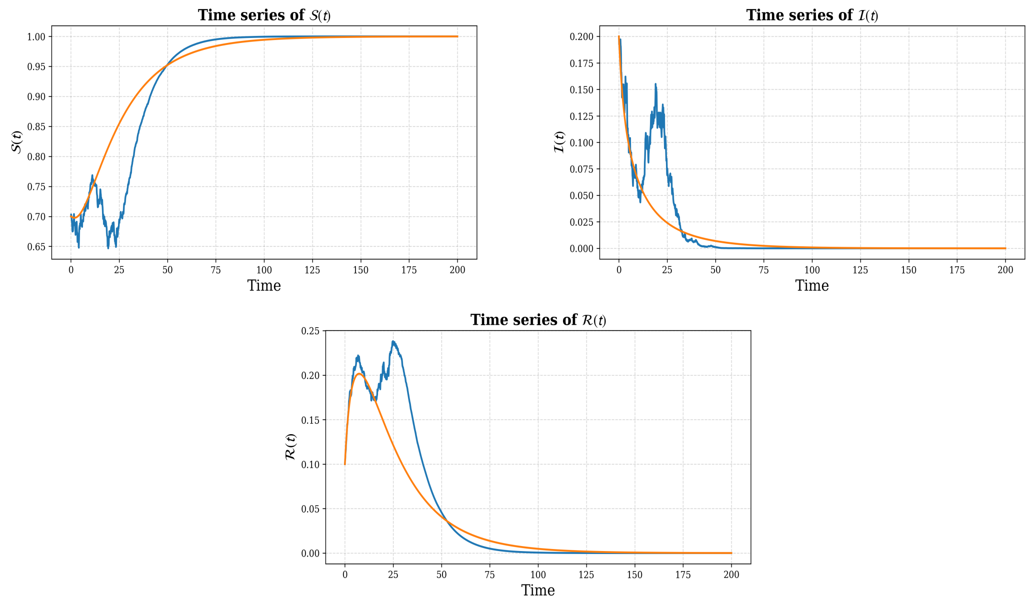

FIGURE 2. The case where $\mathcal{D}_\beta < 1$ is shown, simulations of the system (1.6) will be extinct. The blue trajectories indicate the solution of model (1.6), while the orange trajectories describe the comparable deterministic system.

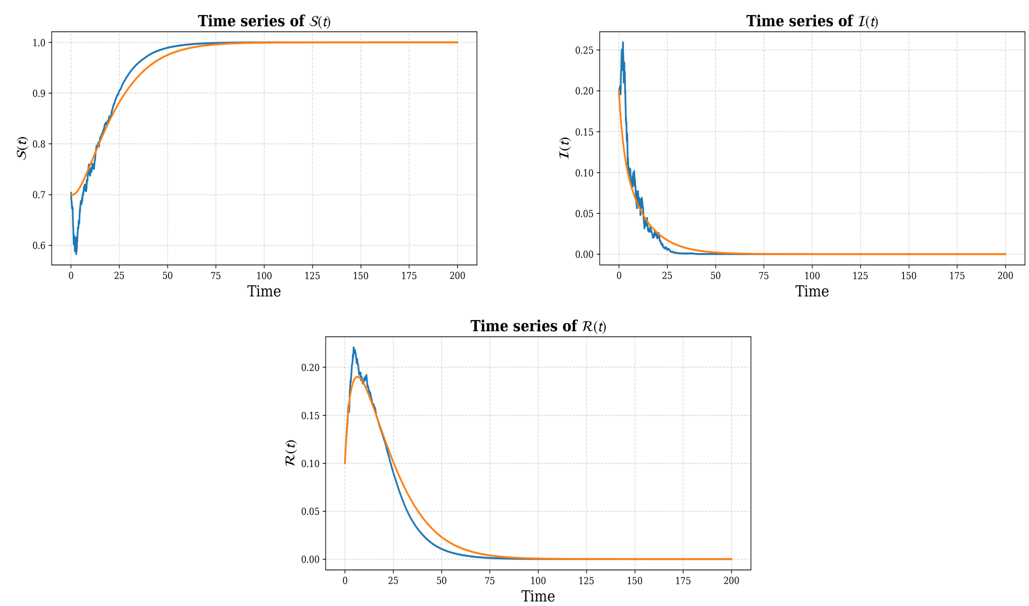

FIGURE 3. When $\mathcal{D}_\alpha < 1$ is revealed, simulations of the system (1.6) will be extinct. The blue trajectories indicate the solution of model (1.6), while the orange trajectories describe the comparable deterministic system.

Theorem 4.2 is verified. It indicates that the disease will remain. This result is illustrated by Figure 4.

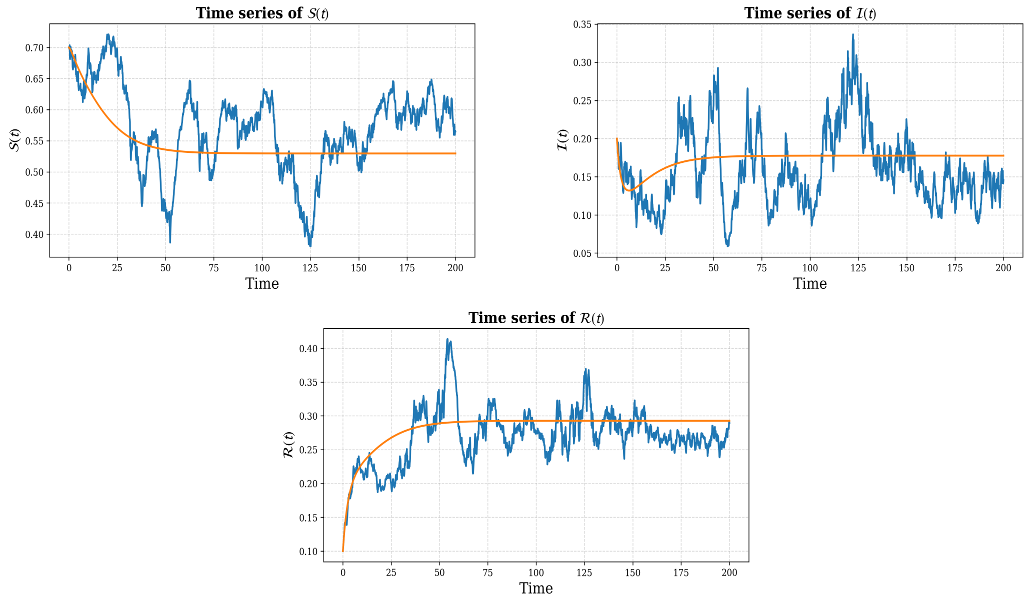

FIGURE 4. Here are the paths of the solution $(\mathcal{S}, \mathcal{I}, \mathcal{R})$ of the system (1.6), based on data from Example 5.3. The stochastic solution is shown by the blue lines. The deterministic solution is shown by the orange lines. As a result, $\mathcal{S}, \mathcal{I}$, and $\mathcal{R}$ persist in the mean.

6. Conclusion

In this study, we examined the dynamics of a stochastic SIR epidemic model with relapse and global incidence rate, including OU process, by perturbing β and α successively. The stochastic system (1.6) determines whether the epidemic disease will terminate or remain active based on two threshold values: $\mathcal{D}_\beta$ and $\mathcal{D}_\alpha$. We find that if $\mathcal{D}_\beta < 1$ or $\mathcal{D}_\alpha < 1$, consequently, with minor supplementary conditions, the disease is almost surely exponential extinction (see Theorem 3.1). However, using Theorem 4.2, we demonstrated that the processes $\mathcal{S}, \mathcal{I}$, and $\mathcal{R}$ are persistent in mean if $\mathcal{D}_\beta > 1$. Importantly, we produce simulations with different parameter values to illustrate and verify our theoretical results. These findings provide crucial insight into the operation of stochastic epidemic models, which has implications for disease management and prevention. They can help us develop focused tactics to stop the spread of infectious illnesses. Furthermore, the successful implementation of the theoretical methods used in this work can help other stochastic epidemic models [27, 30]. The project is currently underway.

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References

1. Bailey N. T. J. (1975) The Mathematical Theory of Infectious Diseases and Its Applications. Second ed Hafner New York.

2. Beretta, E., Takeuchi, Y. (1995). Global stability of a SIR epidemic model with time delay. J. Math. Biol., 33, 250-260. https://doi.org/10.1007/bf00169563.
3. Berrhazi, B., El Fatini, M., Pettersson, R., Laaribi, A. (2019). Media effects on the dynamics of a stochastic SIRI epidemic model with relapse and Lévy noise perturbation. International Journal of Biomathematics, 12, 1950037. https://doi.org/10.1142/s1793524519500372.
4. Cai, Y., Jiao, J., Gui, Z., Liu, Y., Wang, W. (2018). Environmental variability in a stochastic epidemic model. Appl Math Comput., 329, 210-226. https://doi.org/10.1016/j.amc.2018.02.009.
5. Castillo-Garsow, C., Jordan-Salivia, G., Herrera, A.R. (1997). Mathematical models for the dynamics of tobacco use, recovery, and relapse. Technical Report Series BU-1505-M, Cornell University, Ithaca, NY, USA.
6. Dalal, N., Greenhalgh, D., Mao, X. (2007). A stochastic model of AIDS and condom use. J. Math. Anal. Appl., 325, 36-53. https://doi.org/10.1016/j.jmaa.2006.01.055.
7. El Fatini M, Lahrouz A, Pettersson R, Settati A, Taki R. (2018) Stochastic stability and instability of an epidemic model with relapse, Appl Math Comput., 316, 326-341. https://doi.org/10.1016/j.amc.2017.08.037.
8. El Haitami, A. (2024). A stochastic examination of the SIRI epidemic model including nonlinear relapse and the Ornstein-Uhlenbeck process. Commun. Math. Biol. Neurosci., 73. https://doi.org/10.28919/cmbn/8660.
9. El Haitami, A. (2025). A Markovian switching diffusion for an Sirs epidemic model on complex network. Gulf Journal of Mathematics, 20(2), 253-274. https://doi.org/10.56947/gjom.v20i2.3079.
10. El Haitami, A., Settati, A., Lahrouz, A., El Idrissi, A., & El Merzguioui, M. (2024). A stochastic switched SIRI epidemic model integrating nonlinear relapse phenomena. Int. J. Dynam. Control., 12, 1287-1301. https://doi.org/10.1007/s40435-023-01256-9.
11. ElMyr, A. (2025). Threshold dynamics and impulsive treatment in stochastic NEET population model. Gulf Journal of Mathematics, 19(2), 277-292. https://doi.org/10.56947/gjom.v19i2.2743.
12. El Myr, A., Assadouq, A., Omari, L., Settati, A., & Lahrouz, A. (2018). A stochastic SIR epidemic system with a nonlinear relapse. Discrete Dynamics in Nature and Society, 2018, 5493270. https://doi.org/10.1155/2018/5493270.
13. Gray, A., Greenhalgh, D., Mao, X., & Pan, J. (2012). The SIS epidemic model with Markovian switching. J. Math. Anal. Appl., 394, 496-516. https://doi.org/10.1016/j.jmaa.2012.05.029.
14. Higham, D. J. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev., 43, 525-546. https://doi.org/10.1137/s0036144500378302.
15. Lipster, R. (1980). A strong law of large numbers for local martingales. Stochastics, 3, 217-228. https://doi.org/10.1080/17442508008833146.
16. Liu, Q. (2023). Stationary distribution and probability density for a stochastic SISP respiratory disease model with Ornstein–Uhlenbeck process. Commun. Nonl. Sci. Numer. Simul., 119, 107128. https://doi.org/10.1016/j.cnsns.2023.107128.
17. Liu, Q. (2023). Stationary distribution and extinction of a stochastic HLIV model with viral production and Ornstein–Uhlenbeck process. Commun. Nonl. Sci. Numer. Simul., 119, 107111. https://doi.org/10.1016/j.cnsns.2023.107111.
18. Mao, X. (2008). Stochastic Differential Equations and Applications. 2nd ed Chichester Horwood UK.
19. Moreira, H. N., Wang, Y. (1997). Global stability in an SIRI model. SIAM Review, 39(3), 496–502. https://doi.org/10.1137/S0036144595295879.
20. Rajasekar, S., Pitchaimani, M., Zhu, Q. (2019). Dynamic threshold probe of stochastic SIR model with saturated incidence rate and saturated treatment function. Physica A 535-122300. https://doi.org/10.1016/j.physa.2019.122300.

21. Sanchez, F., Wang, X., Castillo-Chavez, C., Gorman, D., Gruenewald, P. (2007). Drinking as an epidemic-a simple mathematical model with recovery and relapse. Therpist's guide to evidence-based relapse prevention, Elsevier, 353-368. https://doi.org/10.1016/B978-012369429-4/50046-X.
22. Settati, A., Lahrouz, A., El Fatini, M., El Haitami, A., El Jarroudi, M., & Erriani, M. (2023). A Markovian switching diffusion for an SIS model incorporating Lévy processes. Discrete and Continuous Dynamical Systems-B, 28(1), 209-229. https://doi.org/10.3934/dcdsb.2022072.
23. Shi, Z., Jiang, D. (2022). Dynamical behaviors of a stochastic HTLV-I infection model with general infection form and Ornstein–Uhlenbeck process. Chaos Solit. Fract., 165, 112789. https://doi.org/10.1016/j.chaos.2022.112789.
24. Tudor, D. (1990). A deterministic model for herpes infections inhuman and animal populations. SIAM Review, 32(1), 136–139. https://doi.org/10.1137/1032003.
25. Wang J, Zhang J, Jin Z. (2010). Analysis of an SIR model with bilinear incidence rate, Nonlinear Anal. RWA, 11, 2390-2402. https://doi.org/10.1016/j.nonrwa.2009.07.012.
26. Xia, P., Zheng, X., Jiang, D. (2013). Persistence and nonpersistence of a nonautonomous stochastic mutualism system. Abstract and Applied Analysis. Hindawi Publishing Corporation, 256249. https://doi.org/10.1155/2013/256249.
27. Zhang, W., Meng, X. (2018). Stochastic analysis of a novel nonautonomous periodic SIRI epidemic system with random disturbances. Physica A., 492, 1290-1301. https://doi.org/10.1016/j.physa.2017.11.057.
28. Zhang, X., Yuan, R. (2021). A stochastic chemostat model with mean-reverting Ornstein–Uhlenbeck process and Monod-Haldane response function. Appl. Math. Comput., 394, 125833. https://doi.org/10.1016/j.amc.2020.125833.
29. Zhou, Y., Jiang, D. (2023). Dynamical behavior of a stochastic SIQR epidemic model with Ornstein–Uhlenbeck process and standard incidence rate after dimensionality reduction. Commun. Nonl. Sci Numer. Simul., 116, 106878. https://doi.org/10.1016/j.cnsns.2022.106878.
30. Zhou, Y., Jiang, D. (2023). Dynamic behavior of infectious diseases influenced by TV and social media advertisement. Chaos Solitons Fractals, 168, 113127. https://doi.org/10.1016/j.chaos.2023.113127.
31. Zhou, B., Jiang, D., Han, B., Hayat, T. (2022). Threshold dynamics and density function of a stochastic epidemic model with media coverage and mean-reverting Ornstein–Uhlenbeck process. Math. Comp. Simul., 196, 15-44. https://doi.org/10.1016/j.matcom.2022.01.014.
32. Zhu, C., Yin, G. (2009). On competitive Lotka–Volterra model in random environments. J. Math. Anal. Appl., 357, 154-170. https://doi.org/10.1016/j.jmaa.2009.03.066.

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