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ON STRUCTURE OF $D_5(G)$ FOR 2-GROUP G

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ABSTRACT. The objective of this paper is to discuss some conditions on a finite group G, under which $D_5(G) = \gamma_5(G)$.

1. INTRODUCTION AND PRELIMINARIES

Let G be a finite group and let $\Delta(G)$ be the augmentation ideal of the integral group ring $\mathbb{Z}G$. For $k \geq 1$, let $\gamma_k(G)$ be the kth term in the lower central series of group G and $D_k(G) = G \cap \{1 + \Delta^k(G)\}$ be the kth dimension subgroup of group G. For $k \geq 1$, the equality of $D_k(G)$ and $\gamma_k(G)$ has been a problem of interest for so many years. G. Higman [4] reduces the problem to p-groups by proving that if for some k, $D_k(G) \neq \gamma_k(G)$, then there exists a finite p-group G for which $D_k(G) \neq \gamma_k(G)$. It is well known that $D_k(G) = \gamma_k(G)$, for all groups G and for $k \leq 3$ (see [2]). It has been proved in [4] that $D_4(G) = \gamma_4(G)$ for p-group G, p odd prime. Rips [5] gave an example of 2-group G for which $D_4(G) \neq \gamma_4(G)$. Further, Tahara [6] gave the structure of $D_5(G)$ and proved that the exponent of $D_5(G)/\gamma_5(G)$ is divisible by 6. It has been proved in [3], that for a metabelian p-group G, p odd prime, $D_5(G) = \gamma_5(G)$. Recently, Gupta [1] discussed certain conditions under which for a metabelian 2-group G, $D_5(G) = \gamma_5(G)$.

In the continuation of the work done in [1], in this paper, we discuss some more conditions on a metabelian 2-group G under which $D_5(G) = \gamma_5(G)$ (Theorem 2.3). Finally, we find some conditions on a finite group G (not necessarily metabelian), under which $D_5(G) = \gamma_5(G)$ (Theorems 2.4, 2.5 and 2.6).

Let G be a finite group of class 4 and $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \gamma_4(G) \supseteq \gamma_5(G) = \{1\}$, be the lower central series of group G. We write $\gamma_1(G)/\gamma_2(G)$ as a sum of s cyclic groups, each generated by $x_{1i}\gamma_2(G)$, say, $x_{1i}^{d(i)} \in \gamma_2(G),\ 1 \leq i \leq s$. Similarly, write $\gamma_2(G)/\gamma_3(G)$ and $\gamma_3(G)/\gamma_4(G)$ as a sum of t and u cyclic groups generated by $x_{2k}\gamma_3(G)$ and $x_{3l}\gamma_4(G)$ respectively, say, $x_{2k}^{e(k)} \in \gamma_3(G),\ 1 \leq k \leq t$ and $x_{3l}^{f(l)} \in \gamma_4(G)),\ 1 \leq l \leq u$.

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Write

$$x_{1i}^{d(i)} = x_{21}^{b_{i1}} x_{22}^{b_{i2}} \dots x_{2t}^{b_{it}} x_{31}^{c_{i1}} x_{32}^{c_{i2}} \dots x_{3u}^{c_{iu}} y_{4i}, \quad y_{4i} \in \gamma_4(G), \ \ 1 \le i \le s; \tag{1.1}$$

$$x_{2k}^{e(k)} = x_{31}^{d_{k1}} x_{32}^{d_{k2}} \dots x_{3u}^{d_{ku}} y'_{4k}, \quad y'_{4k} \in \gamma_4(G), \ \ 1 \le k \le t; \tag{1.2}$$

$$x_{3l}^{f(l)} = x_{41}^{f_{l1}} x_{42}^{f_{l2}} \dots x_{4u}^{f_{lv}}, \ \ 1 \le l \le u; \tag{1.3}$$

$$[x_{1i}^{d(i)}, x_{1j}] = x_{31}^{\alpha_1^{(ij)}} x_{32}^{\alpha_2^{(ij)}} \dots x_{3u}^{\alpha_u^{(ij)}} x_{41}^{\beta_1^{(ij)}} x_{42}^{\beta_2^{(ij)}} \dots x_{4v}^{\beta_v^{(ij)}}, \ 1 \le i < j \le s. \tag{1.4}$$

Lemma 1.1. ([6, Lemma 2.3]). *Let d be a non-negative integer and $x, y, z \in G$, then we have*

$$\begin{aligned}[x,y]^d &= [x,y^d][x,y,y]^{-\binom{d}{2}}[x,y,y,y]^{-\binom{d}{3}} \\ &= [x^d,y][x,y,x]^{-\binom{d}{2}}[x,y,x,x]^{-\binom{d}{3}}\end{aligned}$$

$$\begin{aligned}[x,y,z]^d &= [x^d,y,z][y,x,x,z]^{\binom{d}{2}} \\ &= [x,y^d,z][y,x,y,z]^{\binom{d}{2}} \\ &= [x,y,z^d][y,x,z,z]^{\binom{d}{2}}\end{aligned}$$

We now recall the structure of fifth dimension subgroup given by Tahara in [6] as follows:

Theorem 1.2. ([6, Theorem 6.1]). *Let G be a group of class 4. Then $D_5(G)$ is equal to the subgroup generated by the elements*

$$\prod_{1\le i<j\le s} [{x_{1i}}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \prod_{1\le i\le s,\ 1\le k\le t} \prod_{k<l} [x_{2l}, x_{2k}]^{v_{ik} b_{il}}$$
$$\prod_{1\le i\le j\le k\le s} [{x_{1i}}^{d(i)}, x_{1j}, x_{ik}]^{w_{ijk}},$$

where u_{ij}, v_{ik}, v'_{ik}, w_{ijk}, w'_{ijk} and w''_{ijk} are the integers satisfying following conditions:

$$w_{iii} = 0, \ 1 \le i \le s; \tag{1.5}$$

$$u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2} + w_{iij}d(i) + w''_{iij}d(j) = 0, \ 1 \le i < j \le s; \tag{1.6}$$

$$-\, u_{ij}\binom{d(j)}{2} + w_{ijj}d(i) + w'_{ijj}d(j) = 0, \ 1 \le i < j \le s; \tag{1.7}$$

$$w_{ijk}d(i) + w'_{ijk}d(j) + w''_{ijk}d(k) = 0, \ 1 \le i < j < k \le s; \tag{1.8}$$

$$\sum_{i<h} u_{ih}b_{hk} - \sum_{h<i} u_{hi}\frac{d(i)}{d(h)}b_{hk} + v_{ik}d(i) + v'_{ik}e(k) = 0, \ 1 \le i \le s, \ 1 \le k \le t; \tag{1.9}$$

$$u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{3} + w_{iij}\binom{d(i)}{2} \equiv 0 \pmod{d(i)}, \ 1 \le i < j \le s; \tag{1.10}$$

$$w_{ijj}\binom{d(i)}{2} + w''_{iij}\binom{d(j)}{2} \equiv 0 \pmod{d(i)},\ 1 \leq i < j \leq s; \tag{1.11}$$

$$u_{ij}\binom{d(j)}{3} + w'_{ijj}\binom{d(j)}{2} \equiv 0 \pmod{d(i)},\ 1 \leq i < j \leq s; \tag{1.12}$$

$$w_{ijk}\binom{d(i)}{2},\ w'_{ijk}\binom{d(j)}{2},\ w''_{ijk}\binom{d(k)}{2} \equiv 0 \pmod{d(i)},\ 1 \leq i < j < k \leq s; \tag{1.13}$$

$$v_{ik}\binom{d(i)}{2} - \sum_{h\leq i} w_{hii}b_{hk} - \sum_{i<h} w''_{iih}b_{hk} \equiv 0 \pmod{(d(i),\ e(k))}, \tag{1.14}$$

$$1 \leq i \leq s,\ 1 \leq k \leq t;$$

$$\sum_{h\leq i} w_{hij}b_{hk} + \sum_{i<h\leq j} w'_{ihj}b_{hk} + \sum_{j<h} w''_{ijh}b_{hk} \equiv 0 \pmod{(d(i),\ e(k))}, \tag{1.15}$$

$$1 \leq i < j \leq s,\ 1 \leq k \leq t;$$

$$-\sum_{h<i} u_{hi}\frac{d(i)}{d(h)}{\alpha_l}^{(hi)} + \sum_{i<h} u_{ih}c_{hl} - \sum_{h<i} u_{hi}\frac{d(i)}{d(h)}c_{hl} - \sum_{k} v'_{ik}d_{kl} \tag{1.16}$$

$$-\sum_{g\leq i\leq h} w_{gih}{\alpha_l}^{(gh)} - \sum_{g\leq h\leq i} w_{ghi}{\alpha_l}^{(gh)} - \sum_{i<g\leq h} w'_{igh}{\alpha_l}^{(gh)} \equiv 0 \pmod{(d(i),\ f(l))},$$

$$1 \leq i \leq s,\ 1 \leq l \leq u;$$

$$\sum_i v_{ik}b_{ik} \equiv 0 \pmod{e(k)},\ 1 \leq k \leq t; \tag{1.17}$$

$$\sum_i v_{ik}b_{il} + \sum_i v_{il}b_{ik} \equiv 0 \pmod{e(k)},\ 1 \leq k < l \leq t. \tag{1.18}$$

2. Main results

It has been proved in [3] that $D_5^2(G) \subseteq \gamma_5(G)[\gamma_2(G), \gamma_2(G)]$, which implies that $D_5(G) = \gamma_5(G)$, if G is a metabelian p-group, p odd prime. For a metabelian 2-group G, Gupta [1] proved the following results,

Theorem 2.1. ([1, Theorem 2]). *Let G be a metabelian 2-group with $G/\gamma_2(G)$ as a sum of at most two cyclic groups. Then $D_5(G) = \gamma_5(G)$.*

Theorem 2.2. ([1, Theorem 3]). *Let G be a finite metabelian 2-group and $G/\gamma_2(G) \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where C_i, $1 \leq i \leq n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-1) = 2, d(n) \geq 2^2$. Then $D_5(G) = \gamma_5(G)$.*

In next theorem, we will discuss some more conditions on a metabelian 2-group G, for which $D_5(G) = \gamma_5(G)$.

Theorem 2.3. *Let G be a finite metabelian 2-group and $G/\gamma_2(G) \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where C_i, $1 \leq i \leq n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-2) = 2, d(n-1) = 2^2, d(n) = 2^k, k \geq 2$. Then $D_5(G) = \gamma_5(G)$.*

Proof. It is enough to prove the result for a group G of class 4. It follows from Theorem 1.2 that any element g of $D_5(G)$ is of the form

$$\begin{aligned} g &= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} . \prod_{1\le i\le j\le k\le n} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}} \\ &= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} . \prod_{1\le i<j<k\le n} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}} \\ &\prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{ijj}} \\ &= A.B.C \text{ (say)} \end{aligned} \tag{2.1}$$

where u_{ij} and w_{ijk} are integers satisfying conditions given in Theorem 1.2.

$$\begin{aligned} A &= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \\ &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} . \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\ &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} . \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\ &= A_1.A_2.A_3 \text{ (say)} \end{aligned} \tag{2.2}$$

Now, for $1 \le i < j \le n-2$, condition (1.7) becomes $u_{ij} = w_{ijj}d(i) + w'_{ijj}d(j)$, which implies that,

$$\begin{aligned} A_1 &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \\ &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{w_{ijj}d(i)+w'_{ijj}d(j)} \\ &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{d(j)(w_{ijj}+w'_{ijj})} \\ &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}^{d(j)}]^{w_{ijj}+w'_{ijj}} \\ &\prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-(w_{ijj}+w'_{ijj})\binom{d(j)}{2}} \end{aligned} \tag{2.3}$$

Since G is metabelian, therefore $[x_{1i}^{d(i)}, x_{1j}^{d(j)}] = 1$, as $[x_{1i}^{d(i)}, x_{1j}^{d(j)}] \in [\gamma_2(G), \gamma_2(G)]$. Thus (2.3) reduces to,

$$\begin{aligned} A_1 &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}} \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w'_{ijj}} \\ &= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}}, \end{aligned} \tag{2.4}$$

because, for $1 \le i < j \le n-2$, condition (1.12) becomes, $w'_{ijj} \equiv 0 \pmod{d(j)}$.

For $j = n-1$, condition (1.6) becomes, $u_{in-1}\frac{d(n-1)}{d(i)} = -w_{iin-1}d(i) - w''_{iin-1}d(n-1)$ Thus

$$\begin{aligned}
A_2 &= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-w_{iin-1}d(i) - w''_{iin-1}d(n-1)} \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-w_{iin-1}d(i)} \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-w''_{iin-1}d(n-1)} \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-w_{iin-1}d(i)} \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}^{d(n-1)}]^{-w''_{iin-1}} \\
&\quad \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{w''_{iin-1}\binom{d(n-1)}{2}} \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-2w_{iin-1}} \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{w''_{iin-1}\binom{d(n-1)}{2}} \qquad (2.5)
\end{aligned}$$

For $j = n-1,\ 1 \le i \le n-2$, condition (1.10) implies that $w_{iin-1} \equiv 0 \pmod 2$ i.e. $w_{iin-1} = 2q$, for some integer q. Thus we have

$$\begin{aligned}
\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-2w_{iin-1}} &= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-2^2 q} \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{-d(n-1)q} \qquad [\because d(n-1) = 2^2] \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}^{d(n-1)}]^{-q} \\
&\quad \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-q\binom{d(n-1)}{2}} \\
&= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-q\binom{d(n-1)}{2}}
\end{aligned}$$

Thus (2.5) becomes

$$\begin{aligned}
A_2 &= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{(w''_{iin-1}-q)\binom{d(n-1)}{2}} \\
&= \prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}]^{(w''_{iin-1}-q)d(n-1)(d(n-1)-1)} \\
&\quad \prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1i}, x_{1n-1}]^{-(w''_{iin-1}-q)\binom{d(n-1)}{2}\binom{d(i)}{2}} \quad [\because \text{for } i < n-1,\ d(i) = 2]
\end{aligned}$$

$$= \prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}^{d(n-1)}]^{(w''_{iin-1}-q)(d(n-1)-1)}$$

$$\prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1n-1}, x_{1n-1}]^{(w''_{iin-1}-q)(d(n-1)-1)\binom{d(n-1)}{2}}$$

$$\prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1i}, x_{1n-1}]^{-(w''_{iin-1}-q)\binom{d(n-1)}{2}\binom{d(i)}{2}}$$

Since $[x_{1i}, x_{1n-1}, x_{1n-1}^{d(n-1)}] \in [\gamma_2(G), \gamma_2(G)]$ and $d(n-1) = 2^2$ gives that $\binom{d(n-1)}{2} \equiv 0 \pmod{d(i)}$ and $\binom{d(n-1)}{2}\binom{d(i)}{2} \equiv 0 \pmod{d(i)}$, which implies that

$$A_2 = 1 \tag{2.6}$$

Now

$$A_3 = \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}}$$

$$= \prod_{i<n} \prod_{1\le k\le t} [x_{2k}^{b_{ik}}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \prod_{i<n} \prod_{1\le l\le u} [x_{3l}^{c_{il}}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}}$$

$$= \prod_{i<n} \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}b_{ik}} \prod_{i<n} \prod_{1\le l\le u} [x_{3l}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}b_{ik}} \prod_{1\le l\le u} [x_{3l}^{c_{il}}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}}$$

For $i = n$, $1 \le k \le t$, condition (1.9) gives, $\sum_{i<n} u_{in}\frac{d(n)}{d(i)}b_{ik} = v_{nk}d(n) + v'_{nk}e(k)$, which implies that

$$A_3 = \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{v_{nk}d(n)+v'_{nk}e(k)} \prod_{1\le l\le u} [x_{3l}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}^{d(n)}]^{v_{nk}} \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{1\le k\le t} [x_{2k}^{e(k)}, x_{1n}]^{v'_{nk}}$$

$$\prod_{1\le l\le u} [x_{3l}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{1\le k\le t} [x_{2k}^{e(k)}, x_{1n}]^{v'_{nk}}$$

$$\prod_{1\le l\le u} [x_{3l}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{1\le l\le u} [x_{3l}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}c_{il}+\sum_k v'_{nk}d_{kl}}$$

For $i = n$, condition (1.16) becomes

$\sum_{i<n} u_{in}\frac{d(n)}{d(i)}c_{il}+\sum_k v'_{nk}d_{kl} = -\sum_{i<n} u_{in}\frac{d(n)}{d(i)}\alpha_l^{(in)} - 2\sum_{i<n} w_{inn}\alpha_l^{(in)} - \sum_{i<j<n} w_{ijn}\alpha_l^{(ij)} -$

$\sum_{i<n} w_{iin}\alpha_l^{(ii)} + \mathbb{Z}$-linear combination of $d(n)$ and $f(l)$. Thus, we have

$$A_3 = \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{1\le l\le u} [x_{3l}, x_{1n}]^{-\sum_{i<n} u_{in}\frac{d(n)}{d(i)}\alpha_l^{(in)} - 2\sum_{i<n} w_{inn}\alpha_l^{(in)}}$$
$$\prod_{1\le l\le u} [x_{3l}, x_{1n}]^{-\sum_{i<j<n} w_{ijn}\alpha_l^{(ij)} - \sum_{i<n} w_{iin}\alpha_l^{(ii)}}$$
$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{i<n}[x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-u_{in}\frac{d(n)}{d(i)}} \prod_{i<n}[x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-2w_{inn}}$$
$$\prod_{i<j<n} [x_{1i}^{d(i)}, x_{1j}, x_{1n}]^{-w_{ijn}} \prod_{i<n}[x_{1i}^{d(i)}, x_{1i}, x_{1n}]^{-w_{iin}}$$
$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{i<n} \prod_{1\le k\le t} [x_{2k}^{b_{ik}}, x_{1n}, x_{1n}]^{-u_{in}\frac{d(n)}{d(i)}}$$
$$\prod_{i<n} \prod_{1\le k\le t} [x_{2k}^{b_{ik}}, x_{1n}, x_{1n}]^{-2w_{inn}} \prod_{i<j<n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1n}]^{-w_{ijn}}$$
$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n}]^{-w_{in-1n}}$$
$$= \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} \prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-\sum_{i<n} u_{in}\frac{d(n)}{d(i)} b_{ik}}$$
$$\prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-2\sum_{i<n} w_{inn} b_{ik}} \prod_{i<j<n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1n}]^{-w_{ijn}}$$
$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n}]^{-w_{in-1n}} \tag{2.7}$$

Now we will separately solve the terms of equation (2.7).

For $i = n$, $1 \le k \le t$, condition (1.14) implies that

$$\prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-v_{nk}\binom{d(n)}{2}} = \prod_{i<n}[x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-w_{inn}} \tag{2.8}$$

For $i = n$, condition (1.9) gives, $\sum_{i<n} u_{in}\frac{d(n)}{d(i)} b_{ik} = v_{nk}d(n) + v'_{nk}e(k)$, which implies that

$$\prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-\sum_{i<n} u_{in}\frac{d(n)}{d(i)} b_{ik}} = 1 \tag{2.9}$$

For $i = n$, $1 \le k \le t$, condition (1.14) becomes, $\sum_{i<n} w_{inn} b_{ik} = v_{nk}\binom{d(n)}{2} + \mathbb{Z}$-linear combination of $d(n)$ and $e(k)$, which implies that

$$\prod_{1\le k\le t} [x_{2k}, x_{1n}, x_{1n}]^{-2\sum_{i<n} w_{inn} b_{ik}} = 1 \tag{2.10}$$

For $i < j < n-1$, $d(i) = d(j) = 2$, therefore, condition (1.13) becomes, $w_{ijn} \equiv 0 \pmod{(d(j))}$, which implies that

$$\prod_{i<j<n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1n}]^{-w_{ijn}} = 1 \tag{2.11}$$

From (2.8), (2.9), (2.10) and (2.11), equation (2.7) reduces to

$$A_3 = \prod_{i<n}[x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-w_{inn}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n}]^{-w_{in-1n}} \tag{2.12}$$

Hence, from (2.2), (2.4), (2.6) and (2.12),

$$A = \prod_{1\le i<j\le n-2}[x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}} \prod_{i<n}[x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{-w_{inn}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n}]^{-w_{in-1n}} \tag{2.13}$$

Now, consider

$$\begin{aligned} B &= \prod_{1\le i<j<k\le n}[x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}} \\ &= \prod_{i<j<n-1}[x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n}]^{w_{in-1n}} \end{aligned}$$

As before, condition (1.13) implies that

$$\prod_{i<j<n-1}[x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}} = 1 \tag{2.14}$$

Thus, we have

$$B = \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n}]^{w_{in-1n}} \tag{2.15}$$

Finally, consider

$$\begin{aligned} C &= \prod_{1\le i<j\le n}[x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{ijj}} \\ &= \prod_{1\le i<j\le n-2}[x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{ijj}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{w_{in-1n-1}} \\ &\quad \prod_{i<n}[x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{w_{inn}} \end{aligned} \tag{2.16}$$

For $1 \le i \le n-2$, $j = n-1$, condition (1.11) becomes, $w_{in-1n-1} + w''_{iin-1}\binom{d(n-1)}{2} = d(i)q$, for some integer q, which implies that

$$\begin{aligned} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{w_{in-1n-1}} &= \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-w''_{iin-1}\binom{d(n-1)}{2}} \\ &\quad \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{d(i)q} \end{aligned}$$

$$= \prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}]^{-w''_{iin-1}d(n-1)(d(n-1)-1)} \quad [\because \text{for } i < n-1,\ d(i) = 2]$$

$$\prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1i}, x_{1n-1}]^{w''_{iin-1}\binom{d(n-1)}{2}\binom{d(i)}{2}}$$

$$\prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}]^{d(n-1)q} \quad [\because\ d(n-1) = 2^2]$$

$$\prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1i}, x_{1n-1}]^{-d(i)\binom{d(i)}{2}q}$$

Since, for $i < n-1$, $\binom{d(n-1)}{2}\binom{d(i)}{2} \equiv 0 \,(\mathrm{mod}\,(d(i)))$ and $d(i)\binom{d(i)}{2} \equiv 0 \,(\mathrm{mod}\,(d(i)))$, implies that,

$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{w_{in-1n-1}} = \prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}]^{-w''_{iin-1}d(n-1)(d(n-1)-1)}$$
$$\prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}]^{d(n-1)q}$$

$$= \prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}^{d(n-1)}]^{-w''_{iin-1}(d(n-1)-1)}$$
$$\prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1n-1}, x_{1n-1}]^{-w''_{iin-1}\binom{d(n-1)}{2}(d(n-1)-1)}$$
$$\prod_{i<n-1} [x_{1i}, x_{1n-1}, x_{1n-1}^{d(n-1)}]^{q} \prod_{i<n-1} [x_{1n-1}, x_{1i}, x_{1n-1}, x_{1n-1}]^{\binom{d(n-1)}{2}q}$$
$$= 1, \qquad \text{as} \binom{d(n-1)}{2} \equiv 0 \,(\mathrm{mod}\,(d(i))). \tag{2.17}$$

From (2.16) and (2.17), we get that

$$C = \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{ijj}}$$
$$\prod_{i<n} [x_{1i}^{d(i)}, x_{1n}, x_{1n}]^{w_{inn}} \tag{2.18}$$

Hence, from (2.1), (2.13), (2.15) and (2.18), we get that $g = 1$. □

In next theorems, we will find some conditions on a finite group G (not necessarily metabelian) under which the exponent of $D_5(G)/\gamma_5(G)$ is 1, i.e., $D_5(G) = \gamma_5(G)$.

Theorem 2.4. *Let G be a finite group. Let $G/\gamma_2(G)$ be sum of two cyclic groups and let $\gamma_2(G)/\gamma_3(G)$ be cyclic group. Then $D_5(G) = \gamma_5(G)$.*

Proof. It is enough to prove the result for a group G of class 4. Let $G/\gamma_2(G) \cong C_1 \oplus C_2$, where $C_i,\ 1 \le i \le 2$, is a cyclic group of order $d(i)$. Let $\gamma_2(G)/\gamma_3(G) = < x_{21}\gamma_3(G) >$, where order of $x_{21}\gamma_3(G)$ is $e(1)$. It clearly follows from Theorem 1.2 that an arbitrary element g of $D_5(G)$ is $[x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}}$ $[x_{11}^{d(1)}, x_{12}, x_{12}]^{w_{122}}$,

subject to the conditions given in Theorem 1.2.
Using (1.1) and Lemma 1.1, we get that

$$\begin{aligned}
[x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}} &= [x_{21}^{b_{11}}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}} \prod_{1\le l\le u} [x_{3l}^{c_{1l}}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}} \\
&= [x_{21}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}b_{11}} \prod_{1\le l\le u} [x_{3l}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}c_{1l}}
\end{aligned}$$

For $i = 2$, condition (1.9) becomes, $u_{12}\frac{d(2)}{d(1)}b_{11} = v_{21}d(2) + v'_{21}e(1)$. Thus,

$$\begin{aligned}
[x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}} &= [x_{21}, x_{12}]^{v_{21}d(2)+v'_{21}e(1)} \prod_{1\le l\le u} [x_{3l}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}c_{1l}} \\
&= [x_{21}, x_{12}]^{v_{21}d(2)}.[x_{21}, x_{12}]^{v'_{21}e(1)} \prod_{1\le l\le u} [x_{3l}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}c_{1l}} \\
&= [x_{21}, x_{12}^{d(2)}]^{v_{21}}.[x_{21}, x_{12}, x_{12}]^{-v_{21}\binom{d(2)}{2}}[x_{21}^{e(1)}, x_{12}]^{v'_{21}} \\
&\prod_{1\le l\le u} [x_{3l}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}c_{1l}} \\
&= [x_{21}, x_{12}^{d(2)}]^{v_{21}}.[x_{21}, x_{12}, x_{12}]^{-v_{21}\binom{d(2)}{2}} \prod_{1\le l\le u} [x_{3l}^{d_{1l}}, x_{12}]^{v'_{21}} \\
&\prod_{1\le l\le u} [x_{3l}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}c_{1l}} \\
&= [x_{21}, x_{12}^{d(2)}]^{v_{21}}.[x_{21}, x_{12}, x_{12}]^{-v_{21}\binom{d(2)}{2}} \\
&\prod_{1\le l\le u} [x_{3l}, x_{12}]^{v'_{21}d_{1l}+u_{12}\frac{d(2)}{d(1)}c_{1l}} \\
&= A.B.C \text{ (say)} \qquad (2.19)
\end{aligned}$$

We have,

$$\begin{aligned}
A &= [x_{21}, x_{12}^{d(2)}]^{v_{21}} \\
&= [x_{21}, x_{21}^{b_{21}}]^{v_{21}} \prod_{1\le l\le u} [x_{21}, x_{3l}^{c_{2l}}]^{v_{21}} \\
&= 1 \qquad (2.20)
\end{aligned}$$

For $i = 2$, condition (1.14) becomes, we have $v_{21}\binom{d(2)}{2} = w_{122}b_{11} + \mathbb{Z}$-linear combination of $d(2)$ and $e(1)$.
Thus,

$$\begin{aligned}
B &= [x_{21}, x_{12}, x_{12}]^{-v_{21}\binom{d(2)}{2}} \\
&= [x_{21}, x_{12}, x_{12}]^{-w_{122}b_{11}} \\
&= [x_{21}^{b_{11}}, x_{12}, x_{12}]^{-w_{122}} \\
&= [x_{11}^{d(1)}, x_{12}, x_{12}]^{-w_{122}} \qquad (2.21)
\end{aligned}$$

For $i=2$, condition (1.16) becomes, $u_{12}\frac{d(2)}{d(1)}c_{1l}+v'_{21}d_{1l} = -u_{12}\frac{d(2)}{d(1)}\alpha_l^{(12)}-2w_{122}\alpha_l^{(12)}-w_{112}\alpha_l^{(11)}+\mathbb{Z}$-linear combination of $d(2)$ and $f(l)$, which implies that

$$\begin{aligned}
C &= \prod_{1\le l\le u}[x_{3l},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}\alpha_l^{(12)}-2w_{122}\alpha_l^{(12)}-w_{112}\alpha_l^{(11)}}\\
&= \prod_{1\le l\le u}[x_{3l},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}\alpha_l^{(12)}}\prod_{1\le l\le u}[x_{3l},x_{12}]^{-2w_{122}\alpha_l^{(12)}}\prod_{1\le l\le u}[x_{3l},x_{12}]^{-w_{112}\alpha_l^{(11)}}\\
&= [x_{11}^{d(1)},x_{12},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)},x_{12},x_{12}]^{-2w_{122}}[x_{11}^{d(1)},x_{11},x_{12}]^{-w_{112}}\\
&= [x_{11}^{d(1)},x_{12},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)},x_{12},x_{12}]^{-2w_{122}}\\
&= [x_{21}^{b_{11}},x_{12},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}}[x_{21}^{b_{11}},x_{12},x_{12}]^{-2w_{122}}\\
&= [x_{21},x_{12},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}b_{11}}[x_{21},x_{12},x_{12}]^{-2w_{122}b_{11}}
\end{aligned}$$

For $i=2$, conditions (1.9) and (1.14) yield

$$[x_{21},x_{12},x_{12}]^{-u_{12}\frac{d(2)}{d(1)}b_{11}}=1,\quad [x_{21},x_{12},x_{12}]^{-2w_{122}b_{11}}=1$$

Thus,

$$C=1 \tag{2.22}$$

Now, after putting the values of $A,\ B$ and C from (2.20), (2.21) and (2.22) in equation (2.19), we get that,

$$[x_{11}^{d(1)},x_{12}]^{u_{12}\frac{d(2)}{d(1)}}=[x_{11}^{d(1)},x_{12},x_{12}]^{-w_{122}},$$

Thus, $g=[x_{11}^{d(1)},x_{12}]^{u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)},x_{12},x_{12}]^{w_{122}}=1.$ □

Theorem 2.5. *Let G be a finite 2-group. Let $G/\gamma_2(G)$ be sum of n cyclic groups each of order 2 and let $\gamma_2(G)/\gamma_3(G)$ be cyclic group. Then $D_5(G)=\gamma_5(G)$.*

Proof. It is enough to prove the result for a group G of class 4. It follows from Theorem 1.2 that any element g of $D_5(G)$ is of the form

$$\begin{aligned}
g &= \prod_{1\le i<j\le n}[x_{1i}^{d(i)},x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}}.\prod_{1\le i\le j\le k\le n}[x_{1i}^{d(i)},x_{1j},x_{1k}]^{w_{ijk}}\\
&= \prod_{1\le i<j\le n}[x_{1i}^{d(i)},x_{1j}]^{u_{ij}}.\prod_{1\le i<j<k\le n}[x_{1i}^{d(i)},x_{1j},x_{1k}]^{w_{ijk}}.\\
&\quad\prod_{1\le i<j\le n}[x_{1i}^{d(i)},x_{1j},x_{1j}]^{w_{ijj}}\\
&= A.B.C\ \text{(say)}
\end{aligned} \tag{2.23}$$

Now, for $1 \leq i < j \leq n$, $d(i) = d(j) = 2$, condition (1.7) becomes $u_{ij} = w_{ijj}d(i) + w'_{ijj}d(j)$, which implies that,

$$\begin{aligned}
A &= \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \\
&= \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}]^{w_{ijj}d(i)+w'_{ijj}d(j)} \\
&= \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}]^{d(j)(w_{ijj}+w'_{ijj})} \\
&= \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}^{d(j)}]^{w_{ijj}+w'_{ijj}} \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-(w_{ijj}+w'_{ijj})\binom{d(j)}{2}} \\
&= \prod_{1\leq i<j\leq n} [x_{21}, x_{1j}^{d(j)}]^{b_{i1}(w_{ijj}+w'_{ijj})} \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}}, \quad \text{(using (1.1))} \\
&\prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w'_{ijj}} \\
&= \prod_{1\leq i<j\leq n} [x_{21}, x_{21}]^{b_{i1}b_{j1}(w_{ijj}+w'_{ijj})} \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}} \\
&\prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w'_{ijj}} \\
&= \prod_{1\leq i<j\leq n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{ijj}}, \quad (\because\ w'_{ijj} \equiv 0 \pmod{d(j)} \text{ by condition (1.12)})
\end{aligned} \tag{2.24}$$

Now, using condition (1.13) we get that,

$$B = 1 \tag{2.25}$$

From (2.23), (2.24) and (2.25), we get that $g = 1$. □

Similarly, the following result can be proved.

Theorem 2.6. *Let G be a finite 2-group and $G/\gamma_2(G) \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where C_i, $1 \leq i \leq n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-1) = 2, d(n) \geq 2^2$. Let $\gamma_2(G)/\gamma_3(G)$ be cyclic group. Then $D_5(G) = \gamma_5(G)$.*

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References

1. S. Gupta, *The Fifth Dimension Subgroup for Metabelian 2-Groups*, Chinese Journal of Mathematics (2016) **doi:10.1155/2016/5342926**.
2. G. Losey, *N-series and filtrations of the augmentation ideal*, Canad. J. Math **26** (1974), 962-977.
3. R. Mikhailov, and I. B. S. Passi, Lower central and dimension series of groups, Vol. 1952 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2009.

4. I. B. S. Passi, *Dimension subgroups*, Journal of Algebra **9**, 2 (1968), 152-182.
5. E. Rips *On the fourth integer dimension subgroup*, Israel Journal of Mathematics **12**, 4 (1972), 342-346.
6. K. I. Tahara, *The augmentation quotients of group rings and the fifth dimension subgroups*, Journal of Algebra **71**, 1 (1981), 141-173.

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