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OSCILLATORY FEATURES OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The oscillatory behavior of fourth-order differential equations is examined in this research. We derive novel oscillation criteria using Riccati substitution, which enhance the pertinent findings in the literature. To further highlight the significance of the discovered oscillation criterion, a few examples are provided.

Keywords. Riccati substitution, Delay differential equations, Oscillation criteria, Fourth-order
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1. Introduction

Nowadays, there is a lot of activity in the oscillation theory of differential equations with delay arguments, or delay differential equations (DDEs). This is due to the fact that DDEs have more applications than conventional differential equations. For instance, we see that the significance of this kind of differential equation is clear when analyzing the majority of mathematical models that are used to forecast and evaluate a wide range of real-world scientific phenomena, including dynamic systems, neural network models, electrical engineering, and epidemiology [12, 24, 25]. The DDEs are utilized in epidemiology to ascertain the duration of infection, the stages of the virus life cycle, and the time needed for cell infection and the generation of new viruses (see [13, 17]). Second-order DDEs, on the other hand, are the most common and obvious since they can be used to mathematically simulate a variety of phenomena in engineering, physics, and biology. The voltage control model of oscillating neurons in neuro engineering is one of these models; refer to [26]. Additionally, we direct the reader to the works [20, 3, 1] for biological mathematics models where cross diffusion terms can be used to explain oscillation and/or delay actions.

Consider the fourth-order DEs

$$\left(a\left(s\right)\left(x'''\left(s\right)\right)^{\delta}\right)'+\psi\left(s,x\left(\varsigma_i\left(s\right)\right)\right)=0. \tag{1.1}$$

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Under the following assumptions:

(P_1) $a \in C^1([s_0,\infty))$, $a(s) > 0$, $a'(s) \geq 0$ and

$$\int_{s_0}^{\infty} \frac{1}{a^{1/\delta}(\theta)} \mathrm{d}\theta = \infty. \tag{1.2}$$

(P_2) $\varsigma_i(s) \in C([s_0,\infty),\mathbb{R})$, $\varsigma_i(s) \leq s$, $\lim_{s\to\infty} \varsigma_i(s) = \infty, i = 1,2,3,...$

(P_3) $\psi(s) \in C([s_0,\infty) \times \mathbb{R},\mathbb{R})$ and there exists $h \in C([s_0,\infty),[0,\infty))$ such that

$$\psi(s, x(\varsigma_i(s))) \geq \sum_{i=1}^{m} h_i(s) x^{\gamma}(\varsigma_i(s)),$$

and γ, δ are quotient of odd positive integers.

A function $x(s) \in C([s_x,\infty),\mathbb{R})$ is a solution of (1.1) it has the property $a(s)(x'''(s))^{\delta} \in C^1([s_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[s_x,\infty)$ where $s_x \geq s_0$.The correct solution of (1.1) that satisfies $\sup\{|x(s)| : s \geq s_*\} > 0$ for any $s_* \geq s_x$ was the only one we took into consideration. Such a solution of (1.1) is called oscillatory if it has infinitely many zeros on $[s_x,\infty)$. That is, for any $s_1 \geq s_x$, there exists $s_2 \geq s_1$, $x(s_2) = 0$. Otherwise, the solution is said to be non-oscillatory. Equation (1.1) is oscillatory if all its solutions are oscillatory.

As a part of the qualitative theory of differential equations, the oscillation theory of differential equations has come a long way in the last decade. Studying the features of oscillatory and non-oscillatory solutions was its focus. The oscillation solutions of numerous classes of differential and functional equations have since been the subject of extensive research. Several monographs, such as [9, 19], witness to the interest in this topic.

A well-known and extensively used subclass of functional differential equations (FDEs) are neutral and delay differential equations (NDEs), especially when simulating real-world occurrences. These equations differ from ordinary differential equations (ODEs) in that they incorporate temporal delay arguments into their structure because the derivative of the dependent variable at a particular time is influenced by both its present state and its delayed values. The memory effects present in many physical, biological, and engineering systems are captured by this delay element. As a result, NDEs are essential for accurately modeling and analyzing dynamic systems in which past states have a substantial influence on present and future behavior [7, 8]. The research of the oscillatory behavior of different kinds of DEs has grown and developed remarkably in the last few decades. DEs with delay arguments, which have garnered significant attention in the literature, are among the most frequently researched issues [6, 4]. Similarly, NDEs have emerged as a major field of study, as evidenced by many contributions (see [14, 16, 23]. While even-order DEs have also been the focus of considerable oscillation research, see [21, 5, 15].

Here, we examine several earlier findings that contributed to the development of the theory of oscillation for fourth-order DDEs, as well as some significant prior work that we will compare with our findings to verify their significance and uniqueness. Agarwal et al. [2] studied the even-order DEs

$$\left[a(s)\left(x^{(m-1)}(s)\right)^{\delta}\right]' + h(s)\,x^{\delta}(\varsigma(s)) = 0, \tag{1.3}$$

where $\varsigma(s) \geq s$. They established new criteria for oscillation (1.3) in the canonical and non-canonical cases under various conditions.
Baculikova et al. [10] derived the oscillation criteria of DEs

$$\left[a(s)\left(x^{(m-1)}(s)\right)^{\delta}\right]' + h(s)\,f(x(\varsigma(s))) = 0, \tag{1.4}$$

under the assumption that (1.2) and they proved that the equation (1.4) is oscillatory if

$$q'(s) + h(s)f\left(\frac{\mu\varsigma^{m-1}(s)}{(m-1)!a^{\frac{1}{\delta}}(\varsigma(s))}\right) f\left(q^{\frac{1}{\delta}}(\varsigma(s))\right) = 0,$$

is oscillatory.
Aljawi et al. [22] derived novel oscillation conditions for a class of fourth-order DEs with a superlinear neutral term

$$\left(a(s)\left(x'''(s)\right)^{\delta}\right)' + \sum_{i=1}^{m} h_i(s)\,x^{\delta}(\varsigma_i(s)) = 0. \tag{1.5}$$

They first reduce the nonlinear equation (1.4) to a linear equation and then arrive at criteria for the oscillation of (1.4) via two first-order DDEs whose oscillation properties are known.
The theory of oscillation of fourth-order differential equations was developed through a number of investigations that are reviewed in this work. The new sharpest oscillatory properties for (1.1) are then obtained by providing some oscillatory criteria for the positive solutions. Lastly, we present fundamental oscillation theorems that accomplish the paper's goal and provide some examples to highlight the significance of the study's findings.

2. Preliminary Lemmas

In this section, we present some auxiliary lemmas to make the study of the properties of solutions of (1.1).

Lemma 2.1. [28] *Let $\phi \in C^m([s_0,\infty))$ and $\phi(s) > 0$. Suppose that $\phi^{(m)}(s)$ is of a fixed sign, on $[s_0,\infty)$, $\phi^{(m)}(s)$ not identically zero and that there exists a $s_1 \geq s_0$ such that*

$$\phi^{(m-1)}(s)\,\phi^{(m)}(s) \leq 0.$$

If we have $\lim_{s\to\infty}\phi(s) \neq 0$, then there exists $s_\epsilon \geq s_0$ such that

$$\phi(s) \geq \frac{\epsilon}{(m-1)!}s^{m-1}\left|\phi^{(m-1)}(s)\right|,$$

for every $\epsilon \in (0,1)$ and $s \geq s_\epsilon$.

Lemma 2.2. [27] *Assume that C and D are real, $D > 0$. Then*

$$Cq - Dq^{(\delta+1)/\delta} \leq \frac{\delta^{\delta}}{(\delta+1)^{\delta+1}}C^{\delta+1}D^{-\delta}. \tag{2.1}$$

Lemma 2.3. [18] *If the function x satisfies $x^{(j)}>0$ for all $j=0,1,...,m,$ and $x^{(m+1)}(s)<0,$ then*

$$\frac{m!}{s^{m}}x(s)\geq\frac{(m-1)!}{s^{m-1}}x'(s).$$

Lemma 2.4. [22] *Let $x(s)$ be an eventually positive solution of (1.1). Then, $x(s)$ satisfies one of the following cases:*

$$\begin{array}{lll}(S_1)\ x^{(\kappa)}(s) & > & 0 \quad \text{for } \kappa=0,1,2,3;\\ (S_2)\ x^{(\kappa)}(s) & > & 0 \quad \text{for } \kappa=0,1,3, \text{ and } x''(s)<0.\end{array}$$

To simplify subsequent computations, we employ the following notation:

$$z_1(s):=\frac{a(s)\left(x'''(s)\right)^{\delta}}{x^{\delta}(s)},\ z_2(s):=\frac{x'(s)}{x(s)},\ G_1(s):=\frac{\delta\varepsilon}{2}\frac{s^2}{a^{1/\delta}(s)},$$

and

$$\widetilde{G}(s):=\epsilon^{\gamma/\delta}A_2^{\gamma-\delta}\int_s^{\infty}\left(\frac{1}{a(r)}\int_r^{\infty}\sum_{i=1}^{m}h_i(\theta)\left(\frac{\varsigma_i(\theta)}{\theta}\right)^{\gamma}\mathrm{d}\theta\right)^{1/\delta}\mathrm{d}r,$$

where $\varepsilon,\epsilon\in(0,1)$ and A_2 is a positive constant.

We define a sequence of functions $\{q_n(s)\}_{n=0}^{\infty}$ and $\{z_n(s)\}_{n=0}^{\infty}$ as follows

$$q_n(s)=q_0(s)+\int_s^{\infty}G_1(\theta)\,q_{n-1}^{\frac{\delta+1}{\delta}}(\theta)\,\mathrm{d}\theta \tag{2.2}$$

and

$$z_n(s)=z_0(s)+\int_s^{\infty}z_{n-1}^2(\theta)\,\mathrm{d}\theta, \tag{2.3}$$

where

$$q_0(s)=\int_s^{\infty}A_1^{\gamma-\delta}\sum_{i=1}^{m}h_i(\theta)\frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}}\mathrm{d}\theta$$

and

$$z_0(s)=\int_s^{\infty}\widetilde{G}(\theta)\,\mathrm{d}\theta.$$

3. Main results

In this section, we present some oscillation criteria of solutions of (1.1).

Lemma 3.1. *Let $x(s)$ is an eventually positive solution of equation (1.1) and (S_1) holds, then*

$$z_1'(s)+\sum_{i=1}^{m}h_i(s)\frac{\varsigma_i^{3\delta}(s)}{s^{3\delta}}B^{\gamma-\delta}(\varsigma_i(s))+\frac{\delta\varepsilon}{2}\frac{s^2}{a^{1/\delta}(s)}z_1^{1+1/\delta}(s)\leq 0. \tag{3.1}$$

Proof. Let that $x(s)$ is an eventually positive solution of equation (1.1). Let (S_1) holds. Then, from Lemma 2.1, we get

$$x'(s)\geq\frac{\varepsilon}{2}s^2x'''(s) \tag{3.2}$$

for every $\varepsilon \in (0,1)$ and for all large q. From Lemma 2.3, we have that $x(s) \geq \frac{s}{3}x'(s)$ and hence,

$$\frac{x(\varsigma_i(s))}{x(s)} \geq \frac{\varsigma_i^3(s)}{s^3}. \tag{3.3}$$

Differentiating z_1 and using (1.1), (3.2) and (3.3), we obtain

$$z_1'(s) \leq -\sum_{i=1}^{m} h_i(s)\frac{\varsigma_i^{3\delta}(s)}{s^{3\delta}}x^{\gamma-\delta}(\varsigma_i(s)) - \frac{\delta\varepsilon}{2}\frac{s^2}{a^{1/\delta}(s)}z_1^{1+1/\delta}(s).$$

Since $x'(s) > 0$, there exist a $s_2 \geq s_1$ and $B > 0$ such that $x(s) > B$, for all $s \geq s_2$. Thus, we obtain

$$z_1'(s) \leq -\sum_{i=1}^{m} h_i(s)\frac{\varsigma_i^{3\delta}(s)}{s^{3\delta}}B^{\gamma-\delta}(\varsigma_i(s)) - \frac{\delta\varepsilon}{2}\frac{s^2}{a^{1/\delta}(s)}z_1^{1+1/\delta}(s),$$

Thus, (3.1) is satisfied. The proof is finished. □

Lemma 3.2. *Let $x(s)$ is a positive solution of (1.1) and (S_2) holds, then*

$$z_2'(s) + z_2^2(s) + B^{\gamma-\delta}\widetilde{G}(s) \leq 0. \tag{3.4}$$

Proof. Assume that x is a positive solution of (1.1) and (S_2) holds. We Integrating (1.1) from s to l, we have

$$a(l)(x'''(l))^{\delta} = a(s)(x'''(s))^{\delta} - \int_s^r \sum_{i=1}^{m} h_i(\theta)x^{\gamma}(\varsigma_i(\theta))\,\mathrm{d}\theta. \tag{3.5}$$

When we consider Lemma 2.3, we get

$$x(s) \geq sx'(s). \tag{3.6}$$

Thus, $x(\varsigma_i(s)) \geq (\varsigma_i(s)/s)x(s)$, which with (3.5) and $x'(s) > 0$ gives

$$a(l)(x'''(l))^{\delta} - a(s)(x'''(s))^{\delta} + x^{\gamma}(s)\int_s^l \sum_{i=1}^{m} h_i(\theta)\left(\frac{\varsigma_i(\theta)}{\theta}\right)^{\gamma}\mathrm{d}\theta \leq 0.$$

Letting $l \to \infty$, we obtain

$$x'''(s) \geq \frac{\epsilon^{\gamma/\delta}}{a^{1/\delta}(s)}x^{\gamma/\delta}(s)\left(\int_s^{\infty}\sum_{i=1}^{m} h_i(\theta)\left(\frac{\varsigma_i(\theta)}{\theta}\right)^{\gamma}\mathrm{d}\theta\right)^{1/\delta}.$$

Integrating the above inequality from s to ∞, we obtain

$$\begin{aligned} x''(s) &\leq -\epsilon^{\gamma/\delta}x^{\gamma/\delta}(s)\int_s^{\infty}\left(\frac{1}{a(r)}\int_r^{\infty}\sum_{i=1}^{m} h_i(\theta)\left(\frac{\varsigma_i(\theta)}{\theta}\right)^{\gamma}\mathrm{d}\theta\right)^{1/\delta}\mathrm{d}r \\ &\leq -\widetilde{G}(s)x^{\gamma/\delta}(s). \end{aligned} \tag{3.7}$$

Differentiating z_2 and using (3.7), we get

$$z_2'(s) + z_2^2(s) + B^{\gamma-\delta}\widetilde{G}(s) \leq 0.$$

Thus, the proof is complete. □

Theorem 3.3. *If*

$$\int_{s_0}^{\infty} A_1^{\gamma-\delta} \sum_{i=1}^{m} h_i(\theta) \frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}} \mathrm{d}\theta = \infty \tag{3.8}$$

and

$$\int_{s_0}^{\infty} \widetilde{G}(\theta) \,\mathrm{d}\theta = \infty, \tag{3.9}$$

then (1.1) is oscillatory.

Proof. Assume that x is a positive solution of (1.1). From Lemma 3.1, we see that (3.1) holds, which yields

$$z_1'(s) + A_1^{\gamma-\delta} \sum_{i=1}^{m} h_i(s) \frac{\varsigma_i^{3\delta}(s)}{s^{3\delta}} \leq 0. \tag{3.10}$$

Integrating (3.10) from s_2 to s and using (3.8),we obtain

$$z_1(s) \leq z_1(s_2) - \int_{s_2}^{s} A_1^{\gamma-\delta} \sum_{i=1}^{m} h_i(\theta) \frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}} d\theta \to -\infty \quad \text{as} \quad s \to \infty,$$

which is a contradicts to the fact that $z_1(s) > 0$.
Similarly, in the case where (S_2) holds by Lemma 3.2, (3.9) disagrees, which is why it is excluded here for ease. Thus, the proof is finished. □

Theorem 3.4. *Assume that*

$$\liminf_{s\to\infty} \frac{1}{q_0(s)} \int_{s}^{\infty} G_1(\theta)\, q_0^{\frac{\delta+1}{\delta}}(\theta) \,\mathrm{d}\theta > \frac{\delta}{(\delta+1)^{\frac{\delta+1}{\delta}}} \tag{3.11}$$

and

$$\liminf_{s\to\infty} \frac{1}{z_0(s)} \int_{s}^{\infty} z_0^2(\theta) \,\mathrm{d}\theta > \frac{1}{4}. \tag{3.12}$$

Then, (1.1) is oscillatory.

Proof. Suppose that x is a positive solution of (1.1). Using Lemma 2.4, we get that some cases.
Let case (S_1) holds. Using Lemma 3.1, we obtain (3.1). Integrating (3.1) from s to l, we get

$$z_1(l) - z_1(s) + \int_{s}^{l} A_1^{\gamma-\delta} \sum_{i=1}^{m} h_i(\theta) \frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}} \mathrm{d}\theta + \int_{s}^{l} G_1(\theta)\, z_2^{\frac{\delta+1}{\delta}}(\theta) \,\mathrm{d}\theta \leq 0. \tag{3.13}$$

From(3.13), it is obvious that

$$z_1(l) - z_1(s) + \int_{s}^{l} G_1(\theta)\, z_1(\theta) \,\mathrm{d}\theta \leq 0. \tag{3.14}$$

Then we conclude from (3.14) that

$$\int_{s}^{\infty} G_1(\theta)\, z_1(\theta) \,\mathrm{d}\theta < \infty, \text{ for } s \geq s, \tag{3.15}$$

otherwise,

$$z_1(l) \le z_1(s) - \int_s^\infty G_1(\theta) z_1(\theta) \mathrm{d}\theta \to -\infty.$$

This runs counter to the idea that $z_1(s) > 0$. The $\lim_{s\to\infty} z_1(s) = k \ge 0$ because $z_1(s)$ is positive and declining. We have $k = 0$ by virtue of (3.15). Consequently, from (3.13), we obtain

$$z_1(s) \ge \widetilde{h}(s) + \int_s^\infty G_1(\theta) z_1(\theta) \mathrm{d}\theta = q_0(s) + \int_s^\infty G_1(\theta) z_1(\theta) \mathrm{d}\theta. \tag{3.16}$$

From (3.16), we have

$$\frac{z_1(s)}{q_0(s)} \ge 1 + \frac{1}{q_0(s)} \int_s^\infty G_1(\theta) q_0^{\frac{\delta+1}{\delta}}(\theta) \left(\frac{z_1(\theta)}{q_0(\theta)}\right)^{\frac{\delta+1}{\delta}} \mathrm{d}\theta, \;\; s \ge s. \tag{3.17}$$

If we set $\mu = \inf_{s\ge s} z_1(s)/q_0(s)$, then $\mu \ge 1$. Thus, from (3.11) and (3.17) we see that

$$\delta \left(\frac{\mu}{\delta+1}\right)^{(\delta+1)/\delta} \le \mu - 1$$

or

$$\left(\frac{\mu}{\delta+1}\right)^{(\delta+1)/\delta} \le \frac{\delta+1}{\delta}\left(\frac{\mu}{\delta+1} - \frac{1}{\delta+1}\right).$$

This gives us a contradiction with the value of μ and δ.
Similarly, in case (S_2), if we suppose that $\mu_1 = \inf_{s\ge s_1} z(s)/z_0(s)$ and taking into consideration (3.12), we obtain a contradiction with the value of μ_1. The proof is finished. □

Theorem 3.5. *If*

$$\limsup_{s\to\infty} q_n(s) \left(\frac{\varepsilon}{2} s^2 \int_{s_0}^s a^{-1/\delta}(\theta) \mathrm{d}\theta\right)^\delta > 1 \tag{3.18}$$

and

$$\limsup_{s\to\infty} s z_n(s) > 1, \tag{3.19}$$

hold. Then (1.1) is oscillatory.

Proof. Assume that x is a positive solution of (1.1). From Lemma 2.4, we see that two cases.
Let case (S_1) holds. By Lemma 2.1, we obtain

$$x(s) \ge \frac{\varepsilon}{6} s^3 x'''(s). \tag{3.20}$$

From (3.20), we see

$$\begin{aligned} \frac{1}{z_1(s)} &= \frac{1}{a(s)} \left(\frac{x(s)}{x'''(s)}\right)^\delta \\ &\ge \frac{1}{a(s)} \left(\frac{\varepsilon}{6} s^3\right)^\delta \end{aligned}$$

Thus,

$$z_1(s)\frac{1}{a(s)}\left(\frac{\varepsilon}{6}s^3\right)^{\delta}\leq 1 \tag{3.21}$$

and

$$\limsup_{s\to\infty} z_1(s)\left(\frac{\varepsilon s^3}{6a^{1/\delta}(s)}\right)^{\delta}\leq 1,$$

which contradicts (3.18).
Similarly, in case (S_2), we gives us a contradiction with (3.19). Therefore, the proof is complete. □

Corollary 3.6. *If*

$$\int_T^s A_1^{\gamma-\delta}\sum_{i=1}^m h_i(\theta)\frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}}\exp\left(\int_T^{\theta}G_1(r)\,q_n^{1/\delta}(r)\,\mathrm{d}r\right)\mathrm{d}\theta=\infty \tag{3.22}$$

and

$$\int_T^s \widetilde{G}(\theta)\exp\left(\int_T^{\theta}z_n(r)\,\mathrm{d}r\right)\mathrm{d}\theta=\infty, \tag{3.23}$$

hold, then (1.1) is oscillatory.

Proof. Assume that x is a positive solution of (1.1). From Lemma 2.4, we see that two cases.
Let case (S_1) holds. Following a similar line of reasoning as employed in the proof of Theorem 3.4, we get (3.16). From (3.16), we see

$$z_1(s)\geq q_0(s).$$

Moreover, by induction we can also see that $z_1(s)\geq q_n(s)$ for $s\geq s_0$, $n=1,2,3,....$ Consequently, the sequence $\{q_n(s)\}_0^{\infty}$ converges to $q(s)$ as it is monotonically rising and bounded above. Using the Lebesgues monotone convergence theorem and allowing $n\to\infty$ in (2.2), we get

$$q(s)=q_0(s)+\int_s^{\infty}G_1(\theta)\,q^{\frac{\delta+1}{\delta}}(\theta)\,\mathrm{d}\theta. \tag{3.24}$$

From (3.24), we have that

$$q'(s)=-G_1(s)\,q^{\frac{\delta+1}{\delta}}(s)-A_1^{\gamma-\delta}\sum_{i=1}^m h_i(s)\frac{\varsigma_i^{3\delta}(s)}{s^{3\delta}}. \tag{3.25}$$

Since $q_n(s)\leq q(s)$, it follows from (3.25) that

$$q'(s)\leq -G_1(s)\,q_n^{1/\delta}(s)\,q(s)-A_1^{\gamma-\delta}\sum_{i=1}^m h_i(s)\frac{\varsigma_i^{3\delta}(s)}{s^{3\delta}}.$$

Hence, we get

$$q(s)\leq\exp\left(-\int_T^s G_1(\theta)\,q_n^{1/\delta}(\theta)\,\mathrm{d}\theta\right)\left(q(T)-\int_T^s A_1^{\gamma-\delta}\sum_{i=1}^m h_i(\theta)\frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}}\exp\left(\int_T^{\theta}G_1(r)\,q_n^{1/\delta}(r)\,\mathrm{d}r\right)\mathrm{d}\right.$$

The above inequality follows

$$\int_T^s A_1^{\gamma-\delta} \sum_{i=1}^m h_i(\theta) \frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}} \exp\left(\int_T^\theta G_1(r) q_n^{1/\delta}(r) \mathrm{d}r\right) \mathrm{d}\theta \le q(T) < \infty,$$

which contradicts (3.22).
Similarly, in case (S_2), we gives us a contradiction with (3.23). The proof is finished. □

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4. Examples

Now, we apply the results of this paper to some examples.

Example 4.1. Consider the equation

$$\left((x'''(s))^5\right)' + \frac{h_0}{s} x^5\left(\frac{1}{4}s\right) = 0, \tag{4.1}$$

where $h_0 > 0,\ \delta = \gamma = 5,\ a(s) = 1,\ \varsigma(s) = s/4$ and $h(s) = h_0/s$. Hence, it is easy to see that

$$\int_{s_0}^\infty \frac{1}{a^{1/\delta}(\theta)} \mathrm{d}\theta = \infty.$$

By using Theorem 3.3, we see that the conditions

$$\begin{aligned} &\int_{s_0}^\infty A_1^{\gamma-\delta} \sum_{i=1}^m h_i(\theta) \frac{\varsigma_i^{3\delta}(\theta)}{\theta^{3\delta}} \mathrm{d}\theta, \\ =\ &\int_{s_0}^\infty \frac{h_0}{\theta} \frac{1}{4^{15}} \mathrm{d}\theta = \infty, \end{aligned}$$

and

$$\begin{aligned} &\int_{s_0}^\infty \widetilde{G}(\theta) \mathrm{d}\theta \\ =\ &\int_{s_0}^\infty \epsilon^{\gamma/\delta} A_2^{\gamma-\delta} \int_s^\infty \left(\frac{1}{a(r)} \int_r^\infty \sum_{i=1}^m h_i(\theta) \left(\frac{\varsigma_i(\theta)}{\theta}\right)^\gamma \mathrm{d}\theta\right)^{1/\delta} \mathrm{d}r, \\ =\ &\int_{s_0}^\infty \epsilon \int_s^\infty \left(\int_r^\infty \sum_{i=1}^m \frac{h_0}{\theta}\left(\frac{1}{4}\right)^5 \mathrm{d}\theta\right)^{1/5} \mathrm{d}r, \\ =\ &\infty. \end{aligned}$$

Thus, we see that the equation (4.1) is oscillatory.

Example 4.2. Consider the equation

$$x^{(4)}(s) + \frac{h_0}{s^4} x\left(\frac{1}{2}s\right) = 0, \tag{4.2}$$

where $h_0 > 0$, $\delta = \gamma = 1$, $a(s) = 1$, $\varsigma(s) = s/2$ and $h(s) = h_0/s^4$. Hence, it is easy to see that

$$q_0 = \frac{h_0}{24s}$$

and

$$z_0(s) = \frac{h_0}{2s}.$$

Therefore, if $h_0 > 36$, equation (4.2) is oscillatory according to Theorem 3.4.

5. Conclusion

The oscillatory conditions of fourth-order FDEs was examined in this paper. We developed some criteria that expand and generalize a number of recognized findings in the literature by combining Riccati transformations with some method.

The recursive formulation helped us better capture long-term dynamics impacted by delays, and the analysis relied on auxiliary lemmas that characterize the qualitative behavior of solutions. Even in situations with nonlinear or nonuniform delays, the criteria can be implemented successfully, as demonstrated by the examples given.

Although the findings presented here provide a strong theoretical basis, there are a number of avenues for further research. Adding more broad delay types, expanding the method to higher-order equations, or investigating borderline situations using numerical simulations could all yield insightful results.

All things considered, the results advance oscillation theory for delay differential equations and might encourage more theoretical and practical investigation.

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