

EIGENVALUES FOR THE p -BIHARMONIC DIRICHLET-STEKLOV PROBLEM WITH WEIGHTS

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ABSTRACT. This note is concerned with a weighted Dirichlet-Steklov problem driven by the p -biharmonic operator

$$\begin{cases} \Delta(\tau(x)|\Delta u|^{p-2}\Delta u) = 0, & \text{for } x \in \Omega, \\ \tau(x)|\Delta u|^{p-2}\Delta u = \lambda\xi(x)\left|\frac{\partial u}{\partial v}\right|^{p-2}\frac{\partial u}{\partial v}, & \text{for } x \in \partial\Omega, \\ u = 0, & \text{for } x \in \partial\Omega. \end{cases}$$

Our approach is based on variational method and Ljusternick-Schnirelmann principle, we establish that the above problem admits a non-decreasing sequence of non-negative eigenvalues.

1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$). The aim of this study is to analyse the existence of one non-decreasing sequence of positive eigenvalues of the fourth order nonlinear Dirichlet-Steklov boundary problem

$$\begin{cases} \Delta(\tau(x)|\Delta u|^{p-2}\Delta u) = 0, & \text{for } x \in \Omega, \\ \tau(x)|\Delta u|^{p-2}\Delta u = \lambda\xi(x)\left|\frac{\partial u}{\partial v}\right|^{p-2}\frac{\partial u}{\partial v}, & \text{for } x \in \partial\Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \quad (1.1)$$

where λ is a positive real number, $1 < p < \frac{N}{2}$, $\frac{\partial u}{\partial v}$ denotes the outer normal derivative of u on $\partial\Omega$. We also use the assumptions

(A1) τ is a function which satisfies the condition

$$0 < a \leq \tau(x) \leq b, \text{ where } a \text{ and } b \text{ are positive constants.}$$

(A2) $meas\left\{x \in \partial\Omega \mid \xi(x) > 0\right\} > 0, \xi \in L^{\frac{q}{q-p}}(\partial\Omega)$, for some $p \leq q \leq \frac{(N-1)p}{N-2p}$.

In the sequel $\mathbb{E}_p(\Omega)$ will denote the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, one can show that

$$\|u\| := \left(\int_{\Omega} |\Delta v(x)|^p dx \right)^{\frac{1}{p}}$$

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is a norm on $\mathbb{E}_p(\Omega)$ which is equivalent to the standard norm of $\mathbb{E}_p(\Omega)$ (see [9]).

Define on $\mathbb{E}_p(\Omega)$ the functionals

$$F(u) = \frac{1}{p} \int_{\partial\Omega} \xi(x) \left| \frac{\partial u(x)}{\partial v} \right|^p ds, \quad (1.2)$$

$$G(u) = \frac{1}{p} \int_{\Omega} \tau(x) |\Delta u(x)|^p dx. \quad (1.3)$$

We set

$$\mathfrak{N} := \left\{ u \in \mathbb{E}_p(\Omega) \mid pF(u) = 1 \right\}. \quad (1.4)$$

We say that $\lambda > 0$ is a positive eigenvalue of (1.1) if

$$\int_{\Omega} \tau(x) |\Delta u(x)|^{p-2} \Delta u(x) \Delta \varphi(x) dx = \lambda \int_{\partial\Omega} \xi(x) \left| \frac{\partial u(x)}{\partial v} \right|^{p-2} \frac{\partial u(x)}{\partial v} \frac{\partial \varphi(x)}{\partial v} ds$$

for all $\varphi \in \mathbb{E}_p(\Omega)$, where ds is the $N - 1$ dimensional Hausdorff measure. Then u is called an eigenfunction corresponding to the eigenvalue λ . The pair (u, λ) is called an eigenpair.

Here $\Delta(\tau(x)|\Delta u|^{p-2}\Delta u)$ denotes the operator of fourth order called the p -biharmonic operator with weight. For $p = 2$ and $\tau = 1$, the operator becomes the iterated Laplacian which have been studied by many authors. For example, in [3], the authors proved some results about the first Steklov eigenvalue of the following problem

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = \lambda \frac{\partial u}{\partial v}, & \text{on } \partial\Omega. \end{cases}$$

Elliptic problems with parameters in the boundary conditions have been called Steklov or Steklov-type problems since their first appearance in [13]. Eigenvalue p -biharmonic equations on a bounded domain have been studied extensively and many interesting results have been obtained. (see e.g. [4, 5, 6, 7, 15]).

Beside being of mathematical interest, the study of the p -biharmonic operator is also of interest in micro-electro-mechanical system, in thin film theory, nonlinear surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, phase field models of multi-phase systems, deformation of a nonlinear elastic beam, (see [8, 11, 2] and references therein for discussions of various applications).

Motivated by the contributions cited above and the works of [10], we show that problem (1.1) has at least one non-decreasing sequence positive eigenvalues $(\lambda_k)_{k \geq 1}$ tending to ∞ as $k \rightarrow \infty$, by using a variational technique based on Ljusternik-Schnirelmann theory on C^1 -manifolds [14].

As far as we are aware, nonlinear eigenvalue problems like (1.1) involving the p -biharmonic operator with this special weight have not yet been studied. That is why, at our best knowledge, the present paper is a first contribution in this direction.

Our first result can be described as follow

Theorem 1.1. *(Existence of non-decreasing sequence of eigenvalues) Suppose (A1) and (A2) hold. Then for any integer $k \in \mathbb{N}^*$,*

$$\lambda_k := \inf_{K \in \Gamma_k} \max_{u \in K} p G(u),$$

is a critical value of $\tilde{G}$, where $\tilde{G}$ denotes the restriction of G to the manifold $\mathfrak{N}$. More precisely, there exist $u_k \in K$, such that

$$\lambda_k = p G(u_k) = \sup_{u \in K} p G(u),$$

and u_k is a solution of (1.1) associated to positive eigenvalue λ_k . Moreover,

$$\lambda_k \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

Remark. Here $\Gamma_k := \left\{ K \subset M \mid K \text{ symmetric, compact and } \gamma(K) \geq k \right\}$, where $\gamma(K)$ denote the genus of the set K , i.e.,

$$\gamma(K) := \inf \left\{ j \in \mathbb{N}^* \mid \exists h : K \rightarrow \mathbb{R}^j \setminus \{0\} \text{ such that } h \text{ is continuous and odd map} \right\}.$$

If such a mapping h does not exist for $k > 0$, we set $\gamma(K) = \infty$. From definition, $\gamma(\emptyset) = 0$.

In the second result, we show that the smallest eigenvalue exists and is strictly positive.

Corollary 1.2. *(Existence and positivity of the first eigenvalue) The following assertions hold true*

- 1 $\lambda_1 = \inf \left\{ \|u\|^p, u \in \mathbb{E}_p(\Omega) \mid pF(u) = 1 \right\}.$
- 2 *The spectrum of problem 1.1 consists of a diverging sequence of eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow +\infty.$$

This article consists of three sections. Section 1 contains an introduction and the main results. In Section 2, we state some preliminary properties and an auxiliary results. The proofs of Theorem 1.1 and Corollary 1.2 are given in Section 3.

2. Preliminaries and an auxiliary results

We introduce some notation that will clarify what follows. Thus, when we refer to a Banach space X , we denote by X^* its dual, by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^* , we write $u_n \rightharpoonup u$ and $u_n \rightarrow u$ to denote the weak convergence and strong convergence of sequence u_n in X , respectively, and by $meas(\cdot)$ we denote the Lebesgue measure of a set.

We recall a version of the Ljusternik-Schnirelmann principle (L-S), which was discussed in Corollary 4.1 of [14].

Theorem 2.1. *Suppose that M is a closed symmetric C^1 -manifold of a real Banach space E and $0 \notin M$. Suppose that $f \in C^1(M, \mathbb{R})$ is even and bounded below. Define*

$$c_j = \inf_{K \in \Gamma_j} \sup_{x \in K} f(x),$$

where

$$\Gamma_j = \left\{ K \subset M \mid K \text{ symmetric, compact and } \gamma(K) \geq j \right\},$$

If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies the Palais Smale condition $(PS)_c$ for all $c = c_j, j = 1, \dots, k$, then f has at least k distinct pairs of critical points.

Lemma 2.2. *Let Ω be a domain in $\mathbb{R}^N$ and let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Young function which satisfies a Δ_2 -condition, i.e., there is $c > 0$ such that $\phi(2t) \leq c\phi(t)$ for all $t \geq 0$. If $\{u_n\}$ is a sequence of integrable functions in Ω such that*

$$u(x) = \lim_{n \rightarrow \infty} u_n(x), \text{ a.e. } x \in \Omega \text{ and } \int_{\Omega} \phi(|u|) dx = \lim_{n \rightarrow \infty} \int_{\Omega} \phi(|u_n|) dx,$$

then

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi(|u_n - u|) dx = 0.$$

Proof. See Theorem 12 of [12] for the proof.

Notice that the compact embedding

$$W^{m,p}(\Omega) \hookrightarrow L^q(\partial\Omega),$$

for all $mp < n$ and $p \leq q \leq \frac{(N-1)p}{N-mp}$, (see [1]). Denote by $C > 0$ the best constant for this embedding, that is,

$$\|u\|_{L^q(\partial\Omega)} \leq C \|u\|, \forall u \in W^{m,p}(\Omega).$$

Proposition 2.3. *The following assertions hold true*

- 1 *The functionals F and G are even, and of class C^1 on $\mathbb{E}_p(\Omega)$.*
- 2 *The level set $\mathfrak{N}$ is a closed C^1 -manifold.*

Proof. 1 It is clear that F and G are even and of class C^1 on $\mathbb{E}_p(\Omega)$, with the Gâteaux derivative given by

$$\left\langle G'(u), \varphi \right\rangle = \int_{\Omega} \tau(x) |\Delta u(x)|^{p-2} \Delta u(x) \Delta \varphi(x) dx,$$

and

$$\left\langle F'(u), \varphi \right\rangle = \int_{\partial\Omega} \xi(x) \left| \frac{\partial u(x)}{\partial \nu} \right|^{p-2} \frac{\partial u(x)}{\partial \nu} \frac{\partial \varphi(x)}{\partial \nu} ds.$$

2 In view of 1.2 and 1.4, we state that $\mathfrak{N} = F^{-1}(\{\frac{1}{p}\})$, Then $\mathfrak{N}$ is closed. The derivative operator F' satisfies $F'(u) \neq 0 \forall u \in \mathfrak{N}$, because

$$\langle F'(u), u \rangle = \int_{\partial\Omega} \xi(x) \left| \frac{\partial u(x)}{\partial \nu} \right|^p ds = 1 > 0, \text{ if } u \in \mathfrak{N}.$$

By the fact that $f : t \rightarrow |t|^{p-1}$ is strictly increasing, further $f(0) = 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$. This means that $F'(u)$ is onto $\forall u \in \mathfrak{N}$, so F is a submersion, then $\mathfrak{N}$ is a C^1 -manifold.

Lemma 2.4. *The functional G' satisfies condition (S_+) type, namely*

$$u_n \rightharpoonup u \text{ in } \mathbb{E}_p(\Omega) \text{ and } \overline{\lim}_{n \rightarrow \infty} \langle G'(u_n), u_n - u \rangle \leq 0, \text{ imply } u_n \rightarrow u \text{ in } \mathbb{E}_p(\Omega).$$

Proof. Let $(u_n)_{n \geq 1}$ be a sequence of $\mathbb{E}_p(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } \mathbb{E}_p(\Omega) \text{ and } \overline{\lim}_{n \rightarrow \infty} \langle G'(u_n), u_n - u \rangle \leq 0. \quad (2.1)$$

We have

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} \tau(x) |\Delta u_n(x)|^{p-2} \Delta u(x) dx (u_n - u) dx \\ &= \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} \tau(x) \left[\int_{\Omega} |\Delta u_n(x)|^{p-2} \Delta u_n(x) - |\Delta u(x)|^{p-2} \Delta u(x) \right] (u_n - u) dx. \end{aligned}$$

By the monotonicity of $t \leftarrow |t|^{p-1}$ and by the Hölder inequality, we get

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} \tau(x) \left[\int_{\Omega} |\Delta u_n(x)|^{p-2} \Delta u_n(x) - |\Delta u(x)|^{p-2} \Delta u(x) \right] (u_n - u) dx \\ & \geq a \left[\|u_n(x)\|^{p-1} - \|u(x)\|^{p-1} \right] \left[\|u_n(x)\| - \|u(x)\| \right] \\ & \geq 0. \end{aligned}$$

Hence 2.1 implies that $\|u_n(x)\| \rightarrow \|u(x)\|$. Since $u_n \rightharpoonup u$ in $\mathbb{E}_p(\Omega)$ and $\mathbb{E}_p(\Omega)$ is uniformly convex, $u_n \rightarrow u$ in $\mathbb{E}_p(\Omega)$.

Lemma 2.5. *The functional F' is sequentially weakly-strongly continuous, namely,*

$$u_n \rightharpoonup u \text{ in } \mathbb{E}_p(\Omega) \implies F'(u_n) \rightarrow F'(u) \text{ in } (\mathbb{E}_p(\Omega))^*.$$

Proof. Let $u_n \rightharpoonup u$ in $\mathbb{E}_p(\Omega)$. For any $\varphi \in \mathbb{E}_p(\Omega)$, by Hölder's inequality and the compact embedding $\mathbb{E}_p(\Omega) \hookrightarrow L^q(\partial\Omega)$, it follows that

$$\begin{aligned} & \left| \langle F'(u_n) - F'(u), \varphi \rangle \right| \\ &= \left| \int_{\partial\Omega} \zeta(x) \left(\left| \frac{\partial u_n(x)}{\partial v} \right|^{p-2} \frac{\partial u_n(x)}{\partial v} - \left| \frac{\partial u(x)}{\partial v} \right|^{p-2} \frac{\partial u(x)}{\partial v} \right) \varphi ds \right| \\ &\leq k \left| \zeta(x) \right|_{L^{\frac{q}{q-p}}(\partial\Omega)} \left| \left| \frac{\partial u_n(x)}{\partial v} \right|^{p-2} \frac{\partial u_n(x)}{\partial v} - \left| \frac{\partial u(x)}{\partial v} \right|^{p-2} \frac{\partial u(x)}{\partial v} \right|_{L^{\frac{q}{p-1}}(\partial\Omega)} \left| \varphi \right|_{L^q(\partial\Omega)} \\ &\leq kC \left| \zeta(x) \right|_{L^{\frac{q}{q-p}}(\partial\Omega)} \left| \left| \frac{\partial u_n(x)}{\partial v} \right|^{p-2} \frac{\partial u_n(x)}{\partial v} - \left| \frac{\partial u(x)}{\partial v} \right|^{p-2} \frac{\partial u(x)}{\partial v} \right|_{L^{\frac{q}{p-1}}(\partial\Omega)} \|\varphi\|. \end{aligned}$$

We next show that

$$\left| \frac{\partial u_n(x)}{\partial v} \right|^{p-2} \frac{\partial u_n(x)}{\partial v} \rightarrow \left| \frac{\partial u(x)}{\partial v} \right|^{p-2} \frac{\partial u(x)}{\partial v} \text{ in } L^{\frac{q}{p-1}}(\partial\Omega).$$

To see this, let $w_n = \left| \frac{\partial u_n(x)}{\partial v} \right|^{p-2} \frac{\partial u_n(x)}{\partial v}$ and $w = \left| \frac{\partial u(x)}{\partial v} \right|^{p-2} \frac{\partial u(x)}{\partial v}$. Using the compact embedding $\mathbb{E}_p(\Omega) \hookrightarrow L^q(\partial\Omega)$, for $u_n \rightarrow u$ in $\mathbb{E}_p(\Omega)$, we obtain $u_n \rightarrow u$ in $L^q(\partial\Omega)$. It follows that

$$w_n(x) \rightarrow w(x), \text{ a.e. on } \partial\Omega \text{ and } \int_{\partial\Omega} |w_n(x)|^{\frac{q}{p-1}} ds \rightarrow \int_{\partial\Omega} |w(x)|^{\frac{q}{p-1}} ds.$$

Using Lemma 2.2, we conclude that $w_n(x) \rightarrow w(x)$ in $L^{\frac{q}{p-1}}(\partial\Omega)$. Therefore

$$F'(u_n) \rightarrow F'(u) \text{ in } (\mathbb{E}_p(\Omega))^*.$$

Proposition 2.6. *The functional G satisfies the Palais-Smale condition $(PS)_c$ on $\mathfrak{N}$ for every $c \neq 0$, i.e., for $(u_n)_{n \geq 1} \subset \mathfrak{N}$, if*

$$G(u_n) \rightarrow c \text{ and } \epsilon_n := G'(u_n) - t_n F'(u_n) \rightarrow 0 \text{ as } n \rightarrow +\infty, \quad (2.2)$$

where $(t_n)_{n \geq 1} \subset \mathbb{R}$ such that $t_n = \frac{\langle G'(u_n), u_n \rangle}{\langle F'(u_n), u_n \rangle}$. Then $(u_n)_{n \geq 1}$ has a convergent subsequence in $\mathbb{E}_p(\Omega)$.

Proof. By the definition of G we have

$$\frac{a}{p} \|u_n\|^p \leq G(u_n),$$

then $(u_n)_{n \geq 1}$ is bounded in $\mathbb{E}_p(\Omega)$. Hence, for a subsequence still denoted by $(u_n)_{n \geq 1}$, we have $u_n \rightharpoonup u$ in $\mathbb{E}_p(\Omega)$. As F' is sequentially weakly-strongly continuous, we get $F'(u_n) \rightarrow F'(u)$ in $(\mathbb{E}_p(\Omega))^*$.

On the other hand, by 2.2 we get

$$\langle G'(u_n) - t_n F'(u_n), u_n \rangle = pG(u_n) - t_n \rightarrow 0, \text{ and } t_n \rightarrow pc.$$

It follows that the sequence $(G'(u_n))_{n \geq 1}$ is strongly convergent in $(\mathbb{E}_p(\Omega))^*$. Hence,

$$\int_{\Omega} G'(u_n)(u_n - u) dx \rightarrow 0.$$

Since G' satisfies condition (S_+) , we deduce that $u_n \rightarrow u$. This achieves the proof of the Proposition. $\square$

3. Proof of our results

3.1. Proof of Theorem 1.1. To prove Theorem 1.1, we only need to prove that for any $k \in \mathbb{N}^*$, $\Gamma_k \neq \emptyset$ and the least assertion. Indeed, since $\mathbb{E}_p(\Omega)$ is separable, there exists $(e_i)_{i \geq 1}$ linearly dense in $E_p(\Omega)$ such that

$$\text{supp } e_i \cap \text{supp } e_j = \emptyset \text{ if } i \neq j \text{ and } \int_{\partial\Omega} \xi(x) \left| \frac{\partial e_i}{\partial v} \right|^p ds = 1, \text{ for } i \in \{1, \dots, n\}.$$

Let now $k \in \mathbb{N}^*$ and denote

$$X_k = \text{span}\{e_1, e_2, \dots, e_k\}.$$

It is clear that X_k is a vector subspace with $\dim X_k = k$.

If $v \in X_k$, then there $\exists(\alpha_1, \dots, \alpha_k) \in \mathbb{R}$ such that $v = \sum_{i=1}^k \alpha_i e_i$. Thus

$$F(v) = \sum_{i=1}^k |\alpha_i|^p F(e_i) = \frac{1}{p} \sum_{i=1}^k |\alpha_i|^p > 0.$$

It follows that the map

$$v \mapsto (pF(v))^{\frac{1}{p}} := |v|,$$

defines a norm on X_k . Hence, $S := X_k \cap \mathfrak{N}$ is the unit sphere of X_k which is homeomorphic to the unit sphere of $\mathbb{R}^N$ and this homeomorphism is odd. By the property of a genus (see [14]), we get $\gamma(S) = k$. Finally we obtain that $\Gamma_k \neq \emptyset$.

Now, we show that

$$\lambda_k \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

Since $\mathbb{E}_p(\Omega)$ is separable, there exists a bi-orthogonal system $(e_n, e_j^*)_{n,j}$ such that $e_n \in \mathbb{E}_p(\Omega)$ and $e_j^* \in (\mathbb{E}_p(\Omega))^*$, the e_n 's are linearly dense in $\mathbb{E}_p(\Omega)$ and the e_j^* 's are total for the dual $(\mathbb{E}_p(\Omega))^*$. For $k \in \mathbb{N}^*$, set

$$X_k = \text{span}\{e_1, \dots, e_k\} \quad \text{and} \quad X_k^\perp = \text{span}\{e_{k+1}, e_{k+2}, \dots\}.$$

By the property of a genus (see [14]), we deduce that for any $K \in \Gamma_k$, $K \cap X_{k-1}^\perp \neq \emptyset$. Thus

$$t_k := \inf_{K \in \Gamma_k} \sup_{u \in K \cap X_{k-1}^\perp} pG(u) \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

Indeed, if not, for k is large, there exists $u_k \in X_{k-1}^\perp$ with $\int_{\partial\Omega} \xi(x) \left| \frac{\partial u_k(x)}{\partial \nu} \right|^p ds = 1$ such that

$$t_k \leq pG(u_k) \leq M,$$

for some $M > 0$ independent of k . This implies that $(u_k)_k$ is bounded in $\mathbb{E}_p(\Omega)$. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converges weakly in $\mathbb{E}_p(\Omega)$ and strongly in $L^p(\Omega)$. By our choice of $X_{k-1}^\perp$, we have $u_k \rightharpoonup 0$ in $\mathbb{E}_p(\Omega)$, because $\langle e_n^*, e_k \rangle = 0$, for any $k \geq n$. This contradicts the fact that $\int_{\partial\Omega} \xi(x) \left| \frac{\partial u_k(x)}{\partial \nu} \right|^p ds = 1$ for all k . Since $\lambda_k \geq t_k$ the claim is proved. This achieved the proof of the theorem. $\square$

3.2. Proof of Corollary 1.2. In this subsection, we aim to prove Corollary 1.2. To this aim, we give a direct characterization of λ_1 involving a min-max argument over sets of genus greater than k . We set

$$\lambda_1 = \inf_{u \in \mathbb{E}_p(\Omega), u \neq 0} \frac{\int_{\Omega} \tau(x) |\Delta u|^p dx}{\int_{\partial\Omega} \xi(x) \left| \frac{\partial u}{\partial \nu} \right|^p ds}. \quad (3.1)$$

Remark. Note that the first eigenvalue λ_1 is naturally well defined taking into account the trace embedding $\mathbb{E}_p(\Omega) \hookrightarrow L^p(\partial\Omega)$.

Proof of 1 For $u \in \mathfrak{N}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, G is even and that

$$pG(u) = \max_{K_1} pG(u) \geq \inf_{K \in \Gamma_1} \max_K pG(u).$$

Hence

$$\inf_{u \in \mathfrak{N}} pG(u) \geq \inf_{K \in \Gamma_1} \max_K pG(u) = \lambda_1.$$

On the other hand, $\forall K \in \Gamma_1$, $\forall u \in K$, we have

$$\max_K pG(u) \geq pG(u) \geq \inf_{u \in \mathfrak{N}} pG(u).$$

It follows that

$$\inf_{K \in \Gamma_1} \max_K pG(u) = \lambda_1 \geq \inf_{u \in \mathfrak{N}} pG(u).$$

Thus

$$\lambda_1 = \inf \left\{ \|u\|^p, u \in \mathbb{E}_p(\Omega) \mid pF(u) = 1 \right\}.$$

Proof of 2 For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\lambda_i \geq \lambda_j$. As regards to $\lambda_n \rightarrow \infty$, it proved before in Theorem 1.1. $\square$

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