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BLOW-UP OF SOLUTIONS FOR A PARABOLIC KIRCHHOFF TYPE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This study deals with the parabolic type Kirchhoff equation with logarithmic nonlinearity in a bounded domain. We obtain the finite time blow-up of solutions. This improves and extends some previous studies.

1. Introduction

In this paper, we investigate the parabolic Kirchhoff type equation with logarithmic nonlinearity

$$\begin{cases} u_t - M(\|\nabla u\|^2)\Delta u - \Delta u_t = |u|^{q-2}\, u \ln |u|\,, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $M(s) = 1 + s^\gamma$, $(\gamma > 0)$ and $2 < q < 2 + \frac{4}{n}$.

Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in different areas of physics such as supersymmetric field theories, inflation cosmology, nuclear physics and quantum mechanics [1, 5].

When $M \equiv 1$ and $q = 2$, (1.1) become the following semilinear pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t = u \log |u|\,. \tag{1.2}$$

Chen and Tian [4] studied the global existence, behavior of vacuum isolation and blow-up of solutions at $+\infty$ of the equation (1.2). Without Δu, (1.2) become the following semilinear heat equation

$$u_t - \Delta u_t = u \log |u|\,. \tag{1.3}$$

Chen et al. [3] studied the global existence, decay estimate and blow-up at $+\infty$ of solutions of the equation (1.3).

Peng and Zhou [9] studied the following heat equation

$$u_t - \Delta u_t = |u|^{p-2}\, u \log |u|\,.$$

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They studied the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

He et al. [6] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\log|u|.$$

They obtained the decay and the finite time blow-up for weak solutions of the equation.

Cao and Liu [2] studied the following nonlinear evolution equation with logarithmic source

$$u_t - \Delta u_t - div\left(|\nabla u|^{p-2}\nabla u\right) - k\Delta u_t = |u|^{p-2}u\log|u|.$$

They established the existence of global solutions. Moreover, they considered global boundedness and blowing-up at ∞.

Li and Han [7] studied the following parabolic type p-Kirchhoff initial boundary value problem

$$u_t - (a + b\int_\Omega |\nabla u|^p)\Delta_p u = |u|^{q-1}u.$$

They obtained the global existence and finite time blow up of solutions.

Motivated by the above studies, in this work, we investigate the finite time blow-up of weak solutions for the Eq. (1.1).

This paper is organized as follows: In Section 2, we introduce some lemmas which will be needed later. In Section 3, we prove the blow-up of solutions.

2. Preliminaries

For simplicity, we denote

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \quad \|u\|_{H_0^1(\Omega)} = \left(\|\nabla u\|^2 + \|u\|^2\right)^{\frac{1}{2}},$$

for $1 < s < \infty$.

Let us introduce the energy functional J and Nehari functional I defined on $H_0^1(\Omega)\backslash\{0\}$ as following

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} - \frac{1}{q}\int_\Omega |u|^q \ln|u|\,dx + \frac{1}{q^2}\|u\|_q^q, \tag{2.1}$$

and

$$I(u) = \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} - \int_\Omega |u|^q \ln|u|\,dx. \tag{2.2}$$

By (2.1) and (2.2), we get

$$J(u) = \frac{1}{q}I(u) + \frac{q-2}{2q}\|\nabla u\|^2 + \frac{q-2\gamma-2}{2q\gamma+2q}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{q^2}\|u\|_q^q. \tag{2.3}$$

Let

$$\mathcal{N} = \{u \in H_0^1(\Omega)\backslash\{0\} : I(u) = 0\},$$

be the Nehari manifold. Lemma 2.1, indicates $\mathcal{N}$ is not empty. Thus, we may define

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{2.4}$$

Lemma 2.1. *Let $u \in H_0^1(\Omega)\backslash\{0\}$. Then we possess*
(i) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ and $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$;
(ii) there is a unique $\lambda^* > 0$ such that $j'(\lambda^*) = 0$;
(iii) $j(\lambda)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$ and attains the maximum at λ^*;
(iv) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$.

Proof. For $u \in H_0^1(\Omega)\backslash\{0\}$, by the definition of j, we get

$$\begin{aligned} j(\lambda) &= \frac{1}{2}\|\nabla(\lambda u)\|^2 + \frac{1}{2(\gamma+1)}\|\nabla(\lambda u)\|^{2(\gamma+1)} \\ &\quad -\frac{1}{q}\int_\Omega |\lambda u|^q \ln|\lambda u|\,dx + \frac{1}{q^2}\|\lambda u\|_q^q \\ &= \frac{\lambda^2}{2}\|\nabla u\|^2 + \frac{\lambda^{2(\gamma+1)}}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \\ &\quad -\frac{\lambda^q}{q}\int_\Omega |u|^q \ln|u|\,dx - \frac{\lambda^q}{q}\int_\Omega |u|^q \ln\lambda dx + \frac{\lambda^q}{q^2}\|u\|_q^q \\ &= \frac{\lambda^2}{2}\|\nabla u\|^2 + \frac{\lambda^{2(\gamma+1)}}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \\ &\quad -\frac{\lambda^q}{q}\int_\Omega |u|^q \ln|u|\,dx - \frac{\lambda^q}{q}\ln\lambda\|u\|_q^q + \frac{\lambda^q}{q^2}\|u\|_q^q. \end{aligned} \tag{2.5}$$

It is clear that *(i)* holds due to $\int_\Omega |u|^q\,dx \neq 0$. We have

$$\begin{aligned} \frac{d}{d\lambda}j(\lambda) &= \lambda\|\nabla u\|^2 + \lambda^{2\gamma+1}\|\nabla u\|^{2(\gamma+1)} - \lambda^{q-1}\int_\Omega |u|^q\ln|u|\,dx \\ &\quad -\lambda^{q-1}\ln\lambda\|u\|_q^q - \frac{\lambda^{q-1}}{q}\|u\|_q^q + \frac{\lambda^{q-1}}{q}\|u\|_q^q \\ &= \lambda\|\nabla u\|^2 + \lambda^{2\gamma+1}\|\nabla u\|^{2(\gamma+1)} - \lambda^{q-1}\int_\Omega |u|^q\ln|u|\,dx - \lambda^{q-1}\ln\lambda\|u\|_q^q \\ &= \lambda\left(\|\nabla u\|^2 + \lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)} - \lambda^{q-2}\int_\Omega |u|^q\ln|u|\,dx - \lambda^{q-2}\ln\lambda\|u\|_q^q\right). \end{aligned}$$

Let $\varphi(\lambda) = \lambda^{-1}j'(\lambda)$, thus we obtain

$$\begin{aligned} \varphi(\lambda) &= \lambda^{-1}j'(\lambda) \\ &= \lambda^{-1}\lambda\left(\|\nabla u\|^2 + \lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)} - \lambda^{q-2}\left(\int_\Omega |u|^q\ln|u|\,dx - \ln\lambda\|u\|_q^q\right)\right) \\ &= \|\nabla u\|^2 + \lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)} - \lambda^{q-2}\int_\Omega |u|^q\ln|u|\,dx - \lambda^{q-2}\ln\lambda\|u\|_q^q. \end{aligned}$$

Then

$$\begin{aligned} \varphi'(\lambda) &= 2\gamma\lambda^{2\gamma-1}\|\nabla u\|^{2(\gamma+1)} - (q-2)\lambda^{q-3}\int_\Omega |u|^q\ln|u|\,dx \\ &\quad -(q-2)\lambda^{q-3}\ln\lambda\|u\|_q^q - \lambda^{q-3}\|u\|_q^q \end{aligned}$$

which implies that there exists a $\lambda^*>0$ such that $\varphi'(\lambda)>0$ on $(0,\lambda^*)$, $\varphi'(\lambda)<0$ on $(\lambda^*,+\infty)$ and $\varphi'(\lambda)=0$. So, $\varphi(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$. Since $\lim_{\lambda\to 0^+}\varphi(\lambda)=\|\nabla u\|^2>0$, $\lim_{\lambda\to+\infty}\varphi(\lambda)=-\infty$, there exists a unique $\lambda^*>0$ such that $\varphi(\lambda^*)=0$, i.e., $j'(\lambda^*)=0$. So, *(ii)* holds. Then, $j'(\lambda)=\lambda\varphi(\lambda)$ is positive on $(0,\lambda^*)$, negative on $(\lambda^*,+\infty)$. Thus, $j(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$ and attains the maximum at λ^*. So, *(iii)* holds. From (2.2), we obtain

$$\begin{aligned}
I(\lambda u) &= \|\nabla(\lambda u)\|^2+\|\nabla(\lambda u)\|^{2(\gamma+1)}-\int_\Omega |\lambda u|^q\ln|\lambda u|\,dx\\
&= \lambda^2\|\nabla u\|^2+\lambda^{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\lambda^q\int_\Omega |u|^q\ln|u|\,dx-\lambda^q\int_\Omega |u|^q\ln\lambda dx\\
&= \lambda^2\|\nabla u\|^2+\lambda^{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\lambda^q\int_\Omega |u|^q\ln|u|\,dx-\lambda^q\ln\lambda\,\|u\|_q^q\\
&= \lambda\left(\lambda\|\nabla u\|^2+\lambda^{2\gamma+1}\|\nabla u\|^{2(\gamma+1)}-\lambda^{q-1}\left(\int_\Omega |u|^q\ln|u|\,dx-\ln\lambda\,\|u\|_q^q\right)\right)\\
&= \lambda j'(\lambda).
\end{aligned}$$

Thus, $I(\lambda u)>0$ for $0<\lambda<\lambda^*$, $I(\lambda u)<0$ for $\lambda^*<\lambda<+\infty$ and $I(\lambda^* u)=0$. So, *(iv)* holds. Therefore, the proof is completed. □

Lemma 2.2. *d defined by (2.4) is positive and there exists a positive function $u\in\mathcal{N}$ such that $J(u)=d$.*

Proof. Let $\{u_k\}_k^\infty\subset\mathcal{N}$ be a minimizing sequence for J, which means that

$$\lim_{k\to\infty}J(u_k)=d. \tag{2.6}$$

It is easy to show that $\{|u_k|\}_k\subset\mathcal{N}$ is also a minimizing sequence for J due to $|u_k|\in\mathcal{N}$ and $J(|u_k|)=J(u_k)$. For this reason, we assume that $u_k>0$ a.e. Ω for all $k\in N$.

On the other hand, we have already observed that J is coercive on $\mathcal{N}$ which implies that $\{u_k\}_k^\infty$ is bounded in $H_0^1(\Omega)$. Let $\mu>0$ be sufficiently small such that $q+\mu<\frac{2n}{n-2}$. Since $H_0^1(\Omega)\hookrightarrow L^{q+\mu}(\Omega)$ is compact, there exists a function u and a subsequence of $\{u_k\}_k^\infty$, still denote by $\{u_k\}_k^\infty$, such that

$$\begin{aligned}
u_k&\to u \ \text{ weakly in } \ H_0^1(\Omega),\\
u_k&\to u \ \text{ strongly in } \ L^{q+\mu}(\Omega),\\
u_k(x)&\to u(x) \ \text{ a.e. in } \ \Omega.
\end{aligned}$$

Hence, $u\geq 0$ a.e. in Ω. Firstly, we prove $u\neq 0$. Using the dominated convergence theorem, we get

$$\int_\Omega |u|^q\ln|u|\,dx=\lim_{k\to\infty}\int_\Omega |u_k|^q\ln|u_k|\,dx, \tag{2.7}$$

and

$$\int_\Omega |u|^q\,dx=\lim_{k\to\infty}\int_\Omega |u_k|^q\,dx. \tag{2.8}$$

By the weak lower semicontinuity of $H_0^1(\Omega)$, we get

$$\|\nabla u\|^2 \leq \liminf_{k\to\infty} \|\nabla u_k\|^2, \tag{2.9}$$

$$\|\nabla u\|^{2(\gamma+1)} \leq \lim_{k\to\infty} \|\nabla u_k\|^{2(\gamma+1)}.$$

Using (2.1), (2.6), (2.7), (2.8) and (2.9), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} - \frac{1}{q}\int_\Omega |u|^q \ln|u|\,dx + \frac{1}{q^2}\|u\|_q^q \\ &\leq \liminf_{k\to\infty} \frac{1}{2}\|\nabla u_k\|^2 + \lim_{k\to\infty} \frac{1}{2(\gamma+1)}\|\nabla u_k\|^{2(\gamma+1)} \\ &\quad - \lim_{k\to\infty} \frac{1}{q}\int_\Omega |u_k|^q \ln|u_k|\,dx + \lim_{k\to\infty} \frac{1}{q^2}\|u_k\|_q^q \\ &= \liminf_{k\to\infty} \left(\frac{1}{2}\|\nabla u_k\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u_k\|^{2(\gamma+1)}\right) \\ &\quad - \liminf_{k\to\infty} \left(\frac{1}{q}\int_\Omega |u_k|^q \ln|u_k|\,dx + \frac{1}{q^2}\|u_k\|_q^q\right) \\ &= \liminf_{k\to\infty} J(u_k) = d. \end{aligned} \tag{2.10}$$

Using (2.2), (2.7) and (2.9), we have

$$\begin{aligned} I(u) &= \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} - \int_\Omega |u|^q \ln|u|\,dx \\ &\leq \liminf_{k\to\infty} \|\nabla u_k\|^2 - \lim_{k\to\infty} \|\nabla u_k\|^{2(\gamma+1)} - \lim_{k\to\infty} \int_\Omega |u_k|^q \ln|u_k|\,dx \\ &= \liminf_{k\to\infty} \left(\|\nabla u_k\|^2 - \|\nabla u_k\|^{2(\gamma+1)} - \int_\Omega |u_k|^q \ln|u_k|\,dx\right) \\ &= \liminf_{k\to\infty} I(u_k) = 0. \end{aligned} \tag{2.11}$$

Since $u_k \in \mathcal{N}$, we have $I(u_k) = 0$. So, by using the fact $x^{-\mu}\ln x \leq (e\mu)^{-1}$ for $x \geq 1$ and the Sobolev embedding inequality, we obtain

$$\begin{aligned} \|\nabla u_k\|^2 + \|\nabla u_k\|^{2\delta+2} &= \int_\Omega |u_k|^q \ln|u_k|\,dx \\ &\leq (e\mu)^{-1}\int_\Omega |u_k|^{q+\mu}\,dx \\ &\leq (e\mu)^{-1}\|u_k\|_{q+\mu}^{q+\mu} \\ &\leq C\|\nabla u_k\|_2^{q+\mu}, \end{aligned}$$

for some positive constant C. This implies that

$$\int_\Omega |u_k|^q \ln|u_k|\,dx = \|\nabla u_k\|^2 + \|\nabla u_k\|^{2(\gamma+1)} \geq C.$$

Combining this inequality with (2.7), we obtain

$$\int_\Omega |u|^q \ln|u|\,dx \geq C.$$

Thus, we have $u \in H_0^1(\Omega)\backslash\{0\}$.

If $I(u_k) < 0$, by Lemma 2.1, there exists a λ^* such that $I(\lambda^* u) = 0$ and $0 < \lambda^* < 1$. Thus, $\lambda^* u \in \mathcal{N}$. By (2.3), (2.4), (2.8) and (2.9), we have

$$\begin{aligned}
d &\leq J(\lambda^* u) \\
&= \frac{1}{q} I(\lambda^* u) + \frac{q-2}{2q} \|\nabla(\lambda^* u)\|^2 + \frac{q-2\gamma-2}{2q\gamma+2q} \|\nabla(\lambda^* u)\|^{2(\gamma+1)} + \frac{1}{q^2} \|\lambda^* u\|_q^q \\
&= \frac{q-2}{2q} \|\nabla(\lambda^* u)\|^2 + \frac{q-2\gamma-2}{2q\gamma+2q} \|\nabla(\lambda^* u)\|^{2(\gamma+1)} + \frac{1}{q^2} \|\lambda^* u\|_q^q \\
&= (\lambda^*)^2 \left(\frac{q-2}{2q}\right) \|\nabla u\|^2 + (\lambda^*)^{2(\gamma+1)} \left(\frac{q-2\gamma-2}{2q\gamma+2q}\right) \|\nabla u\|^{2(\gamma+1)} \\
&\quad + (\lambda^*)^q \left(\frac{1}{q^2}\right) \|u\|_q^q \\
&\leq (\lambda^*)^2 \left[\frac{q-2}{2q} \|\nabla u\|^2 + \frac{q-2\gamma-2}{2q\gamma+2q} \|\nabla u\|^{2(\gamma+1)} + \frac{1}{q^2} \|u\|_q^q\right] \\
&\leq (\lambda^*)^2 \liminf_{k\to\infty} \left[\frac{q-2}{2q} \|\nabla u\|^2 + \frac{q-2\gamma-2}{2q\gamma+2q} \|\nabla u\|^{2(\gamma+1)} + \frac{1}{q^2} \|u\|_q^q\right] \\
&= (\lambda^*)^2 \liminf_{k\to\infty} J(u_k) \\
&= (\lambda^*)^2 d,
\end{aligned}$$

which indicates $\lambda^* \geq 1$ by $d > 0$. It contradicts $0 < \lambda^* < 1$. Then, by (2.11), we have $I(u) = 0$. Therefore, $u \in \mathcal{N}$. By (2.6), we have $J(u) \geq d$. By (2.10), we have $J(u) \leq d$. So, $J(u) = d$. □

3. Blow up

As in ([8]), we introduce the follows sets:

$$\begin{aligned}
\mathcal{W}_1 &= \{u \in H_0^1(\Omega)\backslash\{0\} : J(u) < d\}, \quad \mathcal{W}_2 = \{u \in H_0^1(\Omega)\backslash\{0\} : J(u) = d\}, \\
\mathcal{W}_1^+ &= \{u \in \mathcal{W}_1 : I(u) > 0\}, \quad \mathcal{W}_2^+ = \{u \in \mathcal{W}_2 : I(u) > 0\}, \\
\mathcal{W}_1^- &= \{u \in \mathcal{W}_1 : I(u) < 0\}, \quad \mathcal{W}_2^- = \{u \in \mathcal{W}_2 : I(u) < 0\}, \\
\mathcal{W} &= \mathcal{W}_1 \cup \mathcal{W}_2, \ \mathcal{W}^+ = \mathcal{W}_1^+ \cup \mathcal{W}_2^+, \quad \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-.
\end{aligned}$$

Definition 3.1. (Maximal Existence Time). Let $u(t)$ be weak solutions of problem (1.1). We define the maximal existence time $T_{\max}$ as follows

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0, T]\}.$$

Then

(i) If $T_{\max} < +\infty$, we say that $u(t)$ blows up in finite time and $T_{\max}$ is the blow-up time;

(ii) If $T_{\max} = +\infty$, we say that $u(t)$ is global.

Theorem 3.2. *Assume that $u_0 \in W_1^-$, and let $u(t)$ be a unique local weak solution to the problem (1.1). Then $u(t)$ blows up in finite time, that is, there exists $T_* > 0$*

such that

$$\lim_{t\to T_*} \|u(t)\|^2_{H_0^1(\Omega)} = \infty.$$

Proof. Since $u_0 \in W_1^-$, we obtain a unique local solution $u(t)$ of problem (1.1) satisfying the energy inequality

$$\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\, ds + J(u(t)) \le J(u_0) < d, \quad 0 \le t \le T_{\max},$$

where $T_{\max}$ is maximal existence time of solution $u(t)$.

Next, we prove that $u_0 \in W_1^-$ for $t \in [0, T_{\max}]$. We assume $u(t)$ leaves W_1^+ at time $t = t_0$, then there exists a sequence $\{t_n\}$, such that $I(u(t_n)) \le 0$ as $t_n \to t_0^-$. By the lower semicontinuity of $H_0^1(\Omega)$, we have

$$I(u(t_0)) \le \liminf_{n\to\infty} I(u(t_n)) \le 0.$$

We have $I(u(t_0)) = 0$ according to $u(t_0) \notin W_1^+$. However, by variational definition of d and energy inequality, this leads to a contradiction

$$d = \inf_{u\in\mathcal{N}} J(u) \le J(u(t_0)) < d.$$

So, $u(t) \in W_1^-$ for $t \in [0, T_{\max}]$. Now, we prove that $u(t)$ blows up at a finite time. By contradiction, we assume that $u(t)$ is global. We contract a function $\Gamma : [0, \infty) \to \mathbb{R}^+$, and

$$\Gamma(t) = \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds. \tag{3.1}$$

Then, a direct calculation gives

$$\begin{aligned} \Gamma'(t) &= \|u(s)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)} \\ &= 2\int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u)\, dx ds. \end{aligned} \tag{3.2}$$

By (2.2) and (3.2), we get

$$\begin{aligned} \Gamma''(t) &= 2\int_\Omega (u_s u + \nabla u_s \nabla u)\, dx \\ &= 2\int_\Omega u\,(u_s - \Delta u_s)\, dx \\ &= 2\int_\Omega |u|^q \ln|u| - 2\int_\Omega M(\|\nabla u\|^2) u\Delta u \\ &= 2\int_\Omega |u|^q \ln|u| - 2\int_\Omega (1 + \|\nabla u\|^{2\gamma})(\nabla u)^2 \\ &= -2I(u). \end{aligned} \tag{3.3}$$

By (3.3) and $I(u) < 0$, we have $\Gamma''(t) > 0$, so

$$\Gamma'(t) > \Gamma'(0) = \|u_0\|^2_{H_0^1(\Omega)} > 0. \tag{3.4}$$

Thanks to the Hölder's inequality and combining (3.3), we obtain

$$\begin{aligned}\frac{1}{4}\left(\Gamma'(t)-\Gamma'(0)\right)^2 &= \frac{1}{4}\left(\int_0^t \Gamma''(s)ds\right)^2\\ &= \left(\int_0^t\int_\Omega \left(u_s u+\nabla u_s \nabla u\right)dxds\right)^2\\ &\leq \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\,ds\int_0^t \|u_s\|^2_{H_0^1(\Omega)}\,ds. \end{aligned} \tag{3.5}$$

From (2.3) and (3.3) that it follows

$$\begin{aligned}\Gamma''(t) &= -2I(u)\\ &= -2qJ(u)+(q-2)\|\nabla u\|^2+\frac{q-2\gamma-2}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}+\frac{2}{q}\|u\|_q^q\\ &\geq -2qJ(u_0)+2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds\\ &\quad +(q-2)\|\nabla u\|^2+\frac{q-2\gamma-2}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}+\frac{2}{q}\|u\|_q^q. \end{aligned} \tag{3.6}$$

Since $u(t)\in W_1^-$, $t\in[0,T]$, so $I(u)<0$. By Lemma 2.1, there exists a $\lambda^*\in(0,1)$ such that $I(\lambda^* u(t))=0$. Thus, by the definition of d, it follows that

$$\begin{aligned} d &= \inf_{u\in\mathcal{N}} J(u)\leq J(\lambda^* u(t))\\ &\leq \frac{q-2}{2q}\|\nabla u\|^2+\frac{q-2\gamma-2}{2q\gamma+2q}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{q^2}\|u\|_q^q. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}\Gamma''(t) &\geq -2qJ(u_0)+2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds\\ &\quad +(q-2)\|\nabla u\|^2+\frac{q-2\gamma-2}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}+\frac{2}{q}\|u\|_q^q\\ &= -2qJ(u_0)+2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds\\ &\quad +2q\left[\frac{q-2}{2q}\|\nabla u\|^2+\frac{q-2\gamma-2}{2q\gamma+2q}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{q^2}\|u\|_q^q\right]\\ &\geq 2q\left(d-J(u_0)\right)+2q\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds. \end{aligned} \tag{3.8}$$

Using (3.1), (3.5) and (3.8), we get

$$\begin{aligned}\Gamma(t)\Gamma''(t) &\geq 2q\int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\,ds\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds+2q\left(d-J(u_0)\right)\Gamma(t)\\ &\geq 2q\left(d-J(u_0)\right)\Gamma(t)+\frac{q}{2}\left(\Gamma'(t)-\Gamma'(0)\right)^2. \end{aligned} \tag{3.9}$$

Now, fix a $t_0>0$, then from (3.4), we get

$$\Gamma(t)\geq\Gamma(t_0)\geq\|u_0\|^2_{H_0^1(\Omega)}\,t_0>0,\ \text{ for all } t\geq t_0. \tag{3.10}$$

Hence, by (3.9) and (3.10), we get

$$\Gamma(t)\Gamma''(t) - \frac{q}{2}\left(\Gamma'(t) - \Gamma'(0)\right)^2 \geq 2q\left(d - J(u_0)\right)\|u_0\|^2_{H_0^1(\Omega)}\, t_0 > 0, \text{ for all } t \geq t_0. \tag{3.11}$$

Choose $T > t_0$ sufficiently large and let

$$\psi(t) = \Gamma(t) + (T - t)\|u_0\|^2_{H_0^1(\Omega)}\, t_0, \text{ for all } t \in [0, T].$$

Then $\psi(t) \geq \Gamma(t) > 0$, $\psi'(t) = \Gamma'(t) - \Gamma'(0)$ and $\psi''(t) = \Gamma''(t) > 0$, so (3.11) implies

$$\begin{aligned} \psi(t)\psi''(t) - \frac{q}{2}\psi'(t)^2 &\geq \Gamma(t)\Gamma''(t) - \frac{q}{2}\left(\Gamma'(t) - \Gamma'(0)\right)^2 \\ &\geq 2q\left(d - J(u_0)\right)\|u_0\|^2_{H_0^1(\Omega)}\, t_0 > 0, \ t \in [t_0, T]. \end{aligned} \tag{3.12}$$

Let $y(t) = \psi(t)^{-\frac{q-2}{2}}$. Thus,

$$y'(t) = -\frac{q-2}{2}\psi(t)^{-\frac{q}{2}}\psi'(t). \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned} y''(t) &= \frac{q(q-2)}{4}\psi(t)^{-\frac{q+2}{2}}\psi'(t)^2 - \frac{q-2}{2}\psi(t)^{-\frac{q}{2}}\psi''(t) \\ &= \frac{q-2}{2}\psi(t)^{-\frac{q+2}{2}}\left[\frac{q}{2}\psi'(t)^2 - \psi(t)\psi''(t)\right] < 0, \text{ for all } t \in [t_0, T]. \end{aligned}$$

This shows that, for any sufficiently large $T > t_0$, $y(t)$ is a concave function in $[t_0, T]$. Since $y(t_0) > 0$ and $y'(t_0) < 0$, there exists a finite time T_* such that

$$\lim_{t \to T_*^-} y(t) = 0,$$

which implies

$$\lim_{t \to T_*^-} \psi(t) = \infty.$$

Hence, we get

$$\lim_{t \to T_*^-} \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds = \infty,$$

i.e.,

$$\lim_{t \to T_*^-} \|u(s)\|^2_{H_0^1(\Omega)} = \infty.$$

This contradicts with $u(t)$ being a global solution. □

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