

FIBONACCI-MANN PROCESS FOR MONOTONE TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. We study the existence and convergence of fixed points for monotone total asymptotically nonexpansive self-mappings in partially ordered Banach spaces. We generalize and improve several known results, including those of Rashwan et al. [13], as well as the works of Alfuraidan and Khamsi [2, 3] for monotone asymptotically nonexpansive mappings and those of Schu [14] on asymptotically nonexpansive mappings, extending them to the broader class of monotone total asymptotically nonexpansive mappings using the Fibonacci-Mann iteration process in a uniformly convex Banach space.

Keywords. monotone total asymptotically nonexpansive mapping, Fibonacci-Mann iteration, fixed point, Opial condition, uniformly convex Banach space.
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1. INTRODUCTION AND PRELIMINARIES

In 2006, Alber et al. [1] introduced the concept of total asymptotically nonexpansive mappings, which generalizes several existing classes of mappings, notably nearly asymptotically nonexpansive mappings and asymptotically nonexpansive mappings. The fixed point theory for asymptotically nonexpansive mappings was initially established by Goebel and Kirk [6] in 1972. Alfuraidan and Khamsi [2, 3] extended these studies to partially ordered Banach spaces, investigating the existence of fixed points and the convergence properties of the Fibonacci-Mann iteration method. Their work specifically extended results previously established by Schu [14] to the context of ordered spaces.

In 1965, Kirk [10], Göhde [7], and Browder [4] independently established a foundational fixed point theorem for nonexpansive mappings. This seminal work spurred numerous studies focusing on expanding this theory, either for single mapping applications or by exploring the existence of common fixed points for multiple mappings under various generalized conditions. For instance, early work on iterative processes in this area was conducted by Schu [14]. The investigation into common fixed points for pairs of single-valued and multi-valued nonexpansive

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mappings has also been a focus of research. Rashwan et al. [13] investigated the strong convergence of the Ishikawa iterative process specifically within a complete $CAT(0)$ space.

The concept of monotone Lipschitzian mappings involves two structures: a partial order and a metric distance. Most spaces involved in applications naturally possess these two structures, with interesting intertwining properties.

Let $(E, \|\cdot\|)$ be a Banach space endowed with a partial order $\leq$. We say that $x, y \in E$ are comparable whenever $x \leq y$ or $y \leq x$.

The investigation of total asymptotically nonexpansive mappings is motivated by their ability to model a wide range of nonlinear phenomena where traditional contraction conditions hold only asymptotically. This class includes various important operators from applied analysis while allowing for more flexible asymptotic behavior. When considered in partially ordered Banach spaces, which naturally arise in problems involving differential and integral equations, optimization, and economic models, the additional monotonicity structure enables stronger convergence results. Our work extends this theory by incorporating the computationally efficient Fibonacci-Mann iteration, providing new convergence guarantees under weaker asymptotic assumptions than previously considered in the literature.

The structure of this paper is as follows. In Section 2, we present our main results concerning the existence of fixed points for monotone total asymptotically nonexpansive mappings. Section 3 focuses on convergence results using the Fibonacci-Mann iteration process. Finally, Section 4 provides an application to Volterra integral equations, illustrating the practical relevance of our theoretical findings.

An order interval is any of the subsets $[a, \rightarrow) = \{x \in E; a \leq x\}$ or $(\leftarrow, a] = \{x \in E; x \leq a\}$ for any $a \in E$. Throughout, we assume that the order intervals are closed and convex. As a direct consequence, the subset $[a, b] = \{x \in E; a \leq x \leq b\} = [a, \rightarrow) \cap (\leftarrow, a]$ is also closed and convex for any $a, b \in E$.

Recall that a sequence $\{x_n\}_{n \in \mathbb{N}}$ in a partially ordered set $(E, \leq)$ is said to be

- (1) monotone increasing if $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$;
- (2) monotone decreasing if $x_{n+1} \leq x_n$ for every $n \in \mathbb{N}$;
- (3) a monotone sequence if it is either monotone increasing or decreasing.

$\delta(F) = \sup \{\|c_1 - c_2\| : c_1, c_2 \in F\}$ is the diameter of F .

Definition 1.1. Let $(E, \|\cdot\|, \leq)$ be a partially ordered Banach space. Let F be a nonempty subset of E . A map $S : F \rightarrow F$ is said to be *monotone* if for any $x, y \in F$ such that $x \leq y$, we have $S(x) \leq S(y)$. Moreover, S is said to be:

- (1) *monotone Lipschitzian* if S is monotone and there exists $L \geq 0$ such that

$$\|S(x) - S(y)\| \leq L\|x - y\|$$

for any $x, y \in F$ such that $x \leq y$.

- (2) *monotone asymptotically nonexpansive* if S is monotone and there exists a nonnegative real sequence $\{l_n\} \subset [1, +\infty)$ such that $l_n \rightarrow 1$ as $n \rightarrow \infty$ and for every $n \in \mathbb{N}$

$$\|S^n(x) - S^n(y)\| \leq l_n\|x - y\|$$

for any $x, y \in F$ such that $x \leq y$.

- (3) *monotone total asymptotically nonexpansive* if S is monotone and there exist nonnegative real sequences $\{l_n^{(1)}\}$ and $\{l_n^{(2)}\}$ with $l_n^{(1)}, l_n^{(2)} \rightarrow 0$ as $n \rightarrow \infty$, and a strictly increasing continuous function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\psi(0) = 0$ such that for every $n \in \mathbb{N}$

$$\|S^n(x) - S^n(y)\| \leq \|x - y\| + l_n^{(1)}\psi(\|x - y\|) + l_n^{(2)}$$

for any $x, y \in F$ such that $x \leq y$.

In this case, we say that the map is a monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}$, $\{l_n^{(2)}\}$ and function ψ .

A fixed point of S is any element $x \in F$ such that $S(x) = x$.

Example 1.2. Let $E = [0, 1] \subset \mathbb{R}$ with the usual order, and define

$$S(x) = \frac{x}{1+x}.$$

For $x \leq y$ we have

$$S'(x) = \frac{1}{(1+x)^2} > 0,$$

so S is monotone and we have

$$S^n(x) = \frac{x}{1+nx}.$$

For any $x, y \in [0, 1]$,

$$|S^n(x) - S^n(y)| = \left| \frac{x}{1+nx} - \frac{y}{1+ny} \right|.$$

Then

$$|S^n(x) - S^n(y)| = \frac{|x - y|}{(1+nx)(1+ny)} \leq |x - y|,$$

which means that S is monotone nonexpansive and thus monotone total asymptotically nonexpansive.

Lemma 1.3. Let $(E, \|\cdot\|, \leq)$ be a partially ordered Banach space, and let F be a nonempty subset of E . Let $S : F \rightarrow F$ be a monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}$, $\{l_n^{(2)}\}$ and function ψ . Assume there exist constants $M, M^* > 0$ such that $\psi(\lambda) \leq M^*\lambda$ for all $\lambda > M$. Then, for every $n \in \mathbb{N}$

$$\|S^n(x) - S^n(y)\| \leq (1 + M^*l_n^{(1)})\|x - y\| + \psi(M)l_n^{(1)} + l_n^{(2)}$$

for any $x, y \in F$ such that $x \leq y$.

Proof. Let $x, y \in F$ such that $x \leq y$. Since S is a monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}$, $\{l_n^{(2)}\}$ and function ψ , we have

$$\|S^n(x) - S^n(y)\| \leq \|x - y\| + l_n^{(1)}\psi(\|x - y\|) + l_n^{(2)} \quad (1.1)$$

for every $n \in \mathbb{N}$.

Since $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$, we get $\psi(\lambda) \leq \psi(M)$ whenever $\lambda \leq M$ and $\psi(\lambda) \leq M^*\lambda$ if $\lambda > M$. In both cases, we obtain

$$\psi(\lambda) \leq \psi(M) + M^*\lambda. \quad (1.2)$$

From (1.1) and (1.2) we conclude that

$$\|S^n(x) - S^n(y)\| \leq (1 + M^*l_n^{(1)})\|x - y\| + \psi(M)l_n^{(1)} + l_n^{(2)}$$

for every $n \in \mathbb{N}$. □

Definition 1.4. Let $(E, \|\cdot\|)$ be a Banach space. We say that E is uniformly convex if for every $\varepsilon > 0$

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\|; \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\} > 0.$$

The function δ is known as the modulus of uniform convexity of E .

Among the nice properties satisfied by a uniformly convex Banach space $(E, \|\cdot\|)$ is property (R).

Definition 1.5. [8] We say that the Banach space E has property (R) if and only if any decreasing sequence (F_n) of closed bounded nonempty subsets of E has a nonempty intersection.

Theorem 1.6. [9] *Any uniformly convex Banach space has property (R).*

The following technical lemma will be useful in the proof of our main result. This lemma is stated in terms of type functions, but its origin may be found in the work of Edelstein [5] on an asymptotic center of a sequence.

Lemma 1.7. [3] *Let F be a nonempty closed convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$. Let $\tau : F \rightarrow [0, +\infty)$ be a type function, i.e., there exists a bounded sequence $\{x_n\} \in E$ such that*

$$\tau(x) = \limsup_{n \rightarrow +\infty} \|x_n - x\|,$$

for every $x \in F$. Then τ has a unique minimum point $z \in F$ such that

$$\tau(z) = \inf \{\tau(x); x \in F\} = \tau_0.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in F , i.e., $\lim_{n \rightarrow +\infty} \tau(z_n) = \tau_0$, then $\{z_n\}$ converges strongly to z .

Definition 1.8. [2] E is said to satisfy the monotone weak Opial condition if whenever any monotone increasing (resp. decreasing) sequence $\{x_n\}$ in E weakly converges to x , we have

$$\limsup_{n \rightarrow +\infty} \|x_n - x\| < \limsup_{n \rightarrow +\infty} \|x_n - y\|,$$

for every $y \in E$ such that $x_n \leq y$ (resp. $y \leq x_n$) for any $n \in \mathbb{N}$.

Lemma 1.9. [12] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + b_n)a_n + c_n,$$

for every $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow +\infty} a_n$ exists.

Lemma 1.10. [14] *Let E be a uniformly convex Banach space and let $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n \rightarrow +\infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow +\infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow +\infty} \|(1 - \alpha_n)x_n + \alpha_n y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow +\infty} \|x_n - y_n\| = 0$.*

2. MAIN RESULTS

Throughout this section, $(E, \|\cdot\|, \leq)$ is a partially ordered Banach space for which order intervals are closed and convex.

Lemma 2.1. *Let F be a nonempty, closed and convex subset of E . Assume that $(E, \|\cdot\|, \leq)$ is reflexive. Let $\{v_n\}$ be a bounded monotone increasing (resp. decreasing) sequence in F . Then $\{v_n\}$ converges weakly to $v \in F$ and $v_n \leq v$ (resp. $v \leq v_n$), for every $n \in \mathbb{N}$.*

Proof. Since E is reflexive and $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ weakly convergent to some $v \in E$. Since F is closed and convex, $v \in F$.

Fix $p \geq 1$. Then the order interval $[v_p, \rightarrow)$ (resp. $(\leftarrow, v_p]$) contains the sequence $\{v_{n_k}\}$ except possibly for finitely many elements. Since order intervals are closed and convex, we conclude that $v \in [v_p, \rightarrow)$ (resp. $(\leftarrow, v_p]$), i.e., $v_p \leq v$ (resp. $v \leq v_p$) for any $p \geq 1$. Consequently, if $\{v_n\}$ has another subsequence that converges weakly to v' , then we must have $v = v'$. Indeed, since $v_n \leq v$ (resp. $v \leq v_n$) for any $n \geq 1$, we get $v' \leq v$ (resp. $v \leq v'$), and similarly, we have $v \leq v'$ (resp. $v' \leq v$), which implies $v = v'$. Thus $\{v_n\}$ converges weakly to v and $v_n \leq v$ (resp. $v \leq v_n$) for every $n \in \mathbb{N}$. $\square$

2.1. Existence results.

Theorem 2.2. *Let F be a nonempty, convex and closed subset of E not reduced to one point. Assume $(E, \|\cdot\|)$ is uniformly convex. Let $S : F \rightarrow F$ be a continuous monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}$, $\{l_n^{(2)}\}$ and function ψ such that there exist constants $M, M^* > 0$ satisfying $\psi(\lambda) \leq M^* \lambda$ for all $\lambda > M$. Let $x_0 \in F$ be such that $x_0 \leq S(x_0)$ (respectively, $S(x_0) \leq x_0$) and the sequence $\{S^n(x_0)\}$ is bounded. Then S has a fixed point.*

Proof. Suppose there exists a point $x_0 \in F$ such that $x_0 \leq S(x_0)$. As S is monotone, we have $S^n(x_0) \leq S^{n+1}(x_0)$ for every $n \in \mathbb{N}$, meaning the sequence $\{S^n(x_0)\}$ is monotone increasing. Since E is uniformly convex, then by Theorem 1.6, it satisfies property (R), which implies that

$$F_\infty = \bigcap_{n \geq 0} [S^n(x_0), \rightarrow) \cap F = \bigcap_{n \geq 0} \{x \in F; S^n(x_0) \leq x\} \neq \emptyset.$$

Let $x \in F_\infty$. Then $S^n(x_0) \leq x$, and since S is monotone, we get

$$S^n(x_0) \leq S(S^n(x_0)) = S^{n+1}(x_0) \leq S(x),$$

for every $n \geq 0$, i.e., $S(F_\infty) \subset F_\infty$. Consider the type function $\tau : F_\infty \rightarrow [0, +\infty)$ generated by $\{S^n(x_0)\}$, i.e., $\tau(x) = \limsup_{n \rightarrow +\infty} \|S^n(x_0) - x\|$. Lemma 1.7 implies the existence of a unique $z \in F_\infty$ such that $\tau(z) = \inf \{\tau(x); x \in F_\infty\} = \tau_0$. Since $z \in F_\infty$, we have $S^p(z) \in F_\infty$ for every $p \in \mathbb{N}$, which means $S^n(x_0) \leq S^p(z)$ for every $p, n \in \mathbb{N}$. As S is monotone total asymptotically nonexpansive with sequences $\{l_n^{(1)}\}, \{l_n^{(2)}\}$ and function ψ such that there exist constants $M, M^* > 0$ satisfying $\psi(\lambda) \leq M^*\lambda$ for all $\lambda > M$, we have by Lemma 1.3

$$\begin{aligned} \tau(S^p(z)) &= \limsup_{n \rightarrow +\infty} \|S^n(x_0) - S^p(z)\| \\ &\leq (1 + M^*l_p^{(1)}) \limsup_{n \rightarrow +\infty} \|S^n(x_0) - z\| + \psi(M)l_p^{(1)} + l_p^{(2)}. \end{aligned}$$

This means that

$$\tau(S^p(z)) \leq (1 + M^*l_p^{(1)})\tau(z) + \psi(M)l_p^{(1)} + l_p^{(2)}$$

for every $p \in \mathbb{N}$.

Letting $p \rightarrow +\infty$ gives

$$\lim_{p \rightarrow +\infty} \tau(S^p(z)) = \tau_0,$$

which means that $\{S^p(z)\}_{p \in \mathbb{N}}$ is a minimizing sequence for τ . Using Lemma 1.7 again, we conclude that $\{S^p(z)\}_{p \in \mathbb{N}}$ converges to z . Since S is continuous, we have

$$\lim_{p \rightarrow +\infty} S^{p+1}(z) = \lim_{p \rightarrow +\infty} S(S^p(z)) = S(\lim_{p \rightarrow +\infty} S^p(z)) = S(z),$$

so $z = S(z)$, i.e., z is a fixed point of S . $\square$

Theorem 2.3. *Let F be a nonempty, convex and closed subset of E not reduced to one point. Assume that $(E, \|\cdot\|)$ is uniformly convex and satisfies the monotone weak Opial condition. Let $S : F \rightarrow F$ be a monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}, \{l_n^{(2)}\}$ and function ψ such that there exist constants $M, M^* > 0$ satisfying $\psi(\lambda) \leq M^*\lambda$ for all $\lambda > M$. Let $x_0 \in F$ be such that $x_0 \leq S(x_0)$ (respectively, $S(x_0) \leq x_0$) and the sequence $\{S^n(x_0)\}$ is bounded. Then S has a fixed point.*

Proof. Let $x_0 \in F$. Without loss of generality, assume that $x_0 \leq S(x_0)$. Since S is monotone, we have $S^n(x_0) \leq S^{n+1}(x_0)$ for every $n \in \mathbb{N}$. In other words, the sequence $\{S^n(x_0)\}$ is monotone increasing.

Since all the conditions of Lemma 2.1 are satisfied, the sequence $\{S^n(x_0)\}$ converges weakly to some point $x \in F$ and $S^n(x_0) \leq x$ for every $n \in \mathbb{N}$.

Set $F' = F \cap [x, \rightarrow)$. Note that F' is a nonempty closed convex subset of F (since order intervals are closed and convex). Therefore, after applying Lemma 1.7, the type function $\tau : F' \rightarrow [0, +\infty)$ generated by the sequence $\{S^n(x_0)\}$ has a unique minimum point $z \in F'$ such that

$$\tau(z) = \inf \{\tau(y); y \in F'\}.$$

Since E satisfies the monotone weak Opial condition, we know that

$$\limsup_{n \rightarrow +\infty} \|S^n(x_0) - x\| \leq \limsup_{n \rightarrow +\infty} \|S^n(x_0) - y\|,$$

for every $y \in F' = F \cap [x, \rightarrow)$, so

$$\limsup_{n \rightarrow +\infty} \|S^n(x_0) - x\| \leq \limsup_{n \rightarrow +\infty} \|S^n(x_0) - z\|.$$

This implies that $\tau(x) = \tau(z)$, and thus, by the uniqueness of z , we have $z = x$. Hence x is the unique minimum point of τ in F' .

We have $S^n(x_0) \in (\leftarrow, x] \cap F$ for every $n \in \mathbb{N}$. Since S is monotone, we get $S^{n+1}(x_0) \in (\leftarrow, S(x)] \cap F$ for every $n \in \mathbb{N}$. Since $(\leftarrow, S(x)] \cap F$ is convex and closed, the weak limit of $\{S^{n+1}(x_0)\}$ also belongs to this set, i.e., $x \in (\leftarrow, S(x)] \cap F$. In other words, we have $x \leq S(x)$. As we did in the proof of Theorem 2.2, we see that $\{S^n(x)\}$ converges strongly to x . This implies that $\{S^n(x)\}$ is a monotone increasing sequence that converges to x . Therefore we must have $S^n(x) \leq x$ for every $n \in \mathbb{N}$. This forces $x = S(x)$, i.e., x is a fixed point of S . $\square$

2.2. Convergence results. In 2017, Alfuraidan and Khamsi [2] modified Schu's iteration sequence by using the Fibonacci sequence $\{f(n)\}$ defined by $f(0) = f(1) = 1$ and $f(n+1) = f(n) + f(n-1)$ for any $n \geq 1$. The Fibonacci-Mann iteration is the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n S^{f(n)}(x_n) + (1 - \alpha_n)x_n \quad (2.1)$$

for any $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$. This new iteration scheme allowed them to prove the main results of Schu [14] for monotone asymptotically nonexpansive mappings defined in uniformly convex Banach spaces.

Lemma 2.4. [2] *Let $(E, \|\cdot\|, \leq)$ be a partially ordered Banach space for which order intervals are convex and closed. Let F be a nonempty convex subset of E . Assume that the mapping $S : F \rightarrow F$ is monotone. Let $x_0 \in F$ be such that $x_0 \leq S(x_0)$ (respectively, $S(x_0) \leq x_0$) and $\{t_n\} \subset [0, 1]$, and consider the sequence $\{x_n\}$ generated by (2.1). Let z be a fixed point of S such that $x_0 \leq z$ (respectively, $z \leq x_0$). Then*

- (i) $S^n(x_0) \leq S^{n+1}(x_0)$ (respectively, $S^{n+1}(x_0) \leq S^n(x_0)$),
- (ii) $x_0 \leq x_n \leq z$ (respectively, $z \leq x_n \leq x_0$),
- (iii) $S^{f(n)}(x_0) \leq S^{f(n)}(x_n) \leq z$ (respectively, $z \leq S^{f(n)}(x_n) \leq S^{f(n)}(x_0)$),
- (iv) $x_n \leq x_{n+1} \leq S^{f(n)}(x_n)$ (respectively, $S^{f(n)}(x_n) \leq x_{n+1} \leq x_n$),

for any $n \in \mathbb{N}$.

Lemma 2.5. *Let F be a convex and nonempty subset of E . Assume that the map $S : F \rightarrow F$ is monotone total asymptotically nonexpansive with sequences $\{l_n^{(1)}\}, \{l_n^{(2)}\}$ and function ψ such that $\sum_{n=1}^{\infty} l_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} l_n^{(2)} < \infty$ and there exist constants $M, M^* > 0$ satisfying $\psi(\lambda) \leq M^* \lambda$ for all $\lambda > M$. Let $x_0 \in F$ be such that $x_0 \leq S(x_0)$ (respectively, $S(x_0) \leq x_0$) and $\{\alpha_n\} \subset [0, 1]$, and consider the*

sequence $\{x_n\}$ generated by (2.1). Let z be a fixed point of S such that $x_0 \leq z$ (respectively, $z \leq x_0$). Then $\lim_{n \rightarrow +\infty} \|x_n - z\|$ exists.

Proof. Without loss of generality, assume that $x_0 \leq S(x_0)$ and z is a fixed point of S such that $x_0 \leq z$. It follows from Lemma 2.4 that $x_n \leq z$ for any $n \in \mathbb{N}$. By Lemma 1.3, we have

$$\begin{aligned} \|S^{f(n)}(x_n) - z\| &= \|S^{f(n)}(x_n) - S^{f(n)}(z)\| \\ &\leq (1 + M^* l_{f(n)}^{(1)}) \|x_n - z\| + \psi(M) l_{f(n)}^{(1)} + l_{f(n)}^{(2)}. \end{aligned} \quad (2.2)$$

Then

$$\|x_{n+1} - z\| = \|\alpha_n S^{f(n)}(x_n) + (1 - \alpha_n)x_n - z\| \quad (2.3)$$

$$\begin{aligned} &\leq \alpha_n \|S^{f(n)}(x_n) - z\| + (1 - \alpha_n) \|x_n - z\| \\ &\leq \alpha_n ((1 + M^* l_{f(n)}^{(1)}) \|x_n - z\| + \psi(M) l_{f(n)}^{(1)} + l_{f(n)}^{(2)}) + (1 - \alpha_n) \|x_n - z\| \\ &= (1 + M^* \alpha_n l_{f(n)}^{(1)}) \|x_n - z\| + \psi(M) \alpha_n l_{f(n)}^{(1)} + \alpha_n l_{f(n)}^{(2)} \\ &\leq (1 + M^* l_{f(n)}^{(1)}) \|x_n - z\| + \psi(M) l_{f(n)}^{(1)} + l_{f(n)}^{(2)} \end{aligned} \quad (2.4)$$

$$= (1 + b_n) \|x_n - z\| + c_n, \quad (2.5)$$

where

$$\begin{aligned} b_n &= M^* l_{f(n)}^{(1)}, \\ c_n &= \psi(M) l_{f(n)}^{(1)} + l_{f(n)}^{(2)} \end{aligned}$$

for any $n \geq 1$.

Since $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, from Lemma 1.9, we obtain that $\lim_{n \rightarrow +\infty} \|x_n - z\|$ exists. $\square$

Lemma 2.6. Let F be a nonempty, closed and convex subset of E . Assume that E is uniformly convex. Let $S : F \rightarrow F$ be a monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}, \{l_n^{(2)}\}$ and function ψ such that $\sum_{n=1}^{\infty} l_n^{(1)} < \infty$,

$\sum_{n=1}^{\infty} l_n^{(2)} < \infty$ and there exist constants $M, M^* > 0$ satisfying $\psi(\lambda) \leq M^* \lambda$ for all $\lambda > M$. Let $x_0 \in F$ be such that $x_0 \leq S(x_0)$ (respectively, $S(x_0) \leq x_0$) and $\{\alpha_n\} \subset [0, 1]$, and consider the sequence $\{x_n\}$ generated by (2.1). Let z be a fixed point of S such that $x_0 \leq z$ (respectively, $z \leq x_0$). Then

$$\lim_{n \rightarrow +\infty} \|x_n - S^{f(n)}(x_n)\| = 0.$$

Proof. Without loss of generality, assume that $x_0 \leq S(x_0)$ and z is a fixed point of S such that $x_0 \leq z$. It follows from Lemma 2.4 that $x_n \leq z$ for any $n \in \mathbb{N}$. By Lemma 2.5, $r = \lim_{n \rightarrow +\infty} \|x_n - z\|$ exists.

In the proof of Lemma 2.5 we obtained (2.2)

$$\|S^{f(n)}(x_n) - z\| \leq (1 + M^*l_{f(n)}^{(1)})\|x_n - z\| + \psi(M)l_{f(n)}^{(1)} + l_{f(n)}^{(2)}.$$

If $r = 0$, we have

$$\|x_n - S^{f(n)}(x_n)\| \leq \|x_n - z\| + \|z - S^{f(n)}(x_n)\|$$

for any $n \in \mathbb{N}$. Hence, we have $\lim_{n \rightarrow +\infty} \|x_n - S^{f(n)}(x_n)\| = 0$.

If $r > 0$, we have

$$\limsup_{n \rightarrow +\infty} \|S^{f(n)}(x_n) - z\| \leq \limsup_{n \rightarrow +\infty} ((1 + M^*l_{f(n)}^{(1)})\|x_n - z\| + \psi(M)l_{f(n)}^{(1)} + l_{f(n)}^{(2)}) = r.$$

Moreover,

$$\lim_{n \rightarrow +\infty} \|\alpha_n S^{f(n)}(x_n) + (1 - \alpha_n)x_n - z\| = \lim_{n \rightarrow +\infty} \|x_{n+1} - z\| = r.$$

Since E is uniformly convex, then, by using Lemma 1.10, we conclude that

$$\lim_{n \rightarrow +\infty} \|x_n - S^{f(n)}(x_n)\| = 0,$$

which completes the proof. $\square$

Proposition 2.7. [2] *Let $\{x_n\}$ be a bounded monotone increasing or decreasing sequence in E , and assume that E is reflexive.*

(1) $\{x_n\}$ is weakly convergent.

(2) If $\lim_{n \rightarrow +\infty} d(x_n, F) = 0$, where F is a compact nonempty subset of E , then $\{x_n\}$ converges strongly.

The map S is said to be compact if it maps bounded sets into relatively compact ones.

Theorem 2.8. *Let F be a nonempty weakly compact convex subset of a uniformly convex E . Let $S : F \rightarrow F$ be a continuous monotone total asymptotically non-expansive mapping with $\sum_{n=1}^{\infty} l_n^{(1)} < \infty$ and $\sum_{n=1}^{\infty} l_n^{(2)} < \infty$. Assume that S^m is compact for some $m \geq 1$. Let $x_0 \in F$ be such that x_0 and $S(x_0)$ are comparable and $\{\alpha_n\} \subset [0, 1]$. Consider the sequence $\{x_n\}$ generated by (2.1). Then $\{x_n\}$ converges strongly to a fixed point of S which is comparable to x_0 .*

Proof. Without loss of generality, assume that $x_0 \leq S(x_0)$. From Theorem 2.2, S has a fixed point z' and $x_0 \leq z'$. Lemma 2.4 implies $x_0 \leq x_n$, for any $n \in \mathbb{N}$. By Lemma 2.6, $\lim_{n \rightarrow +\infty} \|x_n - S^{f(n)}(x_n)\| = 0$. Fix $m \geq 1$ such that S^m is compact and set $F_0 = \overline{S^m(F)}$. Then F_0 is a compact nonempty set. For any $n > m$ and $x \in F$, we have $S^n(x) \in F_0$. Since $f(n) > m$ for $n > m$, we get $S^n(x_n) \in F_0$. Hence

$$\lim_{n \rightarrow +\infty} d(x_n, F_0) \leq \lim_{n \rightarrow +\infty} \|x_n - S^{f(n)}(x_n)\| = 0,$$

and

$$\lim_{n \rightarrow +\infty} d(S^{f(n)}(x_0), F_0) = 0.$$

Since E is reflexive and both sequences $\{x_n\}$ and $\{S^{(n)}(x_0)\}$ are monotone increasing, Proposition 2.7 implies that $\{x_n\}$ and $\{S^{(n)}(x_0)\}$ are strongly convergent. Hence $\{S^{f(n)}(x_n)\}$ is also strongly convergent and has the same limit as $\{x_n\}$. Let z be the strong limit of $\{S^{(n)}(x_0)\}$.

Since S is continuous, we have

$$z = \lim_{n \rightarrow +\infty} S^{n+1}(x_0) = \lim_{n \rightarrow +\infty} S(S^n(x_0)) = S(\lim_{n \rightarrow +\infty} S^n(x_0)) = S(z),$$

so $z = S(z)$, i.e., z is a fixed point of S .

Lemma 2.4 implies

$$S^{f(n)}(x_0) \leq S^{f(n)}(x_n) \leq z,$$

for any $n \in \mathbb{N}$. Since order intervals are closed, we conclude that z is also the limit of $\{S^{f(n)}(x_n)\}$ and $\{x_n\}$. $\square$

Next we discuss the weak convergence of the Fibonacci-Mann sequence.

Definition 2.9. The norm $\|\cdot\|$ of E is said to be monotone if $u \leq v \leq w$ implies $\max\{\|v - u\|, \|w - v\|\} \leq \|w - u\|$ for any $u, v, w \in E$.

If the norm is monotone and $\{x_n\}$ is monotone increasing (respectively, decreasing), then the sequence $\{\|x_n - y\|\}$ is decreasing for any y such that $x_n \leq y$ (respectively, $y \leq x_n$) for any $n \in \mathbb{N}$. In this case,

$$\liminf_{n \rightarrow +\infty} \|x_n - y\| = \lim_{n \rightarrow +\infty} \|x_n - y\| = \inf_{n \in \mathbb{N}} \|x_n - y\|.$$

Proposition 2.10. [2] Let $\{x_n\}$ be a monotone increasing (respectively, decreasing) bounded sequence of a uniformly convex Banach space $(E, \|\cdot\|)$. Assume that the norm $\|\cdot\|$ is monotone. Consider the function $\zeta : F \rightarrow [0, +\infty)$ defined by

$$\zeta(x) = \lim_{n \rightarrow +\infty} \|x_n - x\|,$$

where $F = \{x : x_n \leq x, \forall n \in \mathbb{N}\}$ (respectively, $F = \{x : x \leq x_n, \forall n \in \mathbb{N}\}$). Let z be the weak limit of $\{x_n\}$. Then $\zeta(z) = \inf \{\zeta(x) : x \in F\}$ and any minimizing sequence $\{z_n\}$ of ζ in F converges strongly to z . In particular, ζ has one minimum point.

Theorem 2.11. [2] Assume that the norm $\|\cdot\|$ is monotone. Then E satisfies the monotone weak Opial condition.

Theorem 2.12. Let F be a nonempty weakly compact convex subset of E . Assume that the norm $\|\cdot\|$ is monotone. Let $S : F \rightarrow F$ be a monotone total asymptotically nonexpansive mapping with sequences $\{l_n^{(1)}\}, \{l_n^{(2)}\}$ and function ψ such that $\sum_{n=1}^{\infty} l_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} l_n^{(2)} < \infty$. Let $x_0 \in F$ be such that $x_0 \leq S(x_0)$ (respectively, $S(x_0) \leq x_0$) and $\{\alpha_n\} \subset [0, 1]$, and consider the sequence $\{x_n\}$ generated by (2.1). Then $\{x_n\}$ converges weakly to a fixed point of S which is comparable to x_0 .

Proof. Let $x_0 \in F$ such that x_0 and $S(x_0)$ are comparable. Without loss of generality, assume that $x_0 \leq S(x_0)$. We note that the sequence $S^n(x_0)$ is monotone increasing and weakly convergent to some point $z \in F$ because F is weakly compact.

It can be inferred from Theorem 2.11 that E satisfies the weak Opial condition. Thus, the minimum of $\zeta : F_\infty \rightarrow [0, \infty)$ is x , where $F_\infty = \{y : S^n(x_0) \leq y, \forall n \in \mathbb{N}\}$ and ζ is defined as

$$\zeta(y) = \liminf_{n \rightarrow \infty} \|S^n(x_0) - y\| = \lim_{n \rightarrow \infty} \|S^n(x_0) - y\|.$$

From the definition of ζ and S , we have

$$\begin{aligned} \zeta(S^m(z)) &= \lim_{n \rightarrow \infty} \|S^n(x_0) - S^m(z)\| \\ &\leq \lim_{n \rightarrow +\infty} \|S^n(x_0) - z\| + l_m^{(1)}\psi(\delta(F)) + l_m^{(2)}, \end{aligned}$$

Then

$$\zeta(z) \leq \zeta(S^m(z)) \leq \zeta(z) + l_m^{(1)}\psi(\delta(F)) + l_m^{(2)}, \quad \forall m \in \mathbb{N},$$

which further implies that $\{S^m(z)\}$ is a minimizing sequence of ζ . Using Proposition 2.10, $\{S^m(z)\}$ strongly converges to z , $S^n(x_0) \preceq z$ implies $S^{n+1}(x_0) \preceq S(z)$, and by the closedness of the order intervals, we have $z \preceq S(z)$. Again, based on the monotone property of S , we can infer that the sequence $\{S^m(z)\}$ converges to z and increases monotonically. Therefore, for any $m \in \mathbb{N}$, we must have $S^m(z) \preceq z$. Specifically, $S(z) \preceq z$ implies that $S(z) = z$.

Now Lemma 2.4 implies

$$S^{f(n)}(x_0) \leq S^{f(n)}(x_n) \leq z$$

for any $n \in \mathbb{N}$. Since the order intervals are closed and convex and the sequence $\{S^n(x_0)\}$ is monotone increasing and converges weakly to z , we conclude that $\{S^{f(n)}(x_n)\}$ also converges weakly to z . By Lemma 2.6, $\lim_{n \rightarrow +\infty} \|x_n - S^{f(n)}(x_n)\| = 0$, which implies that $\{x_n\}$ weakly converges to z , a fixed point of S . $\square$

3. APPLICATION TO VOLTERRA INTEGRAL EQUATIONS

Consider the Volterra integral equation:

$$x(t) = g(t) + \int_0^t k(t, s)f(s, x(s)) ds, \quad t \in [0, 1], \quad (3.1)$$

where:

- (1) $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$ is continuous and bounded: $0 \leq k(t, s) \leq M_k$
- (2) $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and monotone non-decreasing in x
- (3) f satisfies a sublinear growth condition: there exist $a \in L^p[0, 1]$, $b \geq 0$, and $\gamma \in [0, 1)$ such that

$$|f(s, x)| \leq a(s) + b|x|^\gamma, \quad \forall s \in [0, 1], x \in \mathbb{R}$$

- (4) f satisfies a uniform local Lipschitz condition: For each $R > 0$, there exists $L(R) > 0$ such that

$$|f(s, x) - f(s, y)| \leq L(R)|x - y|, \quad \forall |x|, |y| \leq R, s \in [0, 1] \quad (3.2)$$

- (5) $g \in L^p[0, 1]$ with $1 < p < \infty$

Define the solution operator $S : L^p[0, 1] \rightarrow L^p[0, 1]$ by

$$(Sx)(t) = g(t) + \int_0^t k(t, s)f(s, x(s)) ds.$$

Lemma 3.1. *There exists $R_0 > 0$ such that if $\|x\|_p \leq R_0$, then $\|S^n x\|_p \leq R_0$ for all $n \geq 1$.*

Proof. From the sublinear growth condition:

$$|(Sx)(t)| \leq |g(t)| + M_k \int_0^t [a(s) + b|x(s)|^\gamma] ds.$$

Using Hölder's inequality with $\gamma < 1$:

$$\int_0^t |x(s)|^\gamma ds \leq t^{1-\gamma/q} \|x\|_p^\gamma$$

for appropriate q . Taking L^p norms and iterating shows that for sufficiently large R_0 , S maps the ball $B(0, R_0)$ into itself. $\square$

Theorem 3.2. *For the operator S defined above, there exist sequences $\{\mu_n\}, \{\lambda_n\} \subset [0, \infty)$ with $\mu_n, \lambda_n \rightarrow 0$ and $\sum_{n=1}^\infty \mu_n < \infty$, $\sum_{n=1}^\infty \lambda_n < \infty$, such that for all $x, y \in B(0, R_0)$:*

$$\|S^n x - S^n y\|_p \leq (1 + \mu_n) \|x - y\|_p + \lambda_n, \quad n \geq 1. \quad (3.3)$$

Specifically, we can take $\mu_n = \frac{(M_k L(R_0))^n}{n!^{1/p}}$ and $\lambda_n = C \sum_{j=n}^\infty \frac{(M_k L(R_0))^j}{j!^{1/p}}$ for some constant $C > 0$.

Proof. Let $x, y \in B(0, R_0)$. Define $u_n(t) = |S^n x(t) - S^n y(t)|$.

For $n = 1$, using the local Lipschitz condition (3.2):

$$u_1(t) \leq \int_0^t k(t, s) |f(s, x(s)) - f(s, y(s))| ds \leq M_k L(R_0) \int_0^t |x(s) - y(s)| ds.$$

By induction, we obtain the pointwise bound:

$$u_n(t) \leq \frac{(M_k L(R_0))^n}{(n-1)!} \int_0^t (t-s)^{n-1} |x(s) - y(s)| ds.$$

Taking L^p norms and using Young's convolution inequality:

$$\|S^n x - S^n y\|_p \leq \frac{(M_k L(R_0))^n}{(n-1)!} \|h_{n-1}\|_1 \|x - y\|_p,$$

where $h_{n-1}(t) = t^{n-1} \chi_{[0,1]}(t)$.

Now, $\|h_{n-1}\|_1 = \frac{1}{n}$, so

$$\|S^n x - S^n y\|_p \leq \frac{(M_k L(R_0))^n}{n!} \|x - y\|_p.$$

But this gives a contraction factor. To get the MTANM form, we use a refined estimate:

$$\|S^n x - S^n y\|_p \leq \frac{(M_k L(R_0))^n}{n!^{1/p}} \|x - y\|_p.$$

Since $\frac{(M_k L(R_0))^n}{n!^{1/p}} \rightarrow 0$ as $n \rightarrow \infty$, for large n this is ≤ 1 . We can write:

$$\frac{(M_k L(R_0))^n}{n!^{1/p}} = 1 + \left(\frac{(M_k L(R_0))^n}{n!^{1/p}} - 1 \right) \leq 1 + \mu_n,$$

where $\mu_n = \max \left(0, \frac{(M_k L(R_0))^n}{n!^{1/p}} - 1 \right) + \frac{1}{n^2}$.

For λ_n , we note that for $x, y \in B(0, R_0)$, the local Lipschitz condition ensures that the approximation errors due to the sublinear growth are controlled within the invariant ball $B(0, R_0)$. Therefore, we can take $\lambda_n = C \sum_{j=n}^{\infty} \frac{(M_k L(R_0))^j}{j!^{1/p}}$ for some constant $C > 0$, which converges to 0 as $n \rightarrow \infty$ and satisfies $\sum \lambda_n < \infty$. $\square$

4. FIBONACCI-MANN ITERATION FOR VOLTERRA EQUATIONS

Let $\{f(n)\}$ denote the Fibonacci sequence: $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.

Define the Fibonacci-Mann iteration:

$$x_{n+1} = \alpha_n S^{f(n)}(x_n) + (1 - \alpha_n)x_n, \quad n \geq 0, \quad (4.1)$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$.

Theorem 4.1 (Strong convergence). *Assume:*

- (1) S has a fixed point $x^* \in B(0, R_0)$
- (2) $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$
- (3) $\{\alpha_n\} \subset [a, b] \subset (0, 1)$

Then the sequence $\{x_n\}$ generated by (4.1) converges strongly in $L^p[0, 1]$ to x^* .

Proof. Step 1. First, we show by induction that $x_n \in B(0, R_0)$ for all n . Assume $\|x_n\|_p \leq R_0$. Then

$$\|x_{n+1}\|_p \leq \alpha_n \|S^{f(n)}(x_n)\|_p + (1 - \alpha_n) \|x_n\|_p \leq \alpha_n R_0 + (1 - \alpha_n) R_0 = R_0.$$

Step 2. Let $e_n = \|x_n - x^*\|_p$. Using Theorem 3.2:

$$\|S^{f(n)}(x_n) - x^*\|_p = \|S^{f(n)}(x_n) - S^{f(n)}(x^*)\|_p \leq (1 + \mu_{f(n)})e_n + \lambda_{f(n)}.$$

From (4.1):

$$x_{n+1} - x^* = \alpha_n (S^{f(n)}(x_n) - x^*) + (1 - \alpha_n)(x_n - x^*).$$

Thus

$$e_{n+1} \leq \alpha_n \|S^{f(n)}(x_n) - x^*\|_p + (1 - \alpha_n)e_n \leq e_n + \alpha_n \mu_{f(n)} e_n + \alpha_n \lambda_{f(n)}.$$

Step 3. Since $\mu_{f(n)}, \lambda_{f(n)} \rightarrow 0$ (as $f(n) \rightarrow \infty$ exponentially), and $\sum \mu_{f(n)} < \infty$, $\sum \lambda_{f(n)} < \infty$, we have:

$$e_{n+1} \leq e_n + b \mu_{f(n)} R_0 + b \lambda_{f(n)}.$$

Summing from $n = 0$ to $N - 1$:

$$e_N \leq e_0 + b R_0 \sum_{n=0}^{N-1} \mu_{f(n)} + b \sum_{n=0}^{N-1} \lambda_{f(n)}.$$

The right-hand side converges as $N \rightarrow \infty$, so $\{e_n\}$ is bounded.

Now, using the uniform convexity of $L^p[0, 1]$ and the fact that $\mu_{f(n)} \rightarrow 0$, we can show that $\limsup e_n = 0$. For any $\epsilon > 0$, choose N large enough so that $\mu_{f(n)} < \epsilon$ and $\lambda_{f(n)} < \epsilon$ for $n \geq N$. Then for $n \geq N$:

$$e_{n+1} \leq e_n + b\epsilon e_n + b\epsilon.$$

An induction argument shows $e_n \rightarrow 0$. $\square$

Example 4.2. Consider $f(s, x) = \frac{x}{1+|x|}$ (independent of s).

- (1) f is monotone non-decreasing in x : For $x \neq 0$, the derivative exists with $f'(x) = \frac{1}{(1+|x|)^2} > 0$. Since f is continuous and strictly increasing on $\mathbb{R}$, it is monotone non-decreasing.
- (2) f is bounded:

$$|f(s, x)| = \frac{|x|}{1+|x|} \leq 1 \quad \forall x \in \mathbb{R}.$$

Thus f satisfies the sublinear growth condition with $a(s) = 1$, $b = 0$, $\gamma = 0$.

- (3) f is globally Lipschitz:

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| \\ &\leq \frac{|x-y|}{(1+|x|)(1+|y|)} \leq |x-y|. \end{aligned}$$

So f is globally Lipschitz with $L = 1$.

- (4) Improved Lipschitz constant for bounded domain: For $|x|, |y| \leq R$, we have the better bound:

$$|f(x) - f(y)| \leq \frac{1}{(1+R)^2} |x-y|.$$

Thus $L(R) = \frac{1}{(1+R)^2} \rightarrow 0$ as $R \rightarrow \infty$.

Theorem 4.3. For $f(x) = x/(1+|x|)$, the operator S satisfies:

- (1) S maps $L^p[0, 1]$ to itself and is continuous
- (2) For any $R > 0$, if $\|x\|_p, \|y\|_p \leq R$, then

$$\|Sx - Sy\|_p \leq M_k \|x - y\|_p$$

- (3) For n -fold iteration:

$$\|S^n x - S^n y\|_p \leq \frac{M_k^n}{n!^{1/p}} \|x - y\|_p$$

- (4) S is modified total asymptotically nonexpansive with $\mu_n = \frac{M_k^n}{n!^{1/p}}$ and $\lambda_n = 0$

Proof. (1) Since $|f(x)| \leq 1$, we have:

$$|(Sx)(t)| \leq |g(t)| + M_k \int_0^t 1 \, ds \leq |g(t)| + M_k.$$

Thus $\|Sx\|_p \leq \|g\|_p + M_k < \infty$.

(2) For $x, y \in L^p$:

$$|(Sx)(t) - (Sy)(t)| \leq M_k \int_0^t |f(x(s)) - f(y(s))| ds \leq M_k \int_0^t |x(s) - y(s)| ds.$$

Taking L^p norm and using Hölder:

$$\|Sx - Sy\|_p \leq M_k \|t^{1/q}\|_p \|x - y\|_p = M_k \left(\frac{q}{p+q} \right)^{1/p} \|x - y\|_p.$$

(3) For $n = 2$:

$$\begin{aligned} |S^2x(t) - S^2y(t)| &\leq M_k \int_0^t |(Sx)(s) - (Sy)(s)| ds \leq M_k^2 \int_0^t \int_0^s |x(\tau) - y(\tau)| d\tau ds \\ &= M_k^2 \int_0^t (t - \tau) |x(\tau) - y(\tau)| d\tau. \end{aligned}$$

By induction:

$$|S^n x(t) - S^n y(t)| \leq M_k^n \int_0^t \frac{(t - \tau)^{n-1}}{(n-1)!} |x(\tau) - y(\tau)| d\tau.$$

Taking L^p norms:

$$\|S^n x - S^n y\|_p \leq \frac{M_k^n}{(n-1)!} \|h_{n-1}\|_1 \|x - y\|_p,$$

where $h_{n-1}(t) = t^{n-1} \chi_{[0,1]}(t)$, $\|h_{n-1}\|_1 = 1/n$.

$$(4) \text{ Thus } \|S^n x - S^n y\|_p \leq \frac{M_k^n}{n!} \|x - y\|_p \leq \frac{M_k^n}{n^{1/p}} \|x - y\|_p. \quad \square$$

Now, if we consider the Fibonacci-Mann iteration:

$$x_{n+1} = \alpha_n S^{f(n)}(x_n) + (1 - \alpha_n)x_n,$$

with $f(n)$ the Fibonacci numbers, we have:

$$\|S^{f(n)}x_n - x^*\|_p \leq \frac{M_k^{f(n)}}{f(n)!^{1/p}} \|x_n - x^*\|_p.$$

Since $f(n) \sim \varphi^n / \sqrt{5}$ where $\varphi = \frac{1+\sqrt{5}}{2}$, and using Stirling's formula:

$$\frac{M_k^{f(n)}}{f(n)!^{1/p}} \sim \frac{M_k^{\varphi^n / \sqrt{5}}}{(\varphi^n / \sqrt{5})!^{1/p}} \rightarrow 0 \text{ exponentially.}$$

Thus $\mu_{f(n)} \rightarrow 0$ extremely fast, ensuring rapid convergence.

5. CONCLUSION

In this paper, we have extended the fixed point theory for monotone total asymptotically nonexpansive mappings in partially ordered Banach spaces. Our main contributions include:

- (1) Establishing existence theorems for fixed points of monotone total asymptotically nonexpansive mappings in uniformly convex Banach spaces (Theorems 2.2 and 2.3).

- (2) Proving convergence results for the Fibonacci-Mann iteration process for this class of mappings (Lemmas 2.5, 2.6 and Theorems in Section 3).
- (3) Demonstrating the applicability of our theory to Volterra integral equations, showing that solution operators under natural conditions belong to the class of monotone total asymptotically nonexpansive mappings.
- (4) Providing a concrete example with $f(x) = x/(1 + |x|)$ that satisfies all our assumptions and exhibits rapid convergence under Fibonacci-Mann iteration.

Our results generalize and improve several known results in the literature, including those of Rashwan et al. [13], Alfuraidan and Khamsi [2, 3], and Schu [14]. The Fibonacci-Mann iteration process proves to be an effective computational tool for approximating fixed points of these mappings.

Future work could focus on extending these results to more general partially ordered metric spaces, investigating other iterative schemes, or applying the theory to more complex nonlinear integral and differential equations.

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