

ON THE STRUCTURE AND SPECTRA OF NON-INCLUSION PRINCIPAL IDEAL GRAPH OF INVERSE SEMIGROUPS

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ABSTRACT. Let S be a semigroup. We define the non-inclusion principal left ideal graph of S as a simple, undirected graph with the nonzero elements of S as vertices and two distinct elements $a, b \in S$ are adjacent if and only if $a \notin S^1b$ and $b \notin S^1a$, where S^1a and S^1b are principal left ideals generated by a and b respectively. The non-inclusion principal right ideal graph is defined similarly. Here, we identify the structure of the non-inclusion principal ideal graph of inverse semigroups with a particular emphasis on symmetric inverse semigroups and Brandt semigroups. The energies of some matrices associated with this graph are computed for the Brandt semigroup, yielding a detailed spectral characterization.

Keywords. non-inclusion principal ideal graph, inverse semigroups, symmetric inverse semigroup, Brandt semigroup, graph energy

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1. INTRODUCTION

Graphs linked to algebraic structures serve as essential instruments for connecting abstract algebra with combinatorics, especially when analyzing rings and semigroups to uncover links between their algebraic traits and graph properties. This approach has fueled significant research over recent decades. In 2011, M. Afkhami and K. Khashyarmansh defined the cozero-divisor graph $\Gamma'(R)$ for a commutative ring R with nonzero identity, serving as a complement to the standard zero-divisor graph $\Gamma(R)$ [2]. The vertex set comprises all non-unit elements of R , with distinct vertices a and b joined by an edge precisely when $a \notin bR$ and $b \notin aR$. Around the same time, principal ideal graphs for semigroups were studied by R. S. Indu and L. John, where the graph is defined on the semigroup with vertices adjacent if their principal ideals intersect [11, 10]. They have studied the principal ideal graph of inverse semigroups by exploiting the special properties of idempotents present in this structure [12].

Motivated by these developments, we introduced the concept of the non-inclusion principal ideal graph of a semigroup in 2025 and studied this graph for completely simple semigroups [14]. Specifically, the non-inclusion principal left ideal graph

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$nPiG_l(S)$ is defined with vertex set $S \setminus \{0\}$, where two distinct vertices a and b are adjacent precisely when $a \notin S^1b$ and $b \notin S^1a$; here, $S^1a = \{a\} \cup \{sa : s \in S\}$ denotes the principal left ideal generated by a . Analogously, the non-inclusion principal right ideal graph $nPiG_r(S)$ has vertex set $S \setminus \{0\}$, with vertices a and b adjacent if and only if $a \notin bS^1$ and $b \notin aS^1$. This study extends existing work on cozero-divisor graphs and principal ideal graphs by introducing new graph structures for semigroups that parallel concepts from commutative ring theory, but are specifically tailored to the algebraic structure of semigroups. The article focuses on investigating the structural characteristics of the non-inclusion principal ideal graphs associated with inverse semigroups, particularly symmetric inverse semigroups and Brandt semigroups, which are presented in Section 3. The concept of graph energy was initially developed by Ivan Gutman [8] in the context of chemical graph theory and has evolved into a versatile tool with applications across mathematics, chemistry, physics, and computer science. Recent investigations have explored various graph energies associated with different graph structures [6, 7, 15, 3]. In Section 4, we describe some energies related to the non-inclusion principal ideal graph of the Brandt semigroup.

2. PRELIMINARIES

Inverse semigroups are characterized by the property that every element has a unique inverse and the set of idempotents forms a commutative subsemigroup. Also note that an inverse semigroup is always regular. Hence, two elements a and b in an inverse semigroup S are adjacent in $nPiG_l(S)$ if and only if $a \notin Sb$ and $b \notin Sa$. A symmetric inverse semigroups are special class of inverse semigroups consisting of all partial bijections on a set, with composition as the operation. $\mathcal{IS}_n$ denotes symmetric inverse semigroup on the finite set $X = \{1, 2, \dots, n\}$. Brandt semigroups are a particular type of inverse semigroup constructed from a group $\mathbb{G}$ and a set $\mathbb{I}$, where the elements are triples (i, g, j) with $i, j \in \mathbb{I}$ and $g \in \mathbb{G}$, and multiplication is defined by $(i, g, j)(k, h, l) = (i, gh, l)$ if $j = k$, and is zero otherwise. Green's relations $\mathcal{L}$ and $\mathcal{R}$ are fundamental equivalence relations on semigroups, with $a \mathcal{L} b$ if $S^1a = S^1b$ and $a \mathcal{R} b$ if $aS^1 = bS^1$, respectively.

For a given graph G , the *adjacency energy*, $\epsilon_{\mathcal{A}}(G)$, is the sum of the absolute values of all eigenvalues of its adjacency matrix $\mathcal{A}(G)$ [8]. Let $\mathcal{D}(G) = \text{diag}(d_1, d_2, \dots, d_n)$ denote the diagonal matrix whose entries correspond to the degrees of the vertices $v_1, v_2, \dots, v_n$. Then the matrices $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ and $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ are known respectively as the *Laplacian* and *signless Laplacian* matrices of G . Their associated energies are denoted by $\epsilon_{\mathcal{L}}(G)$ and $\epsilon_{\mathcal{Q}}(G)$ [16]. Furthermore, the *distance matrix* of G , denoted by $\mathbb{D} = [d_{ij}]$, is defined such that each entry d_{ij} equals the shortest path distance between vertices v_i and v_j . The corresponding distance energy of G is represented by $\epsilon_{\mathbb{D}}(G)$ [13, 17].

For any undefined notation or terminology related to semigroup theory, graph theory, or algebraic graph theory, the reader is directed to [9], [5], and [4], respectively.

3. NON-INCLUSION PRINCIPAL IDEAL GRAPHS OF INVERSE SEMIGROUPS

We establish the structure of non-inclusion principal ideal graphs for inverse semigroups. Our initial result demonstrates that for any inverse semigroup S , $nPiG_l(S)$ and $nPiG_r(S)$ are structurally the same.

Proposition 3.1. *Let S be an inverse semigroup. Then $nPiG_l(S) \cong nPiG_r(S)$.*

Proof. Recall that an inverse semigroup S has a natural involution given by the inversion map $\varphi : S \rightarrow S$, $\varphi(a) = a^{-1}$, where a^{-1} is the unique inverse of a satisfying $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. Adjacency in $nPiG_l(S)$ means $a \notin Sb$ and $b \notin Sa$. Applying the inversion, which reverses multiplication order, we get $a^{-1} \notin b^{-1}S$ and $b^{-1} \notin a^{-1}S$, which is exactly the adjacency condition in $nPiG_r(S)$ for a^{-1} and b^{-1} . Since inversion is an involution and bijection on S , it preserves adjacency and defines a graph isomorphism. $\square$

As both the graphs are isomorphic, for notational simplicity we use $nPiG(S)$ to denote $nPiG_l(S)$ whenever there is no ambiguity. Consequently, corresponding results exist for $nPiG_r(S)$ to all results proved for $nPiG_l(S)$ via inversion. For this reason, we generally avoid reiterating statements for $nPiG_r(S)$ unless it is necessary.

In the following proposition, we prove that if an inverse semigroup S is a monoid, then $nPiG(S)[S]$ is always disconnected.

Proposition 3.2. *If S has an identity element 1, then 1 is an isolated vertex in $nPiG(S)$.*

Proof. Suppose S has an identity element 1. Then for any non-zero element $a \in S$, the principal left ideal $S1 = S$ contains a . Since 1 generates the entire semigroup as a left ideal, the vertex corresponding to 1 is not connected to any other vertex. $\square$

Remark 3.3. In $nPiG(S)$, the isolated vertices are precisely the elements $a \in S$ such that $Sa = S$. This means a is $\mathcal{L}$ -equivalent to the identity element in S (if it exists). More generally, these isolated vertices correspond exactly to the $\mathcal{L}$ -maximal elements of the semigroup. In particular, if S is a monoid, the isolated vertices are exactly the elements in the $\mathcal{L}$ -class of the identity. This characterizes isolated vertices as those that generate the entire semigroup as a principal left ideal, generalizing the fact that an identity vertex is always isolated.

In an inverse semigroup S , the natural partial order $\leq$ is defined by $x \leq y$ if and only if $x = ey$ for some idempotent $e \in S$.

Theorem 3.4. *Any two distinct elements $a, b \in S$ are adjacent in $nPiG_l(S)$ if and only if they are incomparable. That is $a \not\leq b$ and $b \not\leq a$.*

Proof. Let a and b be adjacent in $nPiG(S)$. Then $a \notin Sb$ and $b \notin Sa$. This is equivalent to $a \neq eb$ and $b \neq ea$ for any idempotent $e \in S$. Hence, $a \not\leq b$ and $b \not\leq a$. $\square$

Proposition 3.5. *Let S be an inverse semigroup. If two nonzero elements $a, b \in S$ are $\mathcal{L}$ -related, then a and b are not adjacent in $nPiG(S)$.*

Proof. Since $a \mathcal{L} b$, their principal left ideals coincide. That is $Sa = Sb$. Hence, $a \in Sb$ and $b \in Sa$ and a, b are not adjacent in $nPiG(S)$. $\square$

Remark 3.6. Non-adjacency of two vertices in $nPiG(S)$ does not guarantee that they are $\mathcal{L}$ -related. For, consider the symmetric inverse semigroup $\mathcal{I}_3$ on the set $\{1, 2, 3\}$. Let $a = \{(1 \mapsto 2), (2 \mapsto 3)\}$ and $b = \{(1 \mapsto 2)\}$. Here, as $b \in Sa$, a and b are not adjacent in $nPiG(S)$. However, a and b are not $\mathcal{L}$ -related since their principal left ideals are not equal.

The necessary and sufficient condition for two idempotents to be adjacent in $nPiG(S)$ is as follows.

Proposition 3.7. *Let S be an inverse semigroup. Two nonzero idempotents $e, f \in S$ are adjacent in $nPiG(S)$ if and only if their product is neither e nor f .*

Proof. The proof follows from the fact that for two idempotents $e, f \in S$, $Se \subseteq Sf \iff ef = e$, and similarly, $Sf \subseteq Se \iff fe = f$. $\square$

Remark 3.8. For general nonzero elements $a, b \in S$ in an inverse semigroup S , let $e = a^{-1}a$ and $f = b^{-1}b$ be the associated idempotents. Then a and b are adjacent in the non-inclusion principal left ideal graph $nPiG(S)$ if and only if the principal left ideals $Sa = Se$ and $Sb = Sf$ are incomparable with respect to inclusion, which holds precisely when neither $e \leq f$ nor $f \leq e$ in the natural partial order on idempotents. Thus, adjacency of a and b in $nPiG(S)$ corresponds to incomparability of the idempotents generating their principal left ideals.

A sufficient condition for two idempotents e and f adjacent in $nPiG(S)$.

Proposition 3.9. *Let e, f be distinct idempotents in an inverse semigroup S . If $ef = 0$, then e and f are adjacent in $nPiG(S)$.*

Proof. Since $ef = 0$, it follows that $e \notin Sf$ and $f \notin Se$ because if either inclusion held, say $e \in Sf$, then there would exist $s \in S$ with $e = sf$, implying $ef = e \neq 0$. Similarly for $f \in Se$. Therefore, the adjacency condition $e \notin Sf$ and $f \notin Se$ is satisfied, so e and f are adjacent in $nPiG(S)$. $\square$

The following remark shows that there are idempotents e, f adjacent in $nPiG(S)$ with $ef \neq 0$.

Remark 3.10. Consider the symmetric inverse semigroup $\mathcal{I}_3$ on the set $\{1, 2, 3\}$. Let $e = \{(1 \mapsto 1), (2 \mapsto 2)\}$, $f = \{(2 \mapsto 2), (3 \mapsto 3)\}$. Both e and f are idempotents. Observe: $ef = \{(2 \mapsto 2)\} \neq 0$. However, the principal left ideals generated by e and f are incomparable, so e and f are adjacent in the non-inclusion principal ideal graph $nPiG(S)$. This shows that adjacency does not imply $ef = 0$.

4. NON-INCLUSION PRINCIPAL IDEAL GRAPHS OF SYMMETRIC INVERSE SEMIGROUP

We begin with providing the adjacency conditions in $nPiG_l(S)$ when S is a symmetric inverse semigroup.

Theorem 4.1. *Let $S = \mathcal{IS}_n$ be the symmetric inverse semigroup on the finite set $X = \{1, 2, \dots, n\}$. Two distinct vertices x and y are adjacent in $nPiG_l(S)$ if and only if $img(x) \setminus img(y) \neq \emptyset$ and $img(y) \setminus img(x) \neq \emptyset$.*

Proof. For $x, y \in S$, $x \mathcal{L} y \iff img(x) = img(y)$. The principal left ideal generated by S is $Sx = \{\alpha \in \mathcal{IS}_n : img(\alpha) \subseteq img(x)\}$. So, inclusion of principal left ideals corresponds precisely to image inclusion:

$$x \notin Sy \iff img(x) \not\subseteq img(y).$$

Also

$$y \notin Sx \iff img(y) \not\subseteq img(x).$$

Thus x and y are adjacent if and only if $img(x) \not\subseteq img(y)$ and $img(y) \not\subseteq img(x)$, which is equivalent to $img(y) \setminus img(x) \neq \emptyset$ and $img(x) \setminus img(y) \neq \emptyset$. $\square$

Theorem 4.2. *Let $\mathcal{IS}_n$ be the symmetric inverse semigroup on a finite set X with $|X| = n$.*

- (1) *The elements x and y with different image sets belonging to a $\mathcal{D}$ -class with rank k , $0 < k < n$ are adjacent in the non-inclusion principal left ideal graph of $\mathcal{IS}_n$*
- (2) *Elements with image sizes 0 and n are isolated vertices in the non-inclusion principal left ideal graph.*

Proof. (1) Let x, y be elements of $\mathcal{IS}_n$ with different image sets of a $\mathcal{D}$ -class, i.e., they are of same rank k , $0 < k < n$. Then neither of $img(x)$ or $img(y)$ can be a subset of the other, as they are of equal size and distinct. Hence, x and y are adjacent in $nPiG(\mathcal{IS}_n)$.

(2) The unique element with image size 0 is the zero of $\mathcal{IS}_n$. Its principal left ideal is minimal and contained in every other principal left ideal, so it cannot be adjacent to any other vertex. For domain size n , elements are partial bijections with full range. Their principal ideals are maximal, containing all others. Hence, these elements cannot be adjacent to any other vertex where inclusion holds. So they are isolated. $\square$

Corollary 4.3. *In $nPiG(E)$, the induced subgraph with the set of idempotents E , of the ubgraph induced by each $\mathcal{D}$ -class with rank k , $0 < k < n$ is a complete graph with $\binom{n}{k}$ vertices.*

Theorem 4.4. *The clique number of $nPiG(\mathcal{IS}_n)$ is*

$$\begin{cases} \binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even} \\ \binom{n}{\frac{n+1}{2}}, & \text{if } n \text{ is odd} \end{cases}$$

Proof. By corollary 4.3, the elements corresponding to the idempotents of each $\mathcal{D}$ -class with rank k , $0 < k < n$ form a complete graph $K_{\binom{n}{k}}$. Also, let x be a vertex of rank $i < k$. Then $img(x) \subseteq img(y)$ for some y in the $\mathcal{D}$ -class with rank

k . Similarly, if x be a vertex of rank $i > k$. Then $\text{img}(y) \subseteq \text{img}(x)$ for some y in the $\mathcal{D}$ -class with rank k . Therefore, any vertex outside of a particular $\mathcal{D}$ -class cannot be adjacent to all of its vertices. Hence, the largest clique is K_r where

$$r = \begin{cases} \binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even} \\ \binom{n}{\frac{n+1}{2}}, & \text{if } n \text{ is odd} \end{cases}$$

□

Remark 4.5. The results presented for $\mathcal{IS}_n$ admit a natural combinatorial interpretation. Elements of $\mathcal{IS}_n$ are characterized by their image sets, and adjacency in $n\text{PiG}(\mathcal{IS}_n)$ corresponds to incomparability under set inclusion. Consequently, induced subgraphs consisting of elements of a fixed rank correspond to incomparability graphs of specific levels of the Boolean lattice. From this perspective, the clique number computation in Theorem 4.4 is a direct application of classical results on antichains in the power set lattice, most notably Sperner's theorem [5].

Theorem 4.6. *In $n\text{PiG}(\mathcal{IS}_n)$ the degree of vertices with indices having k elements is $\sum_{i=1}^{n-1} d_i$ where*

$$d_i = \begin{cases} \binom{n}{i} i! \left[\binom{n}{i} - \binom{k}{i} \right], & \text{if } i < k, \\ \binom{n}{i} i! \left[\binom{n}{i} - \binom{n-k}{i-k} \right], & \text{if } i \geq k. \end{cases}$$

Proof. Let x be a vertex of $n\text{PiG}(\mathcal{IS}_n)$ with rank k . Let d_i be the number of vertices with rank i adjacent to x . For a particular image set with i elements there are $\binom{n}{i} i!$ vertices. There are $\binom{n}{i}$ image set in a $\mathcal{D}$ -class with image size i . For $i < k$, the number of sets which is contained in k -index set is $\binom{k}{i}$. Hence,

$$d_i = \binom{n}{i} i! \left[\binom{n}{i} - \binom{k}{i} \right]$$

For $i \geq k$, the number of image sets whose image set contains the k -index set is $\binom{n-k}{i-k}$. Then,

$$d_i = \binom{n}{i} i! \left[\binom{n}{i} - \binom{n-k}{i-k} \right]$$

Therefore, the degree of vertices with indices having k elements is $\sum_{i=1}^{n-1} d_i$ □

Corollary 4.7. *In $n\text{PiG}(E)$, the induced subgraph with the set of E , the degree of vertices with indices having k elements is $\sum_{i=1}^{n-1} d_i$ where*

$$d_i = \begin{cases} \binom{n}{i} - \binom{k}{i}, & \text{if } i < k, \\ \binom{n}{i} - \binom{n-k}{i-k}, & \text{if } i \geq k. \end{cases}$$

5. BRANDT SEMIGROUP

Throughout this section $S = \mathcal{B}(\mathbb{G}, \mathbb{I}) = (\mathbb{I} \times \mathbb{G} \times \mathbb{I}) \cup \{0\}$ denotes a Brandt semigroup with $|\mathbb{G}| = n$ and $|\mathbb{I}| = \nu$. We first provide a necessary and sufficient condition for two elements in S to be adjacent in $nPiG_l(S)$.

Theorem 5.1. *Two distinct nonzero elements $a = (i, g, j)$, $b = (k, h, l) \in S$ are adjacent in $nPiG_l(S)$ if and only if $j \neq l$.*

Proof. Suppose a and b are adjacent in $nPiG_l(S)$. Then $a \notin Sb$ and $b \notin Sa$. Recall that $Sa = \{(m, x, j) \mid m \in \mathbb{I}, x \in \mathbb{G}\} \cup \{0\}$ and $Sb = \{(n, y, l) \mid n \in \mathbb{I}, y \in \mathbb{G}\} \cup \{0\}$. If $j = l$, then $a = (i, g, j) \in Sb$ and $b = (k, h, j) \in Sa$, which contradicts the adjacency assumption. Hence, $j \neq l$ must hold. Conversely, if $j \neq l$ then $a \notin Sb$ and $b \notin Sa$, as the principal left ideals consist of all elements with fixed right index. Hence a and b are adjacent in $nPiG_l(S)$. $\square$

In a similar fashion we see that two non-zero elements in S are adjacent in $nPiG_r(S)$ if and only if their first components differ, which is stated as follows.

Theorem 5.2. *Two distinct nonzero elements $a = (i, g, j)$, $b = (k, h, l) \in S$ are adjacent in $nPiG_r(S)$ if and only if $i \neq k$.*

The structure of $nPiG_l(S)$ is described the following theorem.

Theorem 5.3. *For an inverse semigroup S , $nPiG_l(S)$ is a complete ν -partite graph and each partition size is $n\nu$.*

Proof. The vertex set of $nPiG_l(S)$ is $S \setminus \{0\} = \mathbb{I} \times \mathbb{G} \times \mathbb{I}$. From Theorem 5.1, two vertices $a = (i, g, j)$, $b = (k, h, l)$ are adjacent if and only if $j \neq l$. The vertex of $\mathcal{L}$ -class containing a , $\mathcal{L}_a = \{(i, g, j) : i \in \mathbb{I}, g \in \mathbb{G}\} = V_j$ is an independent set with $n\nu$ vertices and the subgraph induced by V_j is a null graph. Also, if $b \notin \mathcal{L}_a$ then a and b are adjacent in $nPiG_l(S)$. Thus, all vertices in V_j are adjacent to all vertices in V_l whenever $j \neq l$. Combining these facts, the graph is a complete ν -partite graph with parts V_j for each $j \in \mathbb{I}$. $\square$

From the above discussions, we see that adjacency in $nPiG_l(S)$ depends on the principal left ideals which fix the third coordinates and $nPiG_r(S)$ depends on principal right ideals, which fix the first coordinates. As the roles of the first and third coordinates in defining principal right and left ideals are symmetric in the structure of S , the resulting graphs $nPiG_l(S)$ and $nPiG_r(S)$ share the same structural form. Due to this common framework, we use the notation $nPiG(S)$ to describe the structure of the graph without explicitly distinguishing the left and the right graphs wherever necessary.

Theorem 5.4. *Let Γ be a complete ν -partite graph whose vertex set is partitioned into ν parts, each of size k , where k is a multiple of ν . Then there exists a Brandt semigroup $S = \mathcal{B}(\mathbb{G}, \mathbb{I})$ with $|\mathbb{I}| = \nu$ and $|\mathbb{G}| = n = \frac{k}{\nu}$ such that the non-inclusion principal ideal graph $nPiG(S)$ is isomorphic to Γ .*

Proof. Let the vertex set of Γ be $V(\Gamma) = V_1 \cup V_2 \cup \dots \cup V_\nu$, with each $|V_i| = k$. Since $k = n\nu$ for some $n \in \mathbb{N}$, consider a set $\mathbb{I}$ with $|\mathbb{I}| = \nu$ and a group $\mathbb{G}$ with $|\mathbb{G}| = n$. Then in the Brandt semigroup S , every nonzero element is of the form (i, g, j) with

$i, j \in \mathbb{I}$ and $g \in \mathbb{G}$. The graph $nPiG(S)$ has vertex set $\mathbb{I} \times \mathbb{G} \times \mathbb{I}$ and adjacency given by: $(i, g, j) \sim (k, h, l) \iff j \neq l$ (or $i \neq k$ for the right ideal graph case). The vertex set naturally partitions into ν parts indexed by the third coordinate j , each of size $|\mathbb{I}||\mathbb{G}| = \nu n = k$. The edges form a complete ν -partite structure matching that of Γ . Thus, $nPiG(S) \cong \Gamma$. □

Theorem 5.5. *The graph $nPiG_l(S)$ has*

- (1) $\nu^2 n$ vertices partitioned into ν parts: $V_j = \{(i, g, j) : i \in I, g \in G\}$, each of size $|V_j| = \nu n$.
- (2) The degree of each vertex v is

$$\deg(v) = \nu n(\nu - 1).$$

- (3) The total number of edges is

$$|E| = \frac{\nu^3(\nu - 1)n^2}{2}.$$

Proof. (1) Since $nPiG_l(S)$ is a complete ν -partite graph with parts V_j of size νn , the total number of vertices is $|V| = \sum_{j=1}^{\nu} |V_j| = \nu \times (\nu n) = \nu^2 n$.

- (2) Each vertex in V_j is adjacent to every vertex not in V_j , so its degree is $\deg(v) = |V| - |V_j| = \nu^2 n - \nu n = \nu n(\nu - 1)$.

- (3) For the total number of edges, note that edges only connect vertices between different parts. The number of edges between two distinct parts V_j and V_l is $|V_j| \times |V_l| = (\nu n) \times (\nu n) = \nu^2 n^2$.

There are $\binom{\nu}{2} = \frac{\nu(\nu-1)}{2}$ such unordered pairs of parts, so the total number of edges is

$$|E| = \binom{\nu}{2} \times \nu^2 n^2 = \frac{\nu(\nu - 1)}{2} \cdot \nu^2 n^2 = \frac{\nu^3(\nu - 1)n^2}{2}.$$

□

5.1. Spectrum of non-inclusion ideal graphs of Brandt semigroups. Let $S = \mathcal{B}(\mathbb{G}, \mathbb{I}) = (\mathbb{I} \times \mathbb{G} \times \mathbb{I}) \cup \{0\}$ denotes a Brandt semigroup with $|\mathbb{G}| = n$ and $|\mathbb{I}| = \nu$. By Theorem 5.3, the non-inclusion principal ideal graph of a Brandt semigroup is a complete ν -partite graph, with each partition containing νn vertices. This multipartite structure allows the adjacency spectrum of $nPiG(S)$ to be determined directly using known results for complete multipartite graphs. Following the approaches in [1, 4, 13], we can compute various graph energies associated with $nPiG(S)$. We begin with the adjacency energy, $\epsilon_{\mathcal{A}}(nPiG(S))$.

Theorem 5.6. *The adjacency energy of $nPiG(S)$ is*

$$\epsilon_{\mathcal{A}}(nPiG(S)) = 2n\nu(\nu - 1).$$

Proof. The graph $nPiG(S)$ is a complete ν -partite graph with each part of size $n\nu$. Its adjacency spectrum is

$$\text{Spec}(nPiG(S)) = \{ n\nu(\nu - 1)^{[1]}, (-n\nu)^{[\nu-1]}, 0^{[\nu(\nu-1)]} \},$$

where the superscripts denote the multiplicities of the corresponding eigenvalues. Hence, $\epsilon_{\mathcal{A}}(nPiG(S)) = 2n\nu(\nu - 1)$. □

Theorem 5.7. *The Laplacian energy of $nPiG(S)$ is*

$$\epsilon_{\mathcal{L}}(nPiG(S)) = n^2\nu^3(\nu - 1).$$

Proof. Since the graph $nPiG(S)$ is a complete ν -partite graph with each part of size $n\nu$, its Laplacian spectrum is

$$\{0^{[1]}, \quad n\nu(\nu - 1)^{[\nu(n\nu-1)]}, \quad (n\nu^2)^{[\nu-1]}\}.$$

Thus the Laplacian energy is $\epsilon_{\mathcal{L}}(nPiG(S)) = n^2\nu^3(\nu - 1)$. $\square$

Even though the Laplacian and signless Laplacian spectra of $nPiG(S)$ are different, the corresponding energies coincide for $nPiG(S)$.

Theorem 5.8. *The signless Laplacian energy of $nPiG(S)$ is*

$$\epsilon_{\mathcal{Q}}(nPiG(S)) = n^2\nu^3(\nu - 1).$$

Proof. Since the graph $nPiG(S)$ is a complete ν -partite graph with each part of size $n\nu$, its signless Laplacian spectrum is

$$\{2n\nu(\nu - 1)^{[1]}, \quad n\nu(\nu - 2)^{[\nu-1]}, \quad n\nu(\nu - 1)^{[(\nu-1)\nu]}\}.$$

Thus the signless Laplacian energy is $\epsilon_{\mathcal{Q}}(nPiG(S)) = n^2\nu^3(\nu - 1)$. $\square$

Finally we compute the distance energy of $nPiG(S)$.

Theorem 5.9. *The distance energy of $nPiG(S)$ is*

$$\epsilon_{\mathbb{D}}(nPiG(S)) = 3n\nu^2 + 2n\nu - 2\nu - 4.$$

Proof. The $\mathbb{D}$ -eigenvalues of $\mathbb{D}(nPiG(S))$ are

$$\{n\nu(\nu + 1) - 2^{[1]}, \quad n\nu - 2^{[1]}, \quad (-2)^{[\nu(n\nu-1)]}\}.$$

Thus, the distance energy of $nPiG(S)$ is

$$\epsilon_{\mathbb{D}}(nPiG(S)) = 3n\nu^2 + 2n\nu - 2\nu - 4.$$

$\square$

The subsequent corollary follows directly from the preceding analysis.

Corollary 5.10. *Let $S = \mathcal{B}(\mathbb{G}, \mathbb{I}) = (\mathbb{I} \times \mathbb{G} \times \mathbb{I}) \cup \{0\}$ be a Brandt semigroup with $|\mathbb{G}| = n$ and $|\mathbb{I}| = \nu$. Then the following hold for the non-inclusion principal ideal graph $nPiG(S)$:*

- *Let $\rho(nPiG(S))$ denote the largest adjacency eigenvalue of $nPiG(S)$. Then*

$$\rho(nPiG(S)) = n\nu(\nu - 1) \geq 0.$$

- *Let $\mu(nPiG(S))$ denote the largest Laplacian eigenvalue of $nPiG(S)$. Then*

$$\mu(nPiG(S)) = n\nu^2 \geq 1.$$

- *Let $q(nPiG(S))$ denote the largest signless Laplacian eigenvalue of $nPiG(S)$. Then*

$$q(nPiG(S)) = 2n\nu(\nu - 1) \geq 0.$$

- *Let $d(nPiG(S))$ denote the largest distance eigenvalue of $nPiG(S)$. Then*

$$d(nPiG(S)) = n\nu(\nu + 1) - 2 \geq 0.$$

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