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EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE OF THE SOLUTION TO A HYPERBOLIC EQUATION WITH AN UNKNOWN COEFFICIENT

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Abstract. In this paper, we study the existence, uniqueness, and continuous dependence of the solution for a hyperbolic equation arising in the framework of inverse problems. To achieve this, we first prove the existence and uniqueness of the solution to the associated direct problem using the energy inequality method. Then, by means of fixed point theory, we establish the existence and uniqueness of the solution to the inverse problem.

Keywords. Inverse nonlinear problem, nonlocal integral condition, Fixed point theorem, Nonlinear hyperbolic equation.
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1. Introduction

Recent years have witnessed remarkable progress in the qualitative and numerical analysis of differential equations arising in heat transfer, diffusion, and reaction–diffusion phenomena. In particular, new approximation techniques and transformation-based methods have been developed for fuzzy and nonlinear models [1, 2], while fractional, conformable, and quantum frameworks have significantly broadened the modeling capabilities of evolution equations [3, 4, 5]. These developments have led to important theoretical results concerning existence, stability, and global solvability of reaction–diffusion and heat-type systems [6, 7], as well as their discrete and controlled counterparts [8, 9]. Moreover, advanced numerical schemes, including q-differential approaches and fractional-order reaction–diffusion models, have been successfully applied in various contexts such as biomedical imaging and complex dynamical simulations [10, 11]. Collectively, these contributions highlight the growing importance of modern analytical and computational techniques for studying nonlinear evolution equations and motivate further investigation of inverse problems governed by partial differential equations.

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Since the pioneering works of Euler, d'Alembert, and Laplace in the 18th century, partial differential equations (PDEs) have become a central tool for modeling a wide range of physical and engineering phenomena. Among these, parabolic equations play a prominent role, particularly in connection with diffusion processes. In 1807, Joseph Fourier [12, 13] focused on a fundamental physical phenomenon: the propagation of heat in a metal rod. His objective was clear: to predict how the temperature would evolve over time, given the initial temperature distribution and the boundary conditions. This modeling led to one of the foundational equations in mathematical physics, the heat equation.

The challenge posed by Fourier, now known as the direct problem, was to compute the future temperature of a body from its initial thermal state. However, another question soon emerged, particularly relevant in experimental and industrial contexts: "What if instead, we only had access to the current temperature of a body, could we then infer its initial thermal state?" This seemingly simple question gave rise to what is now called an inverse problem; instead of propagating known data forward in time, one seeks to recover past states from later observations. A particularly striking case of such a problem was analyzed rigorously for the first time by Jacques Hadamard [14] in the early 20th century. He considered the backward heat conduction problem, in which one aims to determine the initial temperature distribution $u(x, 0)$ from a known final state $u(x, T)$, governed by the classical heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \varpi \frac{\partial^2 u}{\partial x^2}, \quad \text{in } (0, L) \times (0, T), \\ u(x, T) &= f(x), \\ u(0, t) &= u(L, t) = 0. \end{aligned}$$

This backward Cauchy problem stands as one of the earliest examples that exposed the necessity for a dedicated mathematical framework for addressing inverse problems. Since the foundational work of Hadamard, who formalized the concept of ill-posedness in the early 20th century, to the development of regularization methods by A. N. Tikhonov [15] in the 1960s, parabolic inverse problems have continued to draw significant attention—both for the theoretical challenges they present and for their wide range of applications.

Over the years, inverse problems involving parabolic equations have drawn considerable research interest. In realistic settings, the available data often come not from classical boundary or initial conditions, but from integral constraints. Far from being auxiliary, these constraints capture essential features of the underlying phenomena and arise naturally in applications such as population dynamics, chemical engineering, thermoelasticity, groundwater flow, and plasma physics (see, e.g., [16, 17, 18, 19, 20, 21]).

Considerable research has been devoted to analyzing whether inverse problems for parabolic equations admit unique solutions, notably when integral overdetermination conditions are imposed. Several researchers have investigated these questions under various assumptions on the data, the geometry of the domain, and the nature of the integral condition. A notable contribution in this direction comes from N. V. Muzyliov [22, 23, 24], who addressed uniqueness results for quasilinear heat equations with unknown thermal conductivity and heat capacity. His work established uniqueness under specific structural assumptions and regularity conditions on the nonlinearities.

In a related direction, I. A. Vasin and V. L. Kamynin [25] analyzed the asymptotic behavior of solutions to inverse parabolic problems with irregular coefficients. Although focused on long-time dynamics, their results are built upon foundational well-posedness properties, particularly in low-regularity regimes. Other authors have examined integral overdetermination conditions directly. Cannon and Lin [26, 27] were among the first to study this setting, followed by Puel and Yamamoto [28, 29], who considered semilinear equations and proved existence, uniqueness, and stability using variational techniques. These works showed that integral constraints are not only natural in practical contexts, but also mathematically advantageous.

Further developments have emerged from the work of Tuan and Lesnic [30, 31], who employed Carleman estimates and functional analytic tools to tackle problems involving integral or final-time data. These techniques are especially effective in counteracting the ill-posedness inherent to inverse parabolic problems. Although most of the literature focuses on parabolic models, similar methods have also been applied to hyperbolic equations, as demonstrated by T. E. Oussaeif and B. Abdelfatah [32], who addressed an inverse hyperbolic problem with integral data and obtained well-posedness results through analytical techniques comparable to the parabolic case. Recent work has addressed related inverse problems for parabolic systems and coefficient identification approaches [33, 34], underscoring the relevance of this study. Collectively, these contributions form a comprehensive body of work that supports both the theoretical aspects and numerical methods used to address inverse problems described by partial differential equations.

The present paper addresses the study of an inverse problem arising from a parabolic equation in the presence of an integral overdetermination condition

$$u_{tt} - a\Delta u + bu + |u|^3 = \sigma(t)h(x,t) \quad (x,t) \in \Omega \times (0,T) \tag{1.1}$$

subject to specified initial conditions

$$u(x,0) = \varphi_1(x) \quad x \in \Omega \tag{1.2}$$

$$u_t(x,0) = \varphi_2(x) \quad x \in \Omega \tag{1.3}$$

The boundary condition

$$u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T) \tag{1.4}$$

together with a nonlocal overdetermination condition

$$\int_\Omega v(x)u(x,t)\,dx = E(t) \quad t \in (0,T). \tag{1.5}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and the functions h, φ_1, φ_2, and E are known functions, while a and b are positive constants given in advance.

2. Preliminaries

Throughout what follows, the space-time domain is denoted by

$$Q = \Omega \times (0, T).$$

In the following, we present the notations and principles that will be adopted in our analysis

$$g^*(t) = \int_\Omega v(x)h(x,t)\,dx.$$

In addition, we employ the widely known Cauchy inequality:

$$2|ab| \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad a, b \in \mathbb{R}.$$

Theorem 2.1 (Sobolev embedding theorem)**.** *Let*

$$\begin{cases} 2 \le q < \infty & \text{for } n = 1, 2, \\ 2 \le q \le \dfrac{2n}{n-2} & \text{for } n \ge 3. \end{cases}$$

Then there exists a constant $c^* = c(q, \Omega)$ such that

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega).$$

For Sobolev spaces $H_0^1(\Omega)$, we use the notation

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}.$$

Let $L^p(0, T; X)$ denote the space of all measurable functions $f : (0, T) \longrightarrow X$ such that the corresponding norm is finite:

$$\begin{cases} \left(\displaystyle\int_0^T \|f(t)\|_X^p\,dt \right)^{\dfrac{1}{p}} & \text{if} \quad 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t \in (0,T)} \|f(t)\|_X < \infty & \text{if} \quad p = \infty. \end{cases} \tag{2.1}$$

Lemma 2.2. *Let $a, b : [0, T] \longrightarrow \mathbb{R}$ be nonnegative differentiable functions for which there exists a positive function $g \in L^1(0, T)$ such that*

$$a(t) \le b(t) + \int_0^t g(s)u(s)\,ds, \quad t \in (0, T).$$

Then, for b non-decreasing we have

$$u(t) \le b(t) \exp(\psi(t)),$$

where

$$\psi(t) = \int_0^t g(s)\,ds, \quad t \in (0, T).$$

3. Existence and Uniqueness Results for the Direct Problem

To analyze the nonlinear problem (1.1)–(1.4), we adopt the energy inequality method, which allows us to prove the existence and uniqueness of a strong solution. The argument is built upon an energy inequality together with the density property of the operator's range, both of which are derived from the problem's detailed examination and the application of techniques that guarantee the solution is unique.

3.0.1. *Formulation Of The Direct Problem.* In the domain $Q = \Omega \times (0, T)$, with $Q = \Omega$ a subset of $\mathbb{R}^n$ and $T > 0$. We study the following nonlinear hyperbolic problem subject to Dirichlet boundary conditions

$$(P) \qquad \begin{cases} u_{tt} - a\Delta u + bu + |u|^3 = f(x,t) & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi_1 & x \in \Omega, \\ u_t(x,0) = \varphi_2 & x \in \Omega. \end{cases} \tag{3.1}$$

where

$$(A_1) \qquad \begin{cases} f \in L^2(Q), \\ \varphi_1 \in H_0^1(\Omega) \cap L^\rho(\Omega) \quad \text{for} \quad \rho = p+2, \\ \varphi_2 \in L^2(\Omega), \end{cases}$$

and a, b are two positive constants. We set

$$\mathcal{L}u = u_{tt} - a\Delta u + bu + |u|^3 = f(x,t), \tag{3.2}$$

where the initial conditions are specified as

$$l_1 u = u(x,0) = \varphi_1 \quad x \in \Omega, \tag{3.3}$$

$$l_2 u = u_t(x,0) = \varphi_2 \quad x \in \Omega. \tag{3.4}$$

The Dirichlet boundary condition is expressed as

$$u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T). \tag{3.5}$$

Let us define the operator $L = (\mathcal{L}, l_1, l_2)$ on

$$D(L) := \left\{ u \in L^2(Q) \text{ such that } u_t, \nabla u, u \in L^2(Q),\ u \in L^4(Q) \right\}$$

where u satisfies the initial condition (3.3) and (3.4). We consider the operator L as a mapping from the Banach space E into F, where E consists of functions $u(x,t)$ with finite norms,

$$\|u\|_E^2 = \|u_t\|_{L^2(Q)}^2 + \|\nabla u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 + \|u\|_{L^4(Q)}^4$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, \varphi_1, \varphi_2)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 + \|\varphi_2\|_{L^2(\Omega)}^2 + \|\nabla\varphi_1\|_{L^2(\Omega)}^2 + \|\varphi_1\|_{L^2(\Omega)}^2 + \|\varphi_1\|_{L^4(\Omega)}^4.$$

is finite. We begin by establishing a priori estimates for the solution to problem (3.2)–(3.5).

3.0.2. *A Priori Estimate.*

Theorem 3.1. *Assume that condition A_1 holds. For every function $u \in D(L)$ there exists a positive constant $C(u)$ such that*

$$\|u\|_E \leq C\|\mathcal{F}\|_F.$$

Proof. We multiply equation (3.2) by the operator $Mu = u_t$ and integrate it over $Q = \Omega \times (0, T)$, we get

$$\begin{aligned}\int_\Omega \int_0^\tau \frac{1}{2}\frac{d}{dt}|u_t|^2\,dt\,dx + \frac{a}{2}\int_\Omega \int_0^\tau \frac{d}{dt}|\nabla u|^2\,dt\,dx + \frac{b}{2}\int_\Omega \int_0^\tau \frac{d}{dt}|u|^2\,dt\,dx \\ +\frac{1}{4}\int_\Omega \int_0^\tau \frac{d}{dt}|u|^4\,dt\,dx = \int_\Omega \int_0^\tau u_t f(x,t)\,dt\,dx. \end{aligned} \tag{3.6}$$

Using Cauchy's inequality, we find

$$\begin{aligned}&\frac{1}{2}\|u_t(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{a}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{b}{2}\|u(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{1}{4}\|u(\cdot,\tau)\|^4_{L^4(\Omega)} \\ &\leq \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\|u_t\|^2_{L^2(Q)} + \frac{1}{2}\|\varphi_2\|^2_{L^2(\Omega)} + \frac{a}{2}\|\nabla\varphi_1\|^2_{L^2(\Omega)} \\ &\quad + \frac{b}{2}\|\varphi_1\|^2_{L^2(\Omega)} + \frac{1}{4}\|\varphi_1\|^4_{L^4(\Omega)}.\end{aligned} \tag{3.7}$$

Integrating (3.7) over $(0, T)$, we get

$$\begin{aligned}&\|u_t\|^2_{L^2(Q)} + \|\nabla u\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^4_{L^4(Q)} \\ &\leq C\Big(\|f\|^2_{L^2(Q)} + \|\varphi_2\|^2_{L^2(\Omega)} + \|\nabla\varphi_1\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{L^2(\Omega)} + \|\varphi_1\|^4_{L^4(\Omega)}\Big),\end{aligned} \tag{3.8}$$

where

$$C = \max\left\{\frac{T}{2\varepsilon}, \frac{aT}{2}, \frac{bT}{2}, \frac{T}{4}, \frac{1}{1-\varepsilon T}, \frac{a}{2}, \frac{b}{2}, \frac{1}{4}\right\}, \qquad (\varepsilon < \tfrac{1}{T}).$$

So, we have

$$\|u\|_E \leq C\|Lu\|_F. \tag{3.9}$$

□

Proposition 3.2. *The operator $L : E \longrightarrow F$ possesses a closure.*

Proof. Consider a sequence $(u_n)_{n\in\mathbb{N}} \subset D(L)$ for which the following conditions hold

$$\begin{aligned} u_n &\longrightarrow 0 \quad \text{in } E \\ Lu_n &\longrightarrow (f, \varphi_1, \varphi_2) \quad \text{in } F \end{aligned} \tag{3.10}$$

Our objective is to prove that

$$f \equiv 0, \quad \varphi_1 \equiv 0 \quad \text{and} \quad \varphi_2 \equiv 0 \quad \text{in } F.$$

The convergence of u_n to 0 in E implies

$$u_n \longrightarrow 0 \quad \text{in } D'(Q) \tag{3.11}$$

By the continuity of the derivative in $D'(Q)$, we conclude that

$$\mathcal{L}u_n \longrightarrow 0 \quad \text{in } D'(Q) \tag{3.12}$$

In addition, the convergence of the sequence Lu_n to f in $L^2(Q)$ leads to

$$\mathcal{L}u_n \longrightarrow f \quad \text{in } D'(Q) \tag{3.13}$$

Since the limit in $D'(Q)$ is unique, we infer from equations (3.12) and (3.13) that

$$f \equiv 0.$$

Subsequently, it is deduced from equation (3.10) that

$$l_1 u_n \longrightarrow \varphi_1, \quad l_2 u_n \longrightarrow \varphi_2 \quad \text{in } L^2(\Omega).$$

On the other hand, we have

$$\|u_n\|_E = \|u_{nt}\|^2_{L^2(Q)} + \|\nabla u_n\|^2_{L^2(Q)} + \|u_n\|^2_{L^2(Q)} + \|u_n\|^4_{L^4(Q)}, \quad \forall t \in (0,T).$$

which implies that

$$\|u_n\|_E \geq \|u_{nt}\|^2_{L^2(Q)}, \quad \|u_n\|_E \geq \|u_n\|^2_{L^2(Q)}, \quad \forall t \in (0,T).$$

So, we have

$$\|u_n\|_E \geq \|\varphi_2\|^2_{L^2(\Omega)}, \quad \|u_n\|_E \geq \|\varphi_1\|^2_{L^2(\Omega)}.$$

Since

$$u_n \longrightarrow 0 \quad \text{in } E,$$

we get

$$\|u_n\|^2_E \longrightarrow 0 \quad \text{in } \mathbb{R}.$$

Hence, we have

$$0 \geq \|\varphi_2\|^2_{L^2(\Omega)}, \quad 0 \geq \|\varphi_1\|^2_{L^2(\Omega)}.$$

So, we infer that

$$\varphi_2 \equiv \varphi_1 \equiv 0.$$

□

Let L be the closure of the given operator, with D as its domain.

Definition 3.3. A function u is said to be a strong solution of the operator equation

$$\overline{L}u = F$$

within the framework of problems (3.2)–(3.5) if it satisfies the equation. Moreover, we can extend the a priori estimate to strong solutions, that is, we get the estimate

$$\|u\|_E \leq C\|\overline{L}u\|_F, \qquad \forall u \in D(\overline{L}) \tag{3.14}$$

Corollary 3.4. *Under hypothesis (H'), with*

$$\rho \leq \frac{2k}{n-2} \qquad (\textit{any finite } \rho \textit{ if } n = 2) \qquad k \in \mathbb{N}^*. \tag{3.15}$$

Then the solution of the problem (3.2)–(3.5) is unique if it exists.

Proof. Let u_1, u_2 be two solutions of the problem (3.2)–(3.5). We put $w = u_1 - u_2$. Then w satisfies

$$(P') \qquad \begin{cases} w_{tt} - a\Delta w + |u_1|^3 - |u_2|^3 + bw = 0 & (x,t) \in \Omega \times (0,T), \\ w(x,0) = 0 & x \in \Omega, \\ w_t(x,0) = 0 & x \in \Omega, \\ w(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{3.16}$$

Multiply the equation of (3.16) by w_t and integrate it into Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |w_t|^2\,dx + \frac{a}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2\,dx + \frac{b}{2}\frac{d}{dt}\int_\Omega |w|^2\,dx \\ &\qquad + \int_\Omega \left(|u_1|^3 - |u_2|^3\right)\left(u_1 - u_2\right)_t dx = 0. \end{aligned} \tag{3.17}$$

Integrating (3.17) over $(0,t)$, we find

$$E(t) - E(0) = -\int_0^t \int_\Omega \left(|u_1|^3 - |u_1|^3\right)\left(u_1 - u_2\right)_t dx\,ds. \tag{3.18}$$

The nonlinear term $f(u) = |u|^3$ is monotone in $\Omega \times (0,T)$. Consequently, the integral

$$-\int_0^t \int_\Omega \left(|u_1|^3 - |u_2|^3\right)\left(u_1 - u_2\right)_t dx\,ds \le 0.$$

As $w(x,0) = w_t(x,0) = 0$, then $E(0) = 0$. That allows us to deduce $u_1 = u_2$. $\square$

Corollary 3.5. *The closed operator $\overline{L}$ has a range $R(\overline{L})$ that is closed in F. Furthermore, it coincides with the closure of the range of L*

$$R(\overline{L}) = \overline{R(L)}.$$

Proof. Assume $z \in \overline{R(L)}$. There exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F consisting of elements from the set $R(L)$ such that

$$\lim_{n\to+\infty} z_n = z.$$

Likewise, because $(z_n)_{n\in\mathbb{N}}$ belongs to $R(L)$, there exists a corresponding sequence $(u_n)_{n\in\mathbb{N}}$ in $D(L)$ such that

$$Lu_n = z_n.$$

Let u_m and $u_{m'}$ be two solutions of (3.1), i.e.,

$$Lu_m = f \quad \text{and} \quad Lu_{m'} = f.$$

Putting $w = u_m - u_{m'}$, then by Corollary 3.4 we get

$$w = 0.$$

which implies that for all $t \in (0,T)$ we have

$$0 \le \|u_m(t) - u_{m'}(t)\|_E \le 0. \tag{3.19}$$

i.e.,

$$\begin{aligned} &\lim_{m,m'\to+\infty} \|u_m(t) - u_{m'}(t)\|_E = 0 \\ &\iff \forall \varepsilon \ge 0,\ \exists n_0 \in \mathbb{N}\ \forall m, m' \ge n_0 :\ \|u_m(t) - u_{m'}(t)\|_E \le \varepsilon. \end{aligned}$$

It follows that $(u_n)_{n\in\mathbb{N}}$ constitutes a Cauchy sequence in E. As E is complete, there exists an element $u \in E$ such that

$$\lim_{n\to+\infty} u_n = u.$$

By the closure property of $\overline{L}$, taking $\lim_{n\to+\infty} u_n = u$ in E and $\lim_{n\to+\infty} Lu_n = \lim_{n\to+\infty} z_n = z$ in F, the closedness of $\overline{L}$ ensures that $\overline{L}u = z$. Accordingly, u satisfies

$$u \in D(\overline{L}), \quad \overline{L}u = z.$$

Then $z \in R(\overline{L})$, so we obtain

$$\overline{R(L)} \subset R(\overline{L}).$$

Furthermore, the fact that $R(\overline{L})$ is a Banach space implies that it is a closed subset of F. It is still necessary to demonstrate the opposite inclusion. Let $z \in R(\overline{L})$. Then there exists a Cauchy sequence $(z_n)_n$ in F consisting of elements of the set $R(\overline{L})$ such that

$$\lim_{n\to+\infty} z_n = z.$$

Since $R(\overline{L})$ is a closed subset of a complete space F, it follows that $R(\overline{L})$ is complete. Therefore, there exists a corresponding sequence $(u_n)_n \subset D(\overline{L})$ such that

$$\overline{L}u_n = z_n.$$

On the other hand, the sequence $(u_n)_n$ constitutes a Cauchy sequence in F, which can be established by applying the same technique employed in the previous step. Consequently, a sequence $(L(u_n))_n \subset R(L)$ exists such that

$$Lu_n = \overline{L}u_n, \quad \forall n \in \mathbb{N}.$$

So, we get

$$\lim_{n\to+\infty} Lu_n = z.$$

As a result $z \in \overline{R(L)}$. Then, we conclude that

$$R(\overline{L}) \subset \overline{R(L)}.$$

□

3.0.3. *Existence Of Solution.* To guarantee the existence of a solution, one must show that $R(L)$, the range of the operator L, is dense in F for every $u \in E$ and for all $\mathcal{F} = (f, \varphi_1, \varphi_2) \in F$.

Proposition 3.6. *Suppose that for every $w \in L^2(Q)$ and for all $u \in D_0(L)$, the following relation holds:*

$$\int_Q (\mathcal{L}u)\, w\, dx\, dt = 0. \tag{3.20}$$

Then $w = 0$ almost everywhere in Q.

Proof. We define the inner product in the space F by

$$(Lu, W)_F = \int_Q \mathcal{L}u\, w\, dx\, dt + \int_\Omega (l_1 u) w_0\, dx\, dt + \int_\Omega (l_2 u) w_1\, dx\, dt, \tag{3.21}$$

where $W = (w, w_0, w_1)$. The equality (3.20) can be expressed as follows:

$$\int_Q u_{tt}(x,t) w(x,t)\, dx\, dt - \int_Q \Delta u\, w(x,t)\, dx\, dt \\ + b \int_Q u(x,t) w(x,t)\, dx\, dt + \int_Q |u|^4 w(x,t)\, dx\, dt = 0. \tag{3.22}$$

We set $w = u_t$, we obtain

$$\int_Q u_{tt}(x,t) u_t(x,t)\, dx\, dt - a \int_Q \Delta u\, u_t(x,t)\, dx\, dt \\ + b \int_Q u(x,t) u_t(x,t)\, dx\, dt + \int_Q |u|^4 u_t(x,t)\, dx\, dt = 0. \tag{3.23}$$

Integrating by parts and utilizing the fact that $l_1 u = l_2 u = 0$ (because $u \in D_0(L)$) we get

$$\frac{1}{2} \int_\Omega |u_t(x,t)|^2\, dx + \frac{a}{2} \int_\Omega |\nabla u|^2\, dx + \frac{b}{2} \int_\Omega |u(x,t)|^2\, dx + \frac{1}{4} \int_\Omega |u(x,t)|^4\, dx = 0. \tag{3.24}$$

We integrate again on $(0, T)$, which yields

$$\frac{1}{2} \|u_t\|^2_{L^2(Q)} + \frac{a}{2} \|\nabla u\|^2_{L^2(Q)} + \frac{b}{2} \|u\|^2_{L^2(Q)} + \frac{1}{4} \|u\|^4_{L^4(Q)} = 0. \tag{3.25}$$

It comes

$$\frac{1}{2} \|u_t\|^2_{L^2(Q)} + \frac{b}{2} \|u\|^2_{L^2(Q)} + \frac{1}{2} \|u\|^4_{L^4(Q)} = -\frac{a}{2} \|\nabla u\|^2_{L^2(Q)}. \tag{3.26}$$

So, we have

$$\frac{1}{2} \|u_t\|^2_{L^2(Q)} + \frac{b}{2} \|u\|^2_{L^2(Q)} + \frac{1}{4} \|u\|^4_{L^4(Q)} \leq 0. \tag{3.27}$$

Then, we obtain

$$\|u_t\|^2_{L^2(Q)} \leq 0$$

means that $u_t \equiv 0$ in Q, hence $w \equiv 0$ in Q. This proves Proposition 3.6. □

Theorem 3.7. *For each $\mathcal{F} \in F$, with $\mathcal{F} = (f, \varphi_1, \varphi_2)$, there exists a unique strong solution $u \in D(\overline{L})$ satisfying the problem (1.2)–(1.5), given by $u = \overline{L}^{-1} \mathcal{F}$.*

Proof. To establish the density of $R(L)$ in F, we consider an element $w \in R^\perp(L)$ and demonstrate that $w = 0$ for all $u \in D(L)$, which implies that $R^\perp(L) = \{0\}$, except for the particular situation in which $D(L)$ reduces to $D_0(L)$, where

$$D_0(L) = \{u \in D(L) : l_1 u = l_2 u = 0\}.$$

We consider a function $W = (w, w_1, w_2) \in R^\perp(L)$ and for all $u \in D(L)$ it holds

$$(Lu, W)_F = \int_Q \mathcal{L}u\, w\, dx\, dt + \int_\Omega (l_1 u) w_1\, dx\, dt + \int_\Omega (l_2 u) w_2\, dx\, dt = 0. \tag{3.28}$$

Assuming that $u \in D_0(L)$, we get

$$\int_Q \mathcal{L}u\, w\, dx\, dt = 0.$$

Using the previous proposition, we deduce that $w \equiv 0$. Thus, equation (3.21) takes the form

$$\int_\Omega (l_1 u) w_1\, dx\, dt + \int_\Omega (l_2 u) w_2\, dx\, dt = 0, \quad u \in D(L). \tag{3.29}$$

Given that the ranges of the trace operators l_1 and l_2 are dense in the Hilbert space, equality (3.29) implies $w_1 = w_2 = 0$. Hence, it follows that $W = 0$, which implies $\overline{R(L)} = F$. As a result, this completes the proof of Theorem 3.7. □

4. Unique Solvability Of The Nonlinear Inverse Problem

Let the functions constituting the data of the problem be measurable and satisfy the following assumptions

$$(H) \qquad \begin{cases} h \in C(0,T; L^2(\Omega)),\ V = \{v, \nabla v \in L^2(\Omega),\, v \in L^4(\Omega)\},\ E \in W_2^2(0,T), \\ \|h(x,t)\| \le m; \quad |g^*(t)| \ge r > 0, \quad \text{for } r \in \mathbb{R}, \text{ and } (x,t) \in Q, \\ \varphi(x) \in W_2^1(\Omega). \end{cases} \tag{4.1}$$

The following operator describes the relationship between σ and u

$$A : L^2(0,T) \longrightarrow L^2(0,T) \tag{4.2}$$

with the values

$$A(\sigma(t)) = \frac{1}{g^*(t)} \left\{ a \int_\Omega \nabla u \cdot \nabla v\, dx + \int |u|^3 v\, dx \right\}. \tag{4.3}$$

Hence, the earlier relation between σ and u gives rise to a second-order linear equation for the function σ in the space $L^2(0,T)$

$$\sigma = A\sigma + W \tag{4.4}$$

where

$$W = \frac{E'' + bE}{g^*(t)}. \tag{4.5}$$

Theorem 4.1. *Suppose that the data functions of the inverse problem (1.1)–(1.5) satisfy condition (H). Then the following assertions are equivalent:*

- **(i):** *If the inverse problem (1.1)–(1.5) admits a solution, then equation (4.4) admits a solution as well.*
- **(ii):** *Conversely, if equation (4.4) has a solution and the compatibility condition*

$$E(0) = \int_\Omega \varphi(x) v(x)\, dx$$

is fulfilled, then the inverse problem (1.1)–(1.5) admits a solution.

Proof. **(i)** Suppose that the problem (1.1)–(1.5) admits a solution, denoted by $\{u, \sigma\}$. Multiplying equation (1.1) by v and integrating over the domain Ω, we obtain

$$\int_\Omega u_{tt} v\,dx + a\int_\Omega \nabla u \nabla v\,dx + b\int_\Omega uv\,dx + \int_\Omega |u|^3 v\,dx = \sigma(t)\int_\Omega h(x,t)v\,dx. \quad (4.6)$$

Using (1.5) and (4.2) it comes

$$\frac{E'' + bE}{g^*} + A\sigma = \sigma,$$

which implies that σ satisfies equation (4.4).

(ii) Under the above assumptions, suppose that equation (4.3) admits a solution, denoted by σ. By substituting σ into equation (1.1), the system (1.1)–(1.4) reduces to a direct problem, which is known to possess a unique solution. It therefore remains to verify that the function u also satisfies the integral overdetermination condition (1.5). From equation (4.6), we obtain

$$\frac{d^2}{dt}\int_\Omega uv\,dx + a\int_\Omega \nabla u\nabla v\,dx + b\int_\Omega uv\,dx + \int_\Omega |u|^3 v\,dx = \sigma(t)g^*(t). \quad (4.7)$$

However, the function u satisfies the following relation

$$E'' + bE + a\int_\Omega \nabla u\nabla v\,dx + \int_\Omega |u|^3 v\,dx = \sigma(t)g^*(t). \quad (4.8)$$

Subtracting (4.7) from (4.8) we get

$$\frac{d^2}{dt}\int_\Omega uv\,dx + b\int_\Omega uv\,dx = E'' + bE. \quad (4.9)$$

Consequently, we conclude that u satisfies the condition defined by the integral equation (1.5). Therefore, it follows that the pair $\{u, \sigma\}$ constitutes a solution to the inverse problem given by (1.1)–(1.5). □

Lemma 4.2. *Under the assumption that condition (H) holds, there exists a positive δ such that A is a contracting operator in $L^2(0,T)$.*

Proof. From (4.3) we get the following estimate

$$\begin{aligned} |A\sigma(t)|^2 &\le \left|\frac{1}{g^*}\left\{a\int_\Omega \nabla u\nabla v\,dx + \int_\Omega |u|^3 v\,dx\right\}\right|^2 \\ &\le \frac{2}{r^2}\left[a^2\left(\int_\Omega \nabla u\nabla v\,dx\right)^2 + \left(\int_\Omega |u|^3|v|\,dx\right)^2\right] \\ &\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \left(\int_\Omega |u|^3|v|\,dx\right)^2\right] \\ &\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \lambda\|u\|^4_{L^4(\Omega)}\|v\|^2_{L^4(\Omega)}\right]. \end{aligned} \quad (4.10)$$

where

$$\lambda = \|u\|^2_{L^\infty(0,T;L^4(\Omega))}.$$

By integrating (4.10) with respect to $(0,T)$, we obtain

$$\int_0^T |A\sigma(t)|^2 dt \le \frac{2}{r^2}\max\Big(a^2\|\nabla v\|^2_{L^2(\Omega)}, \lambda\|v\|^2_{L^4(\Omega)}\Big) \times \left(\int_0^T \|\nabla u\|^2_{L^2(\Omega)} d\tau + \int_0^T \|u(\cdot,\tau)\|^4_{L^4(\Omega)} d\tau\right). \tag{4.11}$$

So, we obtain

$$\|A\sigma\|^2_{L^2(0,T)} \le K\left(\int_0^T \|\nabla u\|^2_{L^2(\Omega)} d\tau + \int_0^T \|u(\cdot,\tau)\|^4_{L^4(\Omega)} d\tau\right),$$

where

$$K = \frac{2}{r^2}\max\Big(a^2\|\nabla v\|^2_{L^2(\Omega)}, \lambda\|v\|^2_{L^4(\Omega)}\Big).$$

Taking the $L^2(Q)$-inner product of equation (1.1) with u_t, performing integration by parts, and invoking Cauchy's inequality, we derive

$$\begin{aligned}
&\frac{1}{2}\|u_t(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{a}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{b}{2}\|u(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{1}{4}\|u(\cdot,\tau)\|^4_{L^4(\Omega)} \\
&\le \frac{\varepsilon}{2}\|u_t\|^2_{L^2(Q)} + \frac{m^2}{2\varepsilon}\|\sigma\|^2_{L^2(Q)} + \frac{1}{2}\|\varphi_2\|^2_{L^2(\Omega)} + \frac{a}{2}\|\nabla\varphi_1\|^2_{L^2(\Omega)} &(4.12)\\
&\quad + \frac{b}{2}\|\varphi_1\|^2_{L^2(\Omega)} + \frac{1}{4}\|\varphi_1\|^4_{L^4(\Omega)}. &(4.13)
\end{aligned}$$

By means of Gronwall's Lemma, we derive

$$\begin{aligned}
&\frac{1}{2}\|u_t(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{a}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{b}{2}\|u(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(\cdot,\tau)\|^4_{L^4(\Omega)} \\
&\le \frac{m^2c'}{2\varepsilon}\|\sigma\|^2_{L^2(Q)} + \frac{c'}{2}\|\varphi_2\|^2_{L^2(\Omega)} + \frac{ac'}{2}\|\nabla\varphi_1\|^2_{L^2(\Omega)} + \frac{bc'}{2}\|\varphi_1\|^2_{L^2(\Omega)} + \frac{c'}{4}\|\varphi_1\|^4_{L^4(\Omega)}.
\end{aligned} \tag{4.14}$$

where

$$c' = \exp\left(\frac{\varepsilon T}{2}\right), \quad \text{and} \quad c' \le 1.$$

Passing to the maximum, we get

$$\begin{aligned}
&\frac{1-c'}{2}\|u_t\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{a-ac'}{2}\|\nabla u\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{b-bc'}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&+ \frac{1-c'}{4}\|u\|^4_{L^\infty(0,T;L^4(\Omega))} \le \frac{m^2c'}{2\varepsilon}\|\sigma\|^2_{L^2(Q)}.
\end{aligned} \tag{4.15}$$

Omitting some terms, we get

$$\|\nabla u\|^2_{L^2(\Omega)} + \|u(\cdot,\tau)\|^4_{L^4(\Omega)} \le k'\|\sigma\|^2_{L^2(Q)}, \tag{4.16}$$

where

$$k' = \frac{\dfrac{m^2c'}{2\varepsilon}}{\min\left(\dfrac{a-ac'}{2}, \dfrac{1-c'}{4}\right)}.$$

We may thus conclude that

$$\|A\sigma\|^2_{L^2(0,T)} \leq \delta \|\sigma\|^2_{L^2(Q)}, \tag{4.17}$$

where

$$\delta = kk', \qquad \delta < 1.$$

□

Theorem 4.3. *Let the compatibility conditions* $E(0) = \int_\Omega \varphi(x)v(x)\,dx$ *and* (H) *be satisfied. Consequently, the following statements are true:*
(i) The inverse problem (1.1)–(1.5) admits a unique solution.
(ii) Given any initial value $\sigma_0 \in L^2(0,T;L^2(0,T))$*, the subsequent approximations*

$$\sigma_{n+1} = \mathcal{A}\sigma_n \tag{4.18}$$

converge to σ *in the* $L^2(0,T;L^2(0,T))$ *norm (see below for the definition of* $\mathcal{A}$*).*

Proof. **(ii)** Employing the nonlinear operator $\mathcal{A}$ defined as

$$\mathcal{A} : L^2(0,T) \longrightarrow L^2(0,T;L^2(0,T)),$$

working in accordance with the equation

$$\mathcal{A}\sigma = A\sigma + \frac{E'' + bE}{g^*}, \tag{4.19}$$

where the operator A and the function g^* are derived from (4.3). According to (4.18) and (4.4), this can be rewritten as

$$\sigma = \mathcal{A}\sigma. \tag{4.20}$$

It suffices to demonstrate that the operator $\mathcal{A}$ has a fixed point in the space $L^2(0,T;L^2(0,T))$:

$$\mathcal{A}\sigma_1 - \mathcal{A}\sigma_2 = A(\sigma_1 - \sigma_2).$$

The estimate (4.17) leads to

$$\|\mathcal{A}\sigma_1 - \mathcal{A}\sigma_2\|_{L^2(0,T)} \leq \delta \|\sigma_1 - \sigma_2\|_{L^2(0,T;L^2(0,T))}. \tag{4.21}$$

We observe that the operator $\mathcal{A}$ defines a contraction on $L^2(0,T;L^2(0,T))$ associated with (4.20) and (4.21). Consequently, by the Banach fixed-point theorem, $\mathcal{A}$ admits a unique fixed point σ in $L^2(0,T;L^2(0,T))$, and the iterative sequence defined by (4.18) converges to σ in the $L^2(0,T;L^2(0,T))$ norm, regardless of the choice of initial iterate $\sigma_0 \in L^2(0,T;L^2(0,T))$.

(i) This establishes the existence and uniqueness of a solution σ to (4.20) and (4.4) in $L^2(0,T;L^2(0,T))$. As a result, the inverse problem (1.1)–(1.5) possesses at least one solution. To demonstrate uniqueness, suppose that there exist two solutions $\{u_1, \sigma_1\}$ and $\{u_2, \sigma_2\}$ to the inverse problem. First consider the case $\sigma_1 \neq \sigma_2$. If instead $\sigma_1 = \sigma_2$, then by the uniqueness of the associated direct problem (2.1)–(3.4), we would have $u_1 = u_2$ almost everywhere in Q. However, if $\sigma_1 \neq \sigma_2$, both pairs satisfy (4.7), which implies the existence of two distinct solutions to (4.20). This contradicts the uniqueness of σ, and therefore the solution to the inverse problem is unique. □

Corollary 4.4. *The solution σ to equation (4.4), under the condition of Theorem 4.3, is continuously dependent on the data W.*

Proof. Suppose that W_1 and W_2 are two sets of data fulfilling the conditions of Theorem 4.3. Let σ_1 and σ_2 be the solutions of equation (4.4) corresponding to W_1 and W_2, respectively. Then, from (4.4), one obtains

$$\sigma_1 = A\sigma_1 + W_1, \quad \sigma_2 = A\sigma_2 + W_2.$$

Let us start by computing the difference $\sigma_1 - \sigma_2$. Using equation (4.17), we get

$$\begin{aligned} \|\sigma_1 - \sigma_2\|_{L^2(0,T;L^2(0,T))} &= \|(A\sigma_1 + W_1) - (A\sigma_2 + W_2)\|_{L^2(0,T)} \\ &\leq \delta\|\sigma_1 - \sigma_2\|_{L^2(0,T;L^2(0,T))} + \|W_1 - W_2\|_{L^2(0,T)}. \end{aligned} \tag{4.22}$$

Thus, we have

$$\|\sigma_1 - \sigma_2\|_{L^2(0,T;L^2(0,T))} \leq \frac{1}{1-\delta}\|W_1 - W_2\|_{L^2(0,T)}.$$

□

5. Conclusion

In this work, we studied an inverse problem associated with a hyperbolic equation containing an unknown coefficient under an integral overdetermination condition. First, we established the well-posedness of the corresponding direct problem by applying the energy inequality method, which allowed us to prove the existence and uniqueness of a strong solution. Then, using fixed-point arguments, we demonstrated the existence of a solution to the inverse problem. Furthermore, we proved the uniqueness and the continuous dependence of the solution on the given data. The obtained results confirmed that the considered inverse problem was well posed in the sense of Hadamard. Therefore, the developed analytical framework provided a reliable approach for determining the unknown coefficient and could serve as a basis for future investigations of more general nonlinear inverse problems.

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