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LIFE SPAN OF NONNEGATIVE SOLUTIONS TO AN EVOLUTION EQUATION

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Abstract. We study the porous medium equation with a fractional integral source. The model is considered in the whole Euclidean space and is supplemented with a nonnegative, bounded and nontrivial initial condition. We show that there exists a critical exponent separating two different behaviors of solutions.When the nonlinearity is below this critical value, the problem admits no global nontrivial solution. This fact allows us to derive an upper bound for the maximal time of existence of solutions. In particular, we prove that solutions exist only on finite time interval and provide an estimate for the lifespan in terms of the size of initial data.

Keywords. Porous medium equation, lifespan, fractional integral, critical exponent, Blow-up
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1. Introduction

In this article, we investigate the maximal interval of existence of the solutions of the problem

$$u_t - \Delta|u|^{m-1}u = \int_0^t (t-s)^{-\gamma}|u|^p u(s)\,ds, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

where $p > 0$, $m > 1$, and $0 < \gamma < 1$ are real numbers with nonnegative, nontrivial, continuous, and bounded initial condition

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0,\ x \in \mathbb{R}^N. \tag{1.2}$$

If we consider (1.1) without nonlinear term, i.e $u_t - \Delta|u|^{m-1}u = 0$, this equation is so-called Porous Medium Equation and have been studied by so many people. In fact, a variety of results concerning this equation have been obtained in the literature.

This model appears in several physical applications, mainly to describe processes involving fluid flow, heat transfer or diffusion. Maybe the best known of them is the description of the flow of an isentropic gas through a porous medium,

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modeled independently by Leibenzon [8] and Muskat [11] around 1930. Indeed, this application was at the base of the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, spread of viscous fluids, boundary layer theory, and other fields. When equation (1.1) is considered with a local nonlinearity of the form $|u|^p u$, it reads

$$u_t - \Delta |u|^{m-1}u = |u|^p u,$$

which is a particular case of (1.1); it corresponds to $\gamma \to 0$. This equation has been considered by H. J. Kuiper [6]. By life span for initial condition u_0, we mean the least upper bound of all values T such that $[0, T)$ is a maximal interval of existence of a solution. He found that the life span $L(\sigma)$ is bounded by $C\sigma^{-(p+1-m)}$ whenever $u(x, 0) = \sigma u_0(x)$, $\sigma > 0$. (see Theorem 3.6 [6]).

The analysis is based on the observation that equation (1.1) can be written in the following form:

$$u_t - \Delta |u|^{m-1}u = \Gamma(\alpha) I^{\alpha}_{0|t}(|u|^p u), \tag{1.3}$$

where the left-sided Riemann-Liouville fractional integral $I^{\alpha}_{0|t}$ of order $\alpha \in (0, 1)$, defined in (2.3), plays a crucial role in the proof of the theorem on life span; we have set in (1.3) $\alpha = 1 - \gamma \in (0, 1)$.

Our article is motivated mathematically by the recent papers [3, 9] which deal with the critical exponent and life span for the parabolic equation with nonlocal in time nonlinearity

$$u_t - \Delta u = \int_0^t (t-s)^{\alpha-1} |u|^p u \, ds, \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.4}$$

where $u_0 \in C_0(\mathbb{R}^N)$, the space of all continuous functions decaying to zero at infinity, which is a particular case of (1.1); it corresponds to $m \to 1$.

The main result of this article is to find the exponent

$$p^* = m - 1 + \frac{N\alpha(m-1) + 2\alpha m + 2}{(N - 2\alpha)_+}$$

such that if $p \le p^*$, the problem (1.1)-(1.2) has no global nontrivial solution. Suppose u_σ is the solution corresponding to the nontrivial, nonnegative initial condition $u_\sigma(x, 0) = \sigma u_0(x)$. Let $[0, T_\sigma)$ be its maximal interval of existence. We obtain a life span of the form

$$L(\sigma) \le C\sigma^{-\frac{p+1-m}{\alpha+1}}.$$

We will show for equation (1.1) a necessary condition for global solution. More precisely, if u is a global solution with initial condition $u(x, 0) = u_0(x)$, then an inequality of the form

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x)\varphi_1(x/R)\, dx \le C\lambda^{\kappa}, \quad \text{for some } S > 0 \text{ and } \kappa > 0,$$

must be satisfied. Here φ_1 is the positive eigenfunction corresponding to the principal eigenvalue of the Dirichlet problem on the unit ball B_1, and normalized such that $\int_{B_1} \varphi_1(\xi)\, d\xi = 1$. The constants C and κ depend on N, m, p and α.

The method used to prove the life span and necessary conditions for global existence is the test function method [1, 2, 4, 10, 13]. The principle of this method is as follows: we assume, by absurd, that the solution is global, then we use the weak formulation of the equation on $[0, T]$, for some $T > 0$ and make an appropriate choice of the test function, then we make a change of variable and finally we take the limit as $T \to \infty$ to get the contradiction.

This paper is organized as follows. In section 2, we present some definitions and properties. In section 3, we find the exponent p^*. Finally, we obtain an upper bound for the maximal interval of existence.

2. Preliminaries

In this section, we present some definitions and results concerning the fractional integrals and fractional derivatives that will be used in the sequel.

If $AC([0, T])$ is the space of all functions which are absolutely continuous on $[0, T]$ with $0 < T < \infty$, then, for $f \in AC([0, T])$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^\alpha_{0|t} f(t)$ and $D^\alpha_{t|T} f(t)$ of order $\alpha \in (0, 1)$ are defined by (see (2.1.5) and (2.1.6) in [5])

$$D^\alpha_{0|t} f(t) \;:=\; D I^{1-\alpha}_{0|t} f(t), \tag{2.1}$$

$$D^\alpha_{t|T} f(t) \;:=\; -\frac{1}{\Gamma(1-\alpha)} D \int_t^T (s-t)^{-\alpha} f(s)\, ds, \tag{2.2}$$

for all $t \in [0, T]$, where $D := d/dt$ is the usual time derivative and

$$I^\alpha_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{2.3}$$

is the Riemann-Liouville fractional integral defined for all $f \in L^q(0, T)$ $(1 \le q \le \infty)$.

Now, for every $f, g \in C([0, T])$, such that $D^\alpha_{0|t} f(t), D^\alpha_{t|T} g(t)$ exist and are continuous, and for all $t \in [0, T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.1.50) in [5])

$$\int_0^T (D^\alpha_{0|t} f)(t) g(t)\, dt \;=\; \int_0^T f(t)(D^\alpha_{t|T} g)(t)\, dt. \tag{2.4}$$

Note also that, for any $f \in AC^2([0, T])$, we have (see (2.1.35) in [5])

$$-D.D^\alpha_{t|T} f = D^{1+\alpha}_{t|T} f, \tag{2.5}$$

where

$$AC^2([0, T]) := \{ f : [0, T] \to \mathbb{R} \text{ such that } Df \in AC([0, T]) \}.$$

Moreover, for all $1 \le q \le \infty$, the following equality (see Lemma 2.4 in [5])

$$D^{\alpha}_{0|t} I^{\alpha}_{0|t} = Id_{L^q(0,T)} \tag{2.6}$$

holds almost everywhere on $[0, T]$.
Later on, we will use the following results [4].

- If $w(t) = (1 - t/T)^{\theta}_{+}$, $t \ge 0$, $T > 0$, and θ sufficiently large, then

$$D^{\alpha}_{t|T} w(t) = C_1 T^{-\alpha} \left(1 - \frac{t}{T}\right)^{\theta-\alpha}_{+}, \tag{2.7}$$

$$D^{\alpha+1}_{t|T} w(t) = C_2 T^{-(\alpha+1)} \left(1 - \frac{t}{T}\right)^{\theta-\alpha-1}_{+}, \tag{2.8}$$

for all $\alpha \in (0, 1)$; so

$$\left(D^{\alpha}_{t|T} w\right)(T) = 0 \quad ; \quad \left(D^{\alpha}_{t|T} w\right)(0) = C_1\, T^{-\alpha}, \tag{2.9}$$

where $C_1 = \dfrac{(1 - \alpha + \theta)\Gamma(\theta + 1)}{\Gamma(2 - \alpha + \theta)}$ and $C_2 = \dfrac{(1 - \alpha + \theta)(\theta - \alpha)\Gamma(\theta + 1)}{\Gamma(2 - \alpha + \theta)}$.

Now, we give the definition of weak solution of (1.3)

Definition 2.1 (Weak solution)**.** A measurable function $u(x, t)$ defined in $S_T = \mathbb{R}^N \times (0, T]$ is called a weak solution of (1.3) if for every bounded open set Ω with smooth boundary $\partial\Omega$,

$$u \in C(0, T : L^1(\Omega)) \cap L^p_{loc}(0, T : W^1_p(\Omega)) \cap L^{\infty}_{loc}(S_T),$$

and

$$\int_{\Omega} u(x, t)\phi(x, t)\, dx + \int_{t_0}^{t} \int_{\Omega} [-u\phi_t + \nabla(|u|^{m-1}u) \cdot \nabla\phi]\, dx\, d\tau$$

$$= \Gamma(\delta) \int_{t_0}^{t} \int_{\Omega} I^{\delta}_{0|\tau}(|u|^p u)\varphi\, dx d\tau + \int_{\Omega} u(x, t_0)\phi(x, t_0)\, dx \tag{2.10}$$

for all $0 \le t_0 < t \le T$ and $\phi \in C^1(\overline{\Omega} \times [0, T])$, $\phi = 0$ near $\partial\Omega \times (0, T)$ where $\delta = 1 - \gamma$.

3. The test function method

Let $t_* > 0$. Suppose that u is a solution of (1.1)-(1.2) on $\mathbb{R}^N \times [0, t_*)$. We assume that

$$1 < (1 + \alpha)m < p + 1.$$

Let λ_R be the principal eigenvalue for the Dirichlet problem on the ball of radius R:

$$-\Delta h(x) = \lambda h(x), \quad x \in B_R,$$
$$h(x) = 0, \quad x \in \partial B_R,$$

where $B_R := \left\{x \in \mathbb{R}^N : |x| < R\right\}$. We note that $\lambda_R = \lambda_1/R^2$ where λ_1 is the principal eigenvalue of the Laplacian on the unit ball B_1. Let φ_1 denote the unique nonnegative eigenfunction corresponding to the principal eigenvalue λ_1

such that $\int_{B_1}\varphi_1(x)dx = 1$. Of course φ_1 is radially symmetric: $\varphi_1(x) = \varphi_1(|x|)$.
We define $\varphi_2(t) = \left(1 - \dfrac{t}{TR^\beta}\right)_+^\theta$, where $1 < T < R$ is such that when $R \to \infty$ we don't have $T \to \infty$ simultaneously, $\theta \gg 1$ is large enough and

$$\beta = \frac{2(p+1) + (m-1)N}{p+2-m} > 0.$$

We also define

$$\widetilde{\varphi}(x,t) := \varphi_1(x/R)\varphi_2(t) \quad , \quad \varphi(x,t) := D^\alpha_{t|TR^\beta}\widetilde{\varphi}(x,t)$$

and for $TR^\beta < t_*$,

$$J_R(T) := \int_0^{TR^\beta}\int_{B_R} I^\alpha_{0|t}(u^{p+1})\varphi(x,t)\,dx\,dt.$$

As u is a weak solution, we have

$$\begin{aligned} J_R(T) &= -\int_{B_R}\varphi_1(x/R)u_0(x)D^\alpha_{t|TR^\beta}\varphi_2(0)\,dx \\ &\quad + \int_0^{TR^\beta}\int_{B_R} uD^{1+\alpha}_{t|TR^\beta}\varphi_2(t)\varphi_1(x/R)\,dx\,dt \\ &\quad + \int_0^{TR^\beta}\int_{B_R} u^m D^\alpha_{t|TR^\beta}\varphi_2(t)R^{-2}\lambda_1\varphi_1(x/R)\,dx\,dt, \end{aligned}$$

where we have used the fact that $\Delta\varphi_1(x/R) = R^{-2}\lambda_1\varphi_1(x/R)$.
Using (2.9) and letting $V_R := \displaystyle\int_{B_R}\varphi_1(x/R)u_0(x)\,dx$, we get

$$\begin{aligned} &J_R(T) + C\,T^{-\alpha}R^{-\alpha\beta}V_R \\ &= \int_0^{TR^\beta}\int_{B_R} uD^{1+\alpha}_{t|TR^\beta}\varphi_2(t)\varphi_1(x/R)\,dx\,dt + \int_0^{TR^\beta}\int_{B_R} u^m D^\alpha_{t|TR^\beta}\varphi_2(t)R^{-2}\lambda_1\varphi_1(x/R)\,dx\,dt \\ &= \int_0^{TR^\beta}\int_{B_R} [u^{p+1}\varphi_1(x/R)\varphi_2(t)]^{\frac{1}{p+1}}\varphi_1(x/R)^{\frac{p}{p+1}}\varphi_2(t)^{-\frac{1}{p+1}}D^{1+\alpha}_{t|TR^\beta}\varphi_2(t)\,dx\,dt \\ &\quad + \lambda_1R^{-2}\int_0^{TR^\beta}\int_{B_R} [u^{p+1}\varphi_1(x/R)\varphi_2(t)]^{\frac{m}{p+1}}\varphi_1(x/R)^{\frac{p+1-m}{p+1}}\varphi_2(t)^{-\frac{m}{p+1}}D^\alpha_{t|TR^\beta}\varphi_2(t)\,dx\,dt \\ &\le J_R(T)^{\frac{1}{p+1}}\left[\int_0^{TR^\beta}\int_{B_R}\varphi_1(x/R)\varphi_2(t)^{-\frac{1}{p}}(D^{1+\alpha}_{t|TR^\beta}\varphi_2(t))^{\frac{p+1}{p}}\,dx\,dt\right]^{\frac{p}{p+1}} \\ &\quad + \lambda_1R^{-2}J_R(T)^{\frac{m}{p+1}}\left[\int_0^{TR^\beta}\int_{B_R}\varphi_1(x/R)\varphi_2(t)^{-\frac{m}{p+1-m}}(D^\alpha_{t|TR^\beta}\varphi_2(t))^{\frac{p+1}{p+1-m}}\,dx\,dt\right]^{\frac{p+1-m}{p+1}}. \end{aligned}$$

We multiply by T^α and use (2.7) and (2.8); we obtain

$$\begin{aligned} H_R(T) + R^{-\alpha\beta}V_R \ \leq\ & H_R(T)^{\frac{1}{p+1}} R^{-\beta(1+\alpha)} \left[\int_0^{TR^\beta}\int_{B_R} T^{-\frac{p+1+\alpha}{p}} \varphi_1(x/R)\varphi_2(t)^{\delta_1}\, dx\, dt\right]^{\frac{p}{p+1}} \\ & + H_R(T)^{\frac{m}{p+1}} \lambda_1 R^{-(2+\alpha\beta)} \left[\int_0^{TR^\beta}\int_{B_R} T^{-\frac{\alpha m}{p+1-m}} \varphi_1(x/R)\varphi_2(t)^{\delta_2}\, dx\, dt\right]^{\frac{p+1-m}{p+1}}, \end{aligned}$$

where

$$H_R(T) = T^\alpha J_R(T), \quad \delta_1 = \frac{p(\theta-1)-\alpha(p+1)-1}{p\theta} \quad \text{and} \quad \delta_2 = \frac{(p+1)(\theta-\alpha)-m\theta}{\theta(p+1-m)}.$$

Making the change of variable $\xi = x/R$ and $\tau = t/R^\beta$, we deduce that

$$H_R(T) + R^{-\alpha\beta}V_R \leq H_R(T)^{\frac{1}{p+1}} R^{s_1} A(T)^{\frac{p}{p+1}} + H_R(T)^{\frac{m}{p+1}} \lambda_1 R^{s_2} B(T)^{\frac{p+1-m}{p+1}}, \quad (3.1)$$

where

$$s_1 = -\beta(1+\alpha) + \frac{p(\beta+N)}{p+1}, \qquad s_2 = -2 + N + \beta(1-\alpha) - \frac{(\beta+N)m}{p+1},$$

$$A(T) = T^{-\frac{p+1+\alpha}{p}} \int_0^T \int_{B_1} \varphi_1(\xi)\varphi_2(\tau)^{\delta_1}\, d\xi\, d\tau,$$

and

$$B(T) = T^{-\frac{\alpha m}{p+1-m}} \int_0^T \int_{B_1} \varphi_1(\xi)\varphi_2(\tau)^{\delta_2}\, d\xi\, d\tau.$$

As

$$\beta = \frac{2(p+1)+(m-1)N}{p+2-m} > 0, \quad (3.2)$$

we have

$$s = s_1 = s_2 = \frac{(p+1-m)N - \alpha[2(p+1)+N(m-1)] - 2}{p+2-m}. \quad (3.3)$$

We aim to use (3.1) in order to obtain information on the relationship between the initial condition and the length of the maximum interval of existence.

Theorem 3.1. *If $s \leq 0$, that is to say*

$$p \leq p^* = m - 1 + \frac{N\alpha(m-1)+2\alpha m+2}{(N-2\alpha)_+},$$

then problem (1.1)-(1.2) has no global nontrivial solution.

Proof. In the case of $s < 0$ (i.e $p < p^*$), we use (2.4), (2.6), and (3.1) to obtain

$$\lim_{R\to\infty} \int_0^{TR^\beta}\int_{B_R} I^\alpha_{0|t}(u^{p+1})\varphi(x,t)\, dx\, dt = \lim_{R\to\infty} \int_0^{TR^\beta}\int_{B_R} u^{p+1}\widetilde{\varphi}(x,t)\, dx\, dt = 0,$$

i.e.

$$\int_0^\infty \int_{\mathbb{R}^N} u^{p+1}\, dx\, dt = 0, \quad (3.4)$$

which implies that $u \equiv 0$ is the only global solution.
If $s = 0$ (i.e $p = p^*$), suppose a nontrivial global solution exists. From monotonicity, we have $\int_0^\infty \int_{\mathbb{R}^N} u^{p^*+1}\,dx\,dt > 0$. But using the critical estimate, we have

$$\int_0^{TR^\beta} \int_{B_R} u^{p^*+1}\,dx\,dt \le ClogR.$$

After dividing by $logR$ and taking the limit as $R \to \infty$, we get

$$\int_0^\infty \int_{\mathbb{R}^N} u^{p^*+1}\,dx\,dt = 0.$$

Hence $u \equiv 0$. Contradiction

□

4. Life Span of a solution

Suppressing argument and subscripts (3.1) becomes

$$H + R^{-\alpha\beta} V \le H^{\frac{1}{p+1}} R^s A^{\frac{1}{p+1}} + H^{\frac{m}{p+1}} \lambda R^s B^{\frac{p+1-m}{p+1}}. \tag{4.1}$$

We will use this to obtain an estimate for V. First we state some lemmas.

Lemma 4.1. *Suppose that $a, b, r,$ and q are positive constants. Define the functions $F(x) := ax^q - bx^r, G(x) := ax^{-q} + bx^r$ on $0 < x < \infty$. Then*

$$\max_{x>0} F(x) = \left(1 - \frac{q}{r}\right) a^{\frac{r}{r-q}} \left(\frac{q}{br}\right)^{\frac{q}{r-q}},$$

$$\min_{x>0} G(x) = \left(1 + \frac{q}{r}\right) a^{\frac{r}{r+q}} \left(\frac{br}{q}\right)^{\frac{q}{r+q}}.$$

Lemma 4.2. *Let $0 < w_1, w_2 < 1, w_1 \ne w_2$. On $[0, \infty)$ define*

$$\Upsilon(x) := max(x^{w_1}, x^{w_2}).$$

Let η be an arbitrary positive number, then

$$\Psi(w_1, w_2; \eta) := \max_x (\eta\Upsilon(x) - x) = \max_i \left((1 - w_i) w_i^{\frac{w_i}{1-w_i}} \eta^{\frac{1}{1-w_i}}\right).$$

For η sufficiently large

$$\Psi(w_1, w_2; \eta) = (1 - \overline{w})\overline{w}^{\frac{\overline{w}}{1-\overline{w}}} \eta^{\frac{1}{1-\overline{w}}}, \tag{4.2}$$

where $\overline{w} = max(w_1, w_2)$.

We will use the notation

$$J_m := \left(1 - \frac{m}{p+1}\right) \left(\frac{m}{p+1}\right)^{\frac{m}{p+1-m}}.$$

Then, for η sufficiently large,

$$\Psi\left(\frac{1}{p+1}, \frac{m}{p+1}, \eta\right) = J_m \eta^{\frac{p+1}{p+1-m}}.$$

Theorem 4.3. *If u is a nonnegative solution of (1.1)-(1.2) on $B_{R_*} \times [0, t_*)$ and s given by (3.3), then for all $(R, T) \in \left\{(\rho, \tau) : R_0 \leq \rho \leq R_*, 0 \leq \tau \leq t_* \rho^{-\beta}\right\}$, we have*

$$R^{-\alpha\beta} \int_{B_R} u_0(x)\varphi_1(x/R)\, dx \leq \Psi\left(\frac{1}{p+1}, \frac{m}{p+1}; \left[A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}\right] R^s\right). \tag{4.3}$$

In particular, if u is a global nonnegative solution then

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x)\varphi_1(x/R)\, dx \leq J_{\overline{m}} \inf_T \left[A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}\right]^{\frac{p+1}{p+1-m}}, \tag{4.4}$$

where

$$S = \frac{s(p+1)}{p+1-m} + \alpha\beta.$$

Proof. For the sake of convenience we define

$$\Theta(T) = A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}.$$

From (4.1) we see that

$$R^{-\alpha\beta} V \leq \Upsilon(J)\Theta(T)R^s - J, \tag{4.5}$$

where

$$\Upsilon(\mu) := max\left\{\mu^{\frac{1}{p+1}}, \mu^{\frac{m}{p+1}}\right\}.$$

Then by Lemma 4.2, we have (4.3). For R sufficiently large we can use equation (4.2) to conclude the validity of (4.4). $\square$

Corollary 4.4. *Suppose that u is a nonnegative global solution. Then*

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x)\varphi_1(x/R)\, dx \leq J_{\overline{m}} K_m^{\frac{p+1}{p+1-m}} \lambda^{\frac{(p+1)(\alpha+1)}{(p+1-m)(p+1-(\alpha+1)(m-1))}},$$

where K_m is a constant depending on m, θ and p.

Proof. We easily obtain

$$A(T) \leq A_0 \equiv \frac{T^{-\frac{1+\alpha}{p}} p}{p\theta - \alpha(p+1) - 1},$$

and

$$B(T) \leq B_0 \equiv \frac{(p+1-m)T^{\frac{(p+1)-m(1+\alpha)}{p+1-m}}}{\theta(p+1-m) + (p+1)(1-\alpha) - m}.$$

Then, from (4.5), we have

$$R^{-\alpha\beta} \int_{B_R} u_0(x)\varphi_1(x/R)\, dx \leq R^s \Theta_0(T)\Upsilon(J) - J,$$

where

$$\Theta(T) \leq \Theta_0(T) := \alpha_0 T^{-\frac{(\alpha+1)}{p+1}} + \beta_0 T^{\frac{p+1-m(1+\alpha)}{p+1}},$$

with

$$\alpha_0 := \left[\frac{p}{\theta p - \alpha(p+1) - 1}\right]^{\frac{p}{p+1}}, \qquad \beta_0 := \lambda\left[\frac{p+1-m}{(p+1-m)\theta + (p+1)(1-\alpha) - m}\right]^{\frac{p+1-m}{p+1}}.$$

By Lemma 4.1, we get

$$\begin{aligned}
\underline{\Theta}_0 \;&:=\; \min \Theta_0(T) \\
&= \left(1 + \frac{\alpha+1}{p+1-m(1+\alpha)}\right)\alpha_0^{\frac{p+1-m(1+\alpha)}{p+1-(\alpha+1)(m-1)}}\left[\beta_0 \frac{(p+1)-m(1-\alpha)}{\alpha+1}\right]^{\frac{\alpha+1}{p+1-(\alpha+1)(m-1)}} \\
&= \left[\frac{p+1-(\alpha+1)(m-1)}{(p+1)-m(1+\alpha)}\right]\left[\frac{p+1-m(\alpha+1)}{\alpha+1}\right]^{\frac{\alpha+1}{p+1-(\alpha+1)(m-1)}} \\
&\quad \times \lambda^{\frac{\alpha+1}{p+1-(\alpha+1)(m-1)}}\left[\frac{p}{\theta p - \alpha(p+1) - 1}\right]^{\frac{p[p+1-m(1+\alpha)]}{(p+1)[p+1-(\alpha+1)(m-1)]}} \\
&\quad \times \left[\frac{p+1-m}{\theta(p+1-m) + (p+1)(1-\alpha) - m}\right]^{\frac{(p+1-m)(\alpha+1)}{(p+1)[p+1-(\alpha+1)(m-1)]}}.
\end{aligned}$$

As $\theta \gg 1$ we have $\underline{\Theta}_0 \le K_m \lambda^{\frac{\alpha+1}{p+1-(m-1)(\alpha+1)}}$. Then after substituting this into equation (4.4), the proof is complete. $\square$

When we are dealing with the problem originally considered by Fujita ($m \to 1$ and $\alpha \to 0$), then $J_1 = p(p+1)^{-\frac{(p+1)}{p}}$ and $K_1 = (p+1)p^{-\frac{p}{p+1}}$, and we see that the above inequality reduces to

$$\limsup_{R\to\infty} R^{-N+2/p} \int_{B_R} u_0(x)\varphi_1(x/R)\,dx \le C\lambda^{1/p}.$$

This is precisely the result found in [7]. Note that if $m > 1$ and $\alpha \to 0$, we recover the case of Kuiper [6]. As it is done in that article, we can deduce the following result.

Corollary 4.5. *When $N \ge S$, Theorem 4.3 and Corollary 4.4 remain valid if we replace*

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x)\varphi_1(x/R)\,dx \qquad \textit{by} \qquad \liminf_{|x|\to\infty}(|x|^{N-S}u_0(x)).$$

Proof. The statement of this corollary follows from the inequalities:

$$\begin{aligned}
\lim_{R\to\infty} R^{-S}V \;&\ge\; \lim_{R\to\infty} R^{-S}\int_{B_R\setminus B_k} \inf_{R\ge|x|\ge k}(|x|^{N-S}u_0(x))R^{S-N}\varphi_1(x/R)\,dx \\
&\ge\; \lim_{R\to\infty} \inf_{R\ge|x|\ge k}(|x|^{N-S}u_0(x))\int_{B_R\setminus B_k} R^{-N}\varphi_1(x/R)\,dx \\
&=\; \lim_{R\to\infty} \inf_{R\ge|x|\ge k}(|x|^{N-S}u_0(x))\int_{B_1\setminus B_{k/R}} \varphi_1(\zeta)\,d\zeta \\
&=\; \inf_{|x|\ge k}(|x|^{N-S}u_0(x)).
\end{aligned}$$

The proof is complete by letting $k \to \infty$. □

Now, we consider the problem (1.1)-(1.2). By the life span for initial condition u_0, we mean the least upper bound of all values T such that $[0, T)$ is the maximal interval of existence of the solution to (1.1)-(1.2). Let us fix $u_0 \not\equiv 0$ and $u_0 \geq 0$ for all $x \in \mathbb{R}^N$. We denote by $L(\sigma)$, $\sigma > 0$, the life span corresponding to initial condition σu_0. There exists a value Λ such that

$$\Lambda R^{-\alpha\beta} V = \Psi(R^s \Theta(T_M)),$$

where T_M is the value of T at which $\Theta(T)$ attains its minimum value.

Let Θ_L denote the restriction of Θ to the interval $[0, T_M)$. If we take $\sigma > \Lambda$, then $L(\sigma) < \infty$, and we have

$$L(\sigma) \leq \Theta_L^{-1}(R^{-s}\Psi^{-1}(\sigma R^{\alpha\beta} V)). \tag{4.6}$$

In the next result we use this inequality to obtain an explicit upper bound for the life span of a solution.

Theorem 4.6. *Let u_0 be a nonnegative, nontrivial, continuous function in $\mathbb{R}^N$. Then, there exists positive constants Λ_m, C and σ_1 so that the life span $L(\sigma)$ corresponding to the initial condition σu_0 with $\sigma > \Lambda_m$ satisfies*

$$L(\sigma) \leq C\sigma^{-\frac{p+1-m}{\alpha+1}}. \tag{4.7}$$

Proof. Decreasing the value of T_M to a value T_m if needed, we may assume that the function Θ_0, introduced above, is decreasing on $(0, T_M)$. We can choose Λ_m such that in the meantime $\Lambda_m R^{-\alpha\beta} V \geq \Psi(R^s\Theta(T_m))$ and $\Lambda_m R^{-\alpha\beta} V \geq C_1$, where C_1 is a sufficiently large constant so that whenever $\sigma > \Lambda_m$, then

$$\Psi^{-1}(\sigma R^{-\alpha\beta} V) = [(1-\overline{w})^{-1}\overline{w}^{-\frac{\overline{w}}{1-\overline{w}}}]^{1-\overline{w}}(\sigma R^{-\alpha\beta} V)^{\frac{p+1-m}{p+1}},$$

with $\overline{w} = m/(p+1)$. We write

$$\Psi^{-1}(\sigma R^{-\alpha\beta} V) = \gamma_0 R^{-\alpha\beta\frac{p+1-m}{p+1}} V^{\frac{p+1-m}{p+1}} \sigma^{\frac{p+1-m}{p+1}},$$

where $\gamma_0 = (p+1)(p+1-m)^{-\frac{p+1-m}{p+1}} m^{-\frac{m}{p+1}}$. Since

$$\Theta(T) \leq \Theta_0(T) \leq \alpha_0 T^{-\frac{\alpha+1}{p+1}} + \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}},$$

on $[0, T_m)$, it follows that

$$\Theta_L^{-1} \leq \left[\frac{\eta - \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}}}{\alpha_0}\right]^{-\frac{p+1}{\alpha+1}}, \quad \text{for} \quad \eta > \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}}.$$

Let $[0, T_{\max})$ be the maximal interval of existence of u and $T \in [0, T_{\max})$. We define

$$G(R, \sigma) := \alpha_0^{p+1}[\gamma_0 R^{-s} R^{-\alpha\beta} V^{\frac{p+1-m}{p+1}} \sigma^{\frac{p+1-m}{p+1}} - \delta_0]^{-\frac{p+1}{\alpha+1}},$$

where $\delta_0 = \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}}$. Whenever $T < L(\sigma)$ we have $T \leq G(R, \sigma)$. Therefore

$$L(\sigma) \leq G(R, \sigma). \tag{4.8}$$

It is easily seen that this implies equation (4.7).

□

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