

ENTROPY SOLUTIONS FOR SOME ELLIPTIC UNILATERAL PROBLEMS WITH DEGENERATE COERCIVITY

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ABSTRACT. In this work, we study a class of elliptic unilateral problems associated with some strongly nonlinear degenerate elliptic equation, where the lower order terms satisfy general growth conditions, together with integrable right-hand side data. We adopt a variational formulation combined with truncation methods to establish the existence of entropy solutions.

Keywords. Degenerated coercivity, entropy solutions, strongly nonlinear elliptic equation, obstacle problems.

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1. INTRODUCTION

The study of anisotropic elliptic equations with lower order terms has received growing attention due to their relevance in the mathematical modeling of physical and mechanical processes arising in anisotropic media. It is well known that lower order terms may significantly affect qualitative properties of solutions, such as existence, uniqueness, regularity and asymptotic behavior; see for instance, [12, 14] and the references therein.

Let Ω be a bounded open subset of $\mathbb{R}^d$ ($d \geq 2$) with Lipschitz boundary $\partial\Omega$. The aim of this paper is to establish the existence of entropy solutions for a class of nonlinear elliptic equations with degenerate coercivity, which can be written in the form:

$$\begin{cases} A(v) + \mathcal{H}(x, v, \nabla v) = f - \sum_1^d D^i \phi_i(x, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $A(v) = - \sum_1^d D^i a_i(x, v, \nabla v)$ is a Leray-Lions operator A , acting from $W_0^{1, \vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1, \vec{p}'(\cdot)}(\Omega)$. In the particular case of $\phi_i \equiv 0$, Porretta et al. have proved in [33] the existence of solutions to problem (1.1) by means

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of rearrangement techniques. Later, Agnese Di Castro has established in [25] the existence and regularity of solution for the problem (1.1) when the datum f belongs to $W^{-1,\vec{p}'}(\Omega)$ or to $L^{(\vec{p}^*)'}(\Omega)$. For the case $f \in L^1(\Omega)$ and $\mathcal{H}(x, v, \nabla v) = |v|^{s-1}v$, we refer the reader to [4, 14, 21, 23, 29]. Within the framework of anisotropic Sobolev spaces. Benboubker et al. have investigated in [13] the existence and regularity results for the problem (1.1) with $f \in L^1(\Omega)$, under some additional assumptions on the Carathéodory function $\mathcal{H}(x, v, \nabla v)$. Boccardo et al. have studied in [14] the anisotropic elliptic Dirichlet problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i-2} \frac{\partial v}{\partial x_i} \right) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

with μ is a measure data. The authors have proved the existence of solutions. In particular, when $\mu = f(x)$ is a measurable such that $|f| \log(1 + |f|) \in L^1(\Omega)$, the existence of a solution in suitable anisotropic Sobolev space was obtained. For existence and regularity for data in $L^m(\Omega)$ then in the Marcinkiewicz space $M^m(\Omega)$, we refer the reader to paper of Di castro in [23].

Elliptic and parabolic equations involving Leray-Lions type operators with degenerate coercivity have been widely investigated in the literature, mainly focusing on the existence, uniqueness, and regularity of their solutions; see for example, [5, 6, 10, 19, 21, 30, 33, 38].

The present paper contributes to this line of research by proving the existence of entropy solutions for a class of unilateral problem associated to the degenerate elliptic equation (1.1) in anisotropic Sobolev spaces. The proof relies on an approximation schema combined with suitable a priori estimates adapted to the entropy framework.

The paper is organized as follows: In section 2, we state the assumptions on the Carathéodory functions $a_i(x, s, \xi)$, $\mathcal{H}(x, s, \xi)$ and $\phi_i(x, s)$ under which our elliptic equation (1.1) has at least one entropy solution. Section 3, is devoted to the definition of entropy solutions and the statement of the main result. Finally, section 4 contains the proof of the existence result, together with some additional regularity properties of the solutions.

2. PRELIMINARIES AND ESSENTIAL ASSUMPTIONS

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with Lipschitz boundary $\partial\Omega$, and let $\mathcal{M}(\overline{\Omega})$ the space of all measurable function $s(\cdot) : \overline{\Omega} \mapsto \mathbb{R}$, we intruduce

$$\mathcal{C}_+(\Omega) = \{s \in \mathcal{M}(\overline{\Omega}) \mid 1 < s^- \leq s^+ < \infty\},$$

where

$$s^- = \text{ess inf}\{s(x) \mid x \in \Omega\} \quad \text{and} \quad s^+ = \text{ess sup}\{s(x) \mid x \in \Omega\}.$$

For each $s \in \mathcal{C}_+(\Omega)$ the variable exponent Lebesgue space $L^{s(\cdot)}(\Omega)$ is defined as

$$L^{s(\cdot)}(\Omega) = \{u \in \mathcal{M}(\Omega) : \rho_{s(\cdot)}(u) = \int_{\Omega} |u|^{s(x)} dx\}$$

The Luxemburg norm on $L^{s(\cdot)}(\Omega)$ is given by

$$\|u\|_{s(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|u|}{\lambda} \right)^{s(x)} dx \leq 1 \right\}.$$

Let $s' \in \mathcal{C}_+(\Omega)$ be the conjugate variable exponent to s that is

$$\frac{1}{s(x)} + \frac{1}{s'(x)} = 1 \quad \text{a.e. } x \in \overline{\Omega},$$

we know that $(L^{s(\cdot)}(\Omega))^* = L^{s'(\cdot)}(\Omega)$ and the space $(L^{s(\cdot)}(\Omega), \|\cdot\|_{s(\cdot)})$ is a reflexive and separable Banach space

Proposition 2.1. (see [37]) *Let $u_n, u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

- (i): $\|u\|_{s(\cdot)} < 1$ (resp, $= 1, > 1$) $\iff \rho_{s(\cdot)}(u) < 1$ (resp, $= 1, > 1$),
- (ii): $\|u\|_{s(\cdot)} > 1 \implies \|u\|_{s(\cdot)}^{s^-} \leq \rho_{s(\cdot)}(u) \leq \|u\|_{s(\cdot)}^{s^+}$ and $\|u\|_{s(\cdot)} < 1 \implies \|u\|_{s(\cdot)}^{s^+} \leq \rho_{s(\cdot)}(u) \leq \|u\|_{s(\cdot)}^{s^-}$,
- (iii): $\|u_n\|_{s(\cdot)} \rightarrow 0 \iff \rho_{s(\cdot)}(u_n) \rightarrow 0$, and $\|u_n\|_{s(\cdot)} \rightarrow \infty \iff \rho_{s(\cdot)}(u_n) \rightarrow \infty$,

For more details on variable exponent Lebesgue spaces we refer the reader to [22].

Now, we introduce the anisotropic Sobolev space with variable exponents:

Let $\vec{p} = (1, p_1(\cdot), \dots, p_d(\cdot))$ be N variable exponents where $p_i(\cdot) \in \mathcal{C}_+(\Omega)$ for $i = 1, \dots, d$. We denote

$$D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, d,$$

Let

$$\underline{p} = \min\{p_1^-, \dots, p_d^-\} \quad \text{and} \quad p_0 = \max\{p_1^-, \dots, p_d^-\}. \quad (2.1)$$

The anisotropic Sobolev space with variable exponents is defined by

$$W^{1, \vec{p}(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for } i = 1, 2, \dots, d\},$$

The space $W^{1, \vec{p}(\cdot)}(\Omega)$ is equipped with the norm

$$\|u\|_{1, \vec{p}(\cdot)} = \|u\|_{1,1} + \sum_{i=1}^d \|D^i u\|_{p_i(\cdot)}. \quad (2.2)$$

The space $(W^{1, \vec{p}(\cdot)}(\Omega), \|u\|_{1, \vec{p}(\cdot)})$ is a reflexive Banach space (see [32]).

We denote by $W_0^{1, \vec{p}(\cdot)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1, \vec{p}(\cdot)}(\Omega)$, and by $W^{-1, \vec{p}(\cdot)}(\Omega)$ its dual space, where $\vec{p}'(\cdot) = (p'_0(\cdot), p'_1(\cdot), \dots, p'_d(\cdot))$ and $\frac{1}{p'_i(\cdot)} + \frac{1}{p_i(\cdot)} = 1$ for $i = 0, 1, \dots, d$.

Remark 2.2. The two norms $\|u\|_{1, \vec{p}(\cdot)} = \sum_{i=1}^d \|D^i u\|_{p_i(\cdot)}$ and $\|u\|_{1, \vec{p}(\cdot)}$ are equivalent on the Sobolev space $W_0^{1, \vec{p}(\cdot)}(\Omega)$.

Lemma 2.3. *We have the following continuous and compact embedding:*

- if $\underline{p} < d$ then $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [\underline{p}, \underline{p}^*[,$ where $\underline{p}^* = \frac{\underline{d}\underline{p}}{d-\underline{p}},$
- if $\underline{p} = d$ then $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, +\infty[,$
- if $\underline{p} > d$ then $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega}).$

Definition 2.4. Let $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

Let $\psi : \Omega \mapsto \overline{\mathbb{R}}$ be a measurable function such that

$$\psi^+ \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad (2.3)$$

We consider the closed convex subset defined by:

$$K_\psi = \{u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ such that } u \geq \psi \text{ a.e. in } \Omega\}. \quad (2.4)$$

3. MAIN RESULTS:

This section is devoted to proof the existence of entropy solution for the unilateral problem associated to the elliptic equation (1.1) under general structural conditions.

Let $p_i(\cdot) \in \mathcal{M}(\overline{\Omega})$ for $i = 1, \dots, d$. We consider the following assumptions.

(H1) For $i = 1, \dots, d$, the functions $a_i(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ are Carathéodory functions and satisfy:

- (i) For any $\xi, \xi' \in \mathbb{R}^d$ with $\xi_i \neq \xi'_i$ for each $i = 1, \dots, d$, and for every $s \in \mathbb{R}$,

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0. \quad (3.1)$$

- (ii) For each $i = 1, \dots, d$, there exists a function $K_i(\cdot) \in L^{p'_i(\cdot)}(\Omega)$, and constants $\beta > 0$ and $0 \leq \sigma < \underline{p} - 1$ such that

$$|a_i(x, s, \xi)| \leq \beta (K_i(x) + |s|^{\frac{p_0}{p'_i(x)} - \sigma} + |\xi_i|^{p_i(x)-1}) \quad (3.2)$$

for almost every $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^d$.

- (iii) There exist some constants $b_0 > 0$ and $\lambda > 0$ such that for almost every $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^d$,

$$a_i(x, s, \xi)\xi_i \geq \frac{b_0|\xi_i|^{p_i(x)}}{(1+|s|)^\lambda} \quad \text{for } i = 1, \dots, d. \quad (3.3)$$

where

$$0 \leq \lambda < \min \left\{ \underline{p} - 1, \frac{p_0 - 1}{p'} \right\}.$$

(H2) For $i = 1, \dots, d$, the functions $g : \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ and $\phi : \Omega \times \mathbb{R} \mapsto \mathbb{R}^d$ are Carathéodory functions and satisfy:

- (i) There exists a positive function $g_0 \in L^1(\Omega)$ and a decreasing function $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that

$$(|\mathcal{H}(x, s, \xi)| \leq g_0(x) + h(|s|) \sum_{i=1}^d |\xi_i|^{p_i(x)}, \quad (3.4)$$

and

$$(1 + |s|)^\lambda h(|s|) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}).$$

and exists a positive constant $C > 0$ such that

$$(1 + |s|)^\lambda h(|s|) \leq \frac{C}{1 + |s|} \quad \text{for any } s \in \mathbb{R}. \quad (3.5)$$

- (ii) There exist some positive functions $c_i \in L^{m_i}(\Omega)$ such that

$$|\phi(x, s)| \leq c_i(x)(1 + |s|)^{\gamma \frac{p_i(x)-1}{p_i(x)}} \quad (3.6)$$

where

$$0 \leq \gamma < \min \left\{ p_0 - \lambda p' - 1, \frac{p_0}{\lambda + 1} - \frac{\lambda}{p - 1} \right\}$$

and

$$m_i(x) \geq \frac{p'_i(x)p_0}{p_0 - \gamma - \lambda p'} \quad \text{a.e. in } \Omega.$$

Definition 3.1. A measurable function v is called an entropy solution for the unilateral problem associated to the degenerated elliptic equation (1.1) if

$$T_k(v) \in K_\psi \quad \text{for any } k > \|\psi^+\|_\infty, \quad \text{and} \quad \mathcal{H}(x, v, \nabla v) \in L^1(\Omega),$$

such that u satisfies the following inequality

$$\begin{aligned} & \sum_{i=1}^d \int_{\Omega} a_i(x, v, \nabla v) \cdot D^i T_k(v - \varphi) dx + \int_{\Omega} \mathcal{H}(x, v, \nabla v) T_k(v - \varphi) dx \\ & \leq \int_{\Omega} f T_k(v - \varphi) dx + \sum_{i=1}^d \int_{\Omega} \phi_i(x, v) D^i T_k(v - \varphi) dx \quad \text{for any } \varphi \in K_\psi \cap L^\infty(\Omega). \end{aligned} \quad (3.7)$$

The main result of this work is the following existence theorem.

Theorem 3.2. *Let $f \in L^1(\Omega)$. Assuming that the conditions (3.1) – (3.6) are satisfied. Then, there exists at least one entropy solution u to the unilateral problem associated to the strongly nonlinear anisotropic elliptic equation (1.1).*

4. PROOF OF THEOREM 3.2

Step 1: Approximate problems.

Let $n \in \mathbb{N}^*$, we consider the following sequence of the approximate problem:

$$\begin{cases} -\sum_{i=1}^d D^i a_i(x, T_n(v_n), \nabla v_n) + \mathcal{H}_n(x, v_n, \nabla v_n) = f_n - \sum_1^N D^i \phi_{n,i}(x, v_n) & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where $\mathcal{H}_n(x, s, \xi) = T_n(\mathcal{H}(x, s, \xi))$, $\phi_{n,i}(x, s) = \phi_i(x, T_n(s))$ and $f_n(x) = T_n(f(x))$. Note that $|f_n(x)| \leq n$ and $f_n(x) \rightarrow f(x)$ strongly in $L^1(\Omega)$.

We consider the operator B_n acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$ defined by

$$\begin{aligned} \langle B_n u, v \rangle &= \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i v \, dx + \int_{\Omega} \mathcal{H}_n(x, u, \nabla u) v \, dx \\ &\quad - \sum_{i=1}^d \int_{\Omega} \phi_{n,i}(x, u) D^i v \, dx, \end{aligned}$$

for any $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Lemma 4.1. *The operator $B_n = A_n + G_n$, acting from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$, is bounded and pseudomonotone. Furthermore B_n satisfies the following coercivity condition:*

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1,\vec{p}(\cdot)}} \rightarrow \infty \quad \text{as} \quad \|v\|_{1,\vec{p}(\cdot)} \rightarrow \infty \quad \text{for} \quad v \in K_{\psi}.$$

where v_0 belongs to K_{ψ} . The proof of Lemma 4.1 follows by applying methods analogous to those in [1].

According to Lemma 4.1 (cf. [31], Theorem 8.2), there exists at least one weak solution $v_n \in K_{\psi}$ for the unilateral problem associated to the approximate problem (4.1), i.e.

$$\begin{aligned} &\sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i (v_n - \varphi) dx + \int_{\Omega} \mathcal{H}_n(x, (x, T_n(v_n), \nabla v_n)) (v_n - \varphi) dx \\ &\leq \int_{\Omega} f_n(v_n - \varphi) dx + \sum_{i=1}^d \int_{\Omega} \phi_i(x, T_n(v_n)) D^i (v_n - \varphi) dx, \end{aligned} \quad (4.2)$$

for any $\varphi \in K_{\psi}$.

Step 2: Weak convergence of truncations.

We set $B(s) = 2 \int_0^s \frac{(1 + |\tau|)^{\lambda} h(|\tau|)}{b_0} d\tau$. Note that $0 \leq B(|s|) \leq B(\infty) < \infty$ is a finite real number.

Let $k \geq \max(1, \|\psi^+\|_{\infty})$ and $M = k + \|\psi^+\|_{\infty}$, we have $v = v_n - \eta T_k(v_n - \psi^+) e^{B(|v_n|)} \in K_{\psi}$ for $\eta > 0$ small enough. Thus v is an admissible test function

for the approximate problem (4.1) and we obtain

$$\begin{aligned}
& \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i T_k(v_n - \psi^+) e^{B(v_n)} dx \\
& + 2 \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& + \int_{\Omega} \mathcal{H}_n(x, (x, v_n, \nabla v_n) T_k(v_n - \psi^+) e^{B(|v_n|)} dx \\
& \leq \int_{\Omega} f_n T_k(v_n - \psi^+) e^{B(|v_n|)} dx + \sum_{i=1}^d \int_{\Omega} \phi_i(x, T_n(v_n)) D^i T_k(v_n - \psi^+) e^{B(|v_n|)} dx \\
& + 2 \sum_{i=1}^d \int_{\Omega} \phi_i(x, T_n(v_n)) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx.
\end{aligned} \tag{4.3}$$

By using (3.3), (3.4) and (3.6) we obtain

$$\begin{aligned}
& \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} a_i(x, T_n(v_n), \nabla v_n) D^i v_n dx + \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq k e^{B(\infty)} \int_{\Omega} (|f_n(x)| + |g_0(x)|) dx + \sum_{i=1}^d \int_{\Omega} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)}} |D^i T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& + \frac{2}{b_0} \sum_{i=1}^d \int_{\Omega} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)} + \lambda} h(|v_n|) |D^i v_n| |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& + \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} a_i(x, T_n(v_n), \nabla v_n) D^i \psi^+ dx.
\end{aligned} \tag{4.4}$$

Applying Young's inequality yields:

$$\begin{aligned}
& \sum_{i=1}^d \int_{\Omega} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)}} |D^i T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq C_1 \int_{\{|v_n - \psi^+| \leq k\}} |c_i(x)|^{p'_i(x)} (1 + |v_n|)^{\gamma + \frac{\lambda}{p_i(x)-1}} dx \\
& + \frac{b_0}{2^{p^+}} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i T_k(v_n - \psi^+)|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx \\
& \leq C_1 k^{\gamma + \frac{\lambda}{p^-}} \sum_{i=1}^d \int_{\Omega} |c_i(x)|^{p'_i(x)} dx + \frac{b_0}{2} \sum_{i=1}^d \int_{\Omega} |D^i \psi^+|^{p_i(x)} dx \\
& + \frac{b_0}{2} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx \\
& \leq C_2 k^{\gamma + \frac{\lambda}{p^-}} + \frac{b_0}{2} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx.
\end{aligned} \tag{4.5}$$

Similarly, using Young's inequality and (3.5), we obtain

$$\begin{aligned}
& \frac{2}{b_0} \sum_{i=1}^d \int_{\Omega} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)} + \lambda} h(|v_n|) |D^i v_n| |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \frac{b_0}{4} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \quad + C_3 \int_{\Omega} |c_i(x)|^{p'_i(x)} (1 + |v_n|)^{\gamma + \lambda p'_i(x)} h(|v_n|) |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \frac{b_0}{4} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \quad + C_3 \int_{\{|v_n| \leq M\}} |c_i(x)|^{p'_i(x)} (1 + |v_n|)^{\gamma + \lambda(p'_i(x)-1)} (1 + |v_n|)^{\lambda} h(|v_n|) |T_k(v_n - \psi^+)| dx \\
& \quad + k C_3 \int_{\{|v_n| > M\}} |c_i(x)|^{p'_i(x)} (1 + |v_n|)^{\gamma + \lambda p'_i(x)} h(|v_n|) dx \\
& \leq \frac{b_0}{4} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \quad + C_4 k^{\gamma + \frac{\lambda}{p-1}} \int_{\{|v_n| \leq M\}} |c_i(x)|^{p'_i(x)} dx + k C_4 \int_{\{|v_n| > M\}} |c_i(x)|^{p'_i(x)} |v_n|^{\gamma + \lambda p'_i(x)} h(|v_n|) dx \\
& \leq \frac{b_0}{4} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx + C_5 k^{\gamma + \frac{\lambda}{p-1}} \\
& \quad + k C_5 \int_{\{|v_n| > M\}} |c_i(x)|^{\frac{p'_i(x)p_0}{p_0 - \gamma - \lambda p'_i(x)}} h(|v_n|) dx + \frac{\varepsilon k b_0}{4} \int_{\{|v_n| > M\}} |v_n|^{p_0} h(|v_n|) dx.
\end{aligned} \tag{4.6}$$

Finally, for the last term in (4.6), we use Poincaré's inequality to obtain

$$\begin{aligned}
& \varepsilon k \int_{\{|v_n| > M\}} |v_n|^{p_0} h(|v_n|) dx \\
& \leq \varepsilon k \int_{\{|v_n| > M\}} (|v_n| - M)^{p_0} h(|v_n|) dx + \varepsilon k \int_{\{|v_n| > M\}} M^{p_0} h(|v_n|) dx \\
& \leq \varepsilon k \int_{\Omega} \left| \int_{|T_M(v_n)|}^{|v_n|} h(s)^{\frac{1}{p_0}} ds \right|^{p_0} dx + \varepsilon C_6 k^{p_0 - \lambda} \\
& \leq \varepsilon k C_{p_0}^{p_0} \int_{\Omega} \left| D^i \int_{|T_M(v_n)|}^{|v_n|} |h(s)|^{\frac{1}{p_0}} ds \right|^{p_0} dx + \varepsilon C_6 k^{p_0 - \lambda} \\
& \leq \varepsilon k C_{p_0}^{p_0} \int_{\{|v_n| > M\}} h(|v_n|) |D^i v_n|^{p_0} dx + \varepsilon C_6 k^{p_0 - \lambda} \\
& \leq \varepsilon C_{p_0}^{p_0} \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_0} |T_k(v_n - \psi^+)| dx + \varepsilon C_6 k^{p_0 - \lambda} \\
& \leq \frac{b_0}{4} \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| dx + \varepsilon C_6 k^{p_0 - \lambda}.
\end{aligned} \tag{4.7}$$

for $\varepsilon > 0$ small enough. Thus, we conclude that

$$\begin{aligned} & \frac{2}{b_0} \sum_{i=1}^d \int_{\Omega} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)} + \lambda} h(|v_n|) |D^i v_n| |T_k(v_n - \psi^+)| e^{B(|v_n|)} dx \\ & \leq \frac{b_0}{2} \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| dx + C_7 (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k). \end{aligned} \quad (4.8)$$

Therefore, combining (4.4) – (4.5) and (4.8) we deduce that

$$\begin{aligned} & \frac{b_0}{2} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx + \frac{b_0}{2} \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| dx \\ & \leq \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} a_i(x, v_n, \nabla v_n) D^i \psi^+ dx + C_8 (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k). \end{aligned} \quad (4.9)$$

By (3.1) and Young's inequality, the first behavior on the right-hand side of (4.4) is estimated as follows

$$\begin{aligned} & e^{B(\infty)} \int_{\{|v_n - \psi^+| \leq k\}} |a_i(x, T_n(v_n), \nabla v_n)| |D^i \psi^+| dx \\ & \leq \frac{b_0}{4\beta} \int_{\{|v_n - \psi^+| \leq k\}} \frac{|a_i(x, T_n(v_n), \nabla v_n)|^{p'_i(x)}}{(1 + |v_n|)^{\lambda}} dx + C_8 \int_{\{|v_n - \psi^+| \leq k\}} |D^i \psi^+|^{p_i(x)} (1 + |v_n|)^{\frac{\lambda}{p_i(x)-1}} dx \\ & \leq \frac{b_0}{4} \int_{\{|v_n - \psi^+| \leq k\}} \frac{|a_0|^{p'_i(x)}}{(1 + |v_n|)^{\lambda}} dx + \frac{b_0}{4} \int_{\{|v_n - \psi^+| \leq k\}} \frac{|v_n|^{\frac{p_0}{p'_i(x)} p'_i(x) - \sigma p'_i(x)}}{(1 + |v_n|)^{\lambda}} dx \\ & \quad + \frac{b_0}{4} \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx + C_9 (1 + k)^{\frac{\lambda}{p-1}} \int_{\{|v_n - \psi^+| \leq k\}} |D^i \psi^+|^{p_i(x)} dx \\ & \leq \frac{b_0}{4} \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx + C_{11} k^{p_0 - \sigma p'_0 - \lambda} + C_{12} k^{\frac{\lambda}{p-1}}. \end{aligned} \quad (4.10)$$

As a result by combining (4.9), and (4.10) we deduce that

$$\begin{aligned} & \frac{b_0}{4} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^{\lambda}} dx + \frac{b_0}{2} \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_k(v_n - \psi^+)| dx \\ & \leq C_9 (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k).. \end{aligned} \quad (4.11)$$

Thanks to (4.11), there exists a constant C_{10} not depending on k and n , such that

$$\sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k\}} |D^i v_n|^{p_i(x)} dx \leq C_{10} (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k). \quad (4.12)$$

On the other hand, we can write: $\{x \in \Omega, |v_n| \leq k\} \subset \{x \in \Omega, |v_n - \psi^+| \leq k + \|\psi^+\|_\infty\}$ and since $\|\psi^+\|_\infty \leq k$, as a result

$$\begin{aligned} \sum_{i=1}^d \int_{\Omega} |D^i T_k v_n|^{p_i(x)} dx &= \sum_{i=1}^d \int_{\{|v_n| \leq k\}} |D^i v_n|^{p_i(x)} dx \\ &\leq \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq k + \|\psi^+\|_\infty\}} |D^i v_n|^{p_i(x)} dx \\ &\leq C_{11} (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k). \end{aligned} \quad (4.13)$$

Hence

$$\|T_k(v_n)\|_{1, \vec{p}(\cdot)} \leq C_{12}(k),$$

Where $C_{12}(\lambda, k)$ denotes a positive constant independent of n . Therefore, the sequence $(T_k(v_n))_{n \in \mathbb{N}^*}$ is uniformly bounded in $W_0^{1, \vec{p}(\cdot)}(\Omega)$, and there exists a subsequence, again denoted by $(T_k(v_n))_{n \in \mathbb{N}}$ and a measurable function $v_k \in W_0^{1, \vec{p}(\cdot)}(\Omega)$ such that:

$$\begin{cases} T_k(v_n) \rightharpoonup v_k & \text{weakly in } W_0^{1, \vec{p}(\cdot)}(\Omega) \\ T_k(v_n) \rightarrow v_k & \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{cases} \quad (4.14)$$

Moreover, using Poincaré's inequality we have

$$\begin{aligned} k^{p_0} \text{meas}\{|v_n| > k\} &= \int_{\{|v_n| > k\}} |T_k(v_n)|^{p_0} dx \\ &\leq C_{p_0}^{p_0} \int_{\Omega} |D^{i_0} T_k(v_n)|^{p_0} dx \\ &\leq C_{p_0}^{p_0} C_{13} (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k). \end{aligned} \quad (4.15)$$

then

$$\begin{aligned} \text{meas}\{|v_n| > k\} &\leq \frac{C_{p_0}^{p_0} C_{13} (k^{\gamma + \frac{\lambda}{p-1}} + k^{p_0 - \lambda} + k)}{k^{p_0}} \rightarrow 0 \\ \text{as } k &\rightarrow \infty \end{aligned} \quad (4.16)$$

Therefore, similarly as in [1], we show that $(v_n)_n$ is a Cauchy in measure. Therefore, there exists a subsequence that convergence almost everywhere to some measurable function u . Using (4.14), we conclude that:

$$\begin{cases} T_k(v_n) \rightharpoonup T_k(v) & \text{weakly in } W_0^{1, \vec{p}(\cdot)}(\Omega) \\ T_k(v_n) \rightarrow T_k(v) & \text{strongly in } L^{p_0(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \quad (4.17)$$

Step 3: The equi-integrability of the sequences $(|v_n|^r h(|v_n|))_n$.

Let $\delta > 0, r > 0$ such that $r < r + \delta < p_0$ thanks to (4.7) and (4.11) then

$$\int_{\Omega} |v_n|^{r+\delta} h(|v_n|) dx \leq C \quad (4.18)$$

In view of (4.18), for every measurable subset $E \subset \Omega$ and for all $h > 0$, we get:

$$\begin{aligned}
\int_E |v_n|^r h(|v_n|) dx &\leq \int_{E \cap \{|v_n| \leq h\}} |v_n|^r h(|v_n|) dx + \int_{E \cap \{|v_n| > h\}} |v_n|^r h(|v_n|) dx \\
&\leq \int_E |T_h(v_n)|^r h(|v_n|) dx + \frac{1}{h^\delta} \int_{E \cap \{|v_n| > h\}} |v_n|^{r+\delta} h(|v_n|) dx \\
&\leq \int_E |T_h(v_n)|^r h(|v_n|) dx + \frac{C}{h^\delta}
\end{aligned} \tag{4.19}$$

On the other hand there exists $\beta(\varepsilon, h) > 0$ such that: for all $E \subset \Omega$ with $\text{meas}(E) \leq \beta(\varepsilon, h)$ we have

$$\int_E |T_h(v_n)|^r h(|v_n|) dx \leq \frac{\varepsilon}{2} \tag{4.20}$$

Since

$$\frac{C}{h^\delta} \longrightarrow 0 \quad \text{as } h \longrightarrow \infty$$

This implies, for each $\varepsilon > 0$, there exists a constant $\beta(\varepsilon) > 0$ satisfying:

$$\int_E |v_n|^r h(|v_n|) dx \leq \varepsilon \quad \text{for any } 0 < r < p_0 \tag{4.21}$$

Then

$$|v_n|^r h(|v_n|) \longrightarrow |u|^r h(|v|) \quad \text{in } L^1(\Omega) \quad \text{for any } 0 < r < p_0. \tag{4.22}$$

Step 4: Some regularity results.

We define $\varepsilon_i(n)$ for $i = 1, 2, \dots$ some various real valued function verifying $\lim_{n \rightarrow \infty} \varepsilon_i(n) = 0$. Similarly, we define $\varepsilon_i(h)$ $\varepsilon_i(n, h)$.

We will show that

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq h\}} a_i(x, T_h(v_n), \nabla v_n) D^i v_n dx = 0. \tag{4.23}$$

Indeed, let $h \geq \max(\|\psi^+\|_\infty, 1)$ we set

$$\varphi = v_n - \eta \frac{T_h(v_n - \psi^+)}{h} e^{B(|v_n|)} \in K_\psi$$

We can use φ as a test function in (4.1) and we obtain

$$\begin{aligned}
& \frac{1}{h} \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \\
& + \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} \left| \frac{T_h(v_n - \psi^+)}{h} \right| e^{B(|v_n|)} dx \\
& + \int_{\Omega} \mathcal{H}_n(x, (x, v_n, \nabla v_n)) \left(\frac{T_h(v_n - \psi^+)}{h} e^{B(|v_n|)} \right) dx \\
& \leq \int_{\Omega} f_n \frac{T_h(v_n - \psi^+)}{h} e^{B(|v_n|)} dx \\
& + \sum_{i=1}^d \int_{\Omega} \phi_i(x, v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} \left| \frac{T_h(v_n - \psi^+)}{h} \right| e^{B(|v_n|)} dx \\
& + \frac{1}{h} \sum_{i=1}^d \int_{\Omega} \phi_i(x, v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx
\end{aligned} \tag{4.24}$$

By (3.4) we have

$$\begin{aligned}
& \frac{1}{h} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq h\}} a_i(x, T_n(v_n), \nabla v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \\
& + \frac{1}{h} \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \frac{1}{h} \int_{\Omega} (|f(x)| + |g_0(x)|) |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& + \frac{1}{h} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq h\}} \phi_i(x, v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \\
& + \frac{1}{h} \sum_{i=1}^d \int_{\Omega} \phi_i(x, v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx
\end{aligned} \tag{4.25}$$

We know that $\lim_{h \rightarrow \infty} |\{|v_n| > h\}| = 0$ and that $\psi^+ \in L^\infty(\Omega)$. Consequently,

$$\frac{|T_h(v_n - \psi^+)|}{h} \xrightarrow{*} 0 \quad \text{weak in } L^\infty(\Omega) \text{ as } h \rightarrow \infty.$$

Hence,

$$\begin{aligned}
& \frac{1}{h} \int_{\Omega} (|f| + |g_0(x)|) |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \frac{e^{B(\infty)}}{h} \int_{\Omega} (|g_0(x)| + |f|) |T_h(v_n - \psi^+)| dx \longrightarrow 0 \quad \text{as } h \rightarrow \infty.
\end{aligned} \tag{4.26}$$

Similarly to (4.5) we obtain

$$\begin{aligned}
& \left| \frac{1}{h} \int_{\{|v_n - \psi^+| \leq h\}} \phi_i(x, v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \right| \\
& \leq \frac{1}{h} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq h\}} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)}} D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \\
& \leq \frac{C_1}{h} \int_{\{|v_n - \psi^+| \leq h\}} |c_i(x)|^{p'_i(x)} (1 + |v_n|)^{\gamma + \frac{\lambda}{p_i(x)-1}} dx \\
& \quad + \frac{b_0}{4h} \int_{\{|v_n - \psi^+| \leq h\}} \frac{|D^i T_h(v_n - \psi^+)|^{p_i(x)}}{(1 + |v_n|)^\lambda} dx \\
& \leq \frac{Ch^{\frac{\lambda}{q-1}}}{h} \int_{\{|v_n - \psi^+| \leq h\}} |c_i(x)|^{\frac{p'_i(x)p_0}{p_0 - \gamma - \lambda p'_i(x)}} dx + \frac{\varepsilon}{h(1+h)^\lambda} \int_{\{|v_n - \psi^+| \leq h\}} |v_n|^{p_0} dx \\
& \quad + \frac{b_0}{4h} \int_{\{|v_n - \psi^+| \leq h\}} \frac{|D^i v_n|^{p_i(x)}}{(1 + |v_n|)^\lambda} |T_h(v_n - \psi^+)| dx
\end{aligned} \tag{4.27}$$

Since $|c_i(x)|^{\frac{p'_i(x)p_0}{p_0 - \gamma - \lambda p'_i(x)}}$ belongs to $L^1(\Omega)$, we obtain

$$\varepsilon(h) = \frac{Ch^{\frac{\lambda}{q-1}}}{h} \int_{\{|v_n - \psi^+| \leq h\}} |c_i(x)|^{p'_i(x)q'} dx \longrightarrow 0 \quad \text{as } h \rightarrow \infty. \tag{4.28}$$

One other hand, thanks to Poincaré's inequality, and since $p_0 = \max\{p_1^-, \dots, p_N^-\}$, then there exists some $i_0 = \{1, 2, \dots, N\}$ such that $p_0 = p_{i_0}^-$ and thanks to Poincaré's inequality, it's clear that we get

$$\begin{aligned}
& \frac{\varepsilon}{h(1+h)^\lambda} \int_{\{|v_n - \psi^+| \leq h\}} |v_n|^{p_0} dx \\
& \leq \frac{\varepsilon}{h(1+h)^\lambda} \int_{\{|v_n - \psi^+| \leq h\}} |T_h(v_n - \psi^+)|^{p_0} dx + \varepsilon_2(h) \\
& \leq \frac{\varepsilon C_{p_0}^{p_0}}{h(1+h)^\lambda} \int_{\Omega} |D^{i_0} T_h(v_n - \psi^+)|^{p_{i_0}^-} dx + \varepsilon_2(h) \\
& \leq \frac{b_0}{4h(1+h)^\lambda} \int_{\Omega} |D^{i_0} T_h(v_n - \psi^+)|^{p_{i_0}(x)} dx + \varepsilon_3(h)
\end{aligned} \tag{4.29}$$

Combining (4.27) – (4.29), we deduce that

$$\begin{aligned}
& \left| \frac{1}{h} \int_{\{|v_n - \psi^+| \leq h\}} \phi_i(x, v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \right| \\
& \leq \frac{b_0}{4h(1+h)^\lambda} \int_{\Omega} |D^i T_h(v_n - \psi^+)|^{p_i(x)} dx + \varepsilon_4(n, h).
\end{aligned} \tag{4.30}$$

By assumption (3.6) and Young's inequality, it follows that

$$\begin{aligned}
& \left| \frac{1}{h} \sum_{i=1}^d \int_{\Omega} \phi_i(x, v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \right| \\
& \leq \frac{1}{b_0 h} \int_{\Omega} c_i(x) (1 + |v_n|)^{\frac{\gamma(p_i(x)-1)}{p_i(x)} + \lambda} h(|v_n|) |D^i v_n| |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \frac{C}{h} \int_{\Omega} |c_i(x)|^{p'_i(x)} (1 + |v_n|)^{\gamma + \lambda p'_i(x)} h(|v_n|) |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \quad + \frac{1}{2h} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \frac{C}{h} \int_{\Omega} |c_i(x)|^{p'_i(x)q'_\varepsilon} |T_h(v_n - \psi^+)| dx + \frac{1}{h} \int_{\Omega} |v_n|^{p_0 - \varepsilon} h(|v_n|) |T_h(v_n - \psi^+)| dx \\
& \quad + \frac{1}{2h} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_h(v_n - \psi^+)| dx
\end{aligned} \tag{4.31}$$

where $q_\varepsilon = \frac{p_0 - \varepsilon}{\gamma + \frac{\lambda}{p-1}}$ such that ε small enough. In view of (4.22) and since $|c_i(x)|^{p'_i(x)q'_\varepsilon}$ belong to $L^1(\Omega)$ we deduce that

$$\begin{aligned}
& \frac{C}{h} \int_{\Omega} |c_i(x)|^{p'_i(x)q'_\varepsilon} |T_h(v_n - \psi^+)| dx + \frac{1}{h} \int_{\Omega} |v_n|^{p_0 - \varepsilon} h(|v_n|) |T_h(v_n - \psi^+)| dx \\
& \quad \longrightarrow 0 \quad \text{as } n, h \longrightarrow \infty.
\end{aligned} \tag{4.32}$$

Hence

$$\begin{aligned}
& \left| \frac{1}{h} \sum_{i=1}^d \int_{\Omega} \phi_i(x, v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \right| \\
& \leq \varepsilon_5(n, h) + \frac{b_0}{2h} \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |T_h(v_n - \psi^+)| dx
\end{aligned} \tag{4.33}$$

By combining (4.25), (4.26), (4.30) and (4.33), we deduce that

$$\begin{aligned}
& \frac{1}{h} \sum_{i=1}^d \int_{\{|v_n - \psi^+| \leq h\}} a_i(x, T_n(v_n), \nabla v_n) D^i T_h(v_n - \psi^+) e^{B(|v_n|)} dx \\
& + \frac{1}{h} \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} |T_h(v_n - \psi^+)| e^{B(|v_n|)} dx \\
& \leq \varepsilon_6(n, h)
\end{aligned} \tag{4.34}$$

Now, by letting h go to infinity in (4.34), we obtain

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \sum_{i=1}^d \int_{\{|v_n| \leq h\}} a_i(x, T_n(v_n), \nabla v_n) D^i v_n dx = 0. \tag{4.35}$$

Furthermore, we have:

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{|v_n| > h\}} h(|v_n|) |D^i T_h v_n|^{p_i(x)} dx = 0, \tag{4.36}$$

and in view of (4.30) and (4.35) we obtain

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \int_{\{|v_n - \psi^+| \leq h\}} |\phi_i(x, v_n)| |D^i v_n| dx = 0. \quad (4.37)$$

Step 5: Almost everywhere convergence of the gradients.

We set

$$S_h(s) = 1 - \frac{|T_{2h}(s) - T_h(s)|}{h} \quad \text{and} \quad \varphi(s) = s \cdot \exp\left(\frac{\rho^2 s^2}{2}\right) \quad \text{for any } s \in \mathbb{R},$$

where $h > k \geq \max\{1, \|\psi^+\|_\infty\}$ and $\rho = 2 \left\| (1 + |s|)^\lambda h(|s|) \right\|_{L^\infty(\mathbb{R})}$. Evidently,

$$|\varphi'(s) - \rho \varphi(s)| \geq \frac{1}{2} \quad \text{for any } s \in \mathbb{R}.$$

Let η small enough such that $\varphi = v_n - \eta \varphi(T_k(v_n) - T_k(v)) S_h(v_n) e^{B(|v_n|)}$ belongs to K_ψ , by taking v as a test function in the approximate problem (4.2), we obtain

$$\mathcal{K}_1^{n,k} + \mathcal{K}_2^{n,k} + \mathcal{K}_3^{n,k} + \mathcal{K}_4^{n,k} = \mathcal{K}_5^{n,k} + \mathcal{K}_6^{n,k} + \mathcal{K}_7^{n,k} + \mathcal{K}_8^{n,k}, \quad (4.38)$$

where

$$\begin{aligned} \mathcal{K}_1^{n,k} &= \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) S_h(v_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_2^{n,k} &= -\frac{1}{h} \sum_{i=1}^d \int_{\{h < |v_n| \leq 2h\}} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \text{sign}(v_n) \varphi(\omega_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_3^{n,k} &= \sum_{i=1}^d \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} \text{sign}(v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_4^{n,k} &= \int_{\Omega} \mathcal{H}_n(x, (x, v_n, \nabla v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_5^{n,k} &= \sum_{i=1}^d \int_{\Omega} \phi_{n,i}(x, v_n) (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) S_h(v_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_6^{n,k} &= -\frac{1}{h} \sum_{i=1}^d \int_{\{h < |v_n| \leq 2h\}} \phi_{n,i}(x, v_n) D^i v_n \text{sign}(v_n) \varphi(\omega_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_7^{n,k} &= \sum_{i=1}^d \int_{\Omega} \phi_{n,i}(x, v_n) D^i v_n \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} \text{sign}(v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx. \\ \mathcal{K}_8^{n,k} &= \int_{\Omega} f_n(x) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx. \end{aligned}$$

For the first behavior $\mathcal{K}_1^{n,k}$:

We have $S_h(v_n) = 1$ on the set $\{|v_n| \leq h\}$, and since $a_i(x, s, 0) = 0$, we obtain

$$\begin{aligned}
\mathcal{K}_1^{n,k} &= \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v_n)) \cdot (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) e^{B(|v_n|)} dx \\
&\quad - \sum_{i=1}^d \int_{\{|v_n| > k\}} a_i(x, T_n(v_n), \nabla v_n) D^i T_k(v) \varphi'(\omega_n) S_h(v_n) e^{B(|v_n|)} dx \\
&\geq \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\
&\quad \times (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) e^{B(|v_n|)} dx \\
&\quad - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v)) (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) e^{B(|v_n|)} dx \\
&\quad - \sum_{i=1}^d \int_{\{|v_n| > k\}} a_i(x, T_{2h}(v_n), \nabla T_{2h}(v_n)) D^i T_k(v) \varphi'(\omega_n) e^{B(|v_n|)} dx.
\end{aligned} \tag{4.39}$$

It manifest to see from $T_k(v_n) \rightarrow T_k(v)$ in $L^{p_i(\cdot)}(\Omega)$ and dominated convergence theorem That

$$|a_i(x, T_k(v_n), \nabla T_k(v))| \longrightarrow a_i(x, T_k(v), \nabla T_k(v)) \quad \text{strongly in } L^{p'_i}(\Omega),$$

and since $D^i T_k(v_n)$ converges to $D^i T_k(v)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\begin{aligned}
\varepsilon_8(n) &= \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v)) (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) e^{B(|v_n|)} dx \\
&\longrightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{4.40}$$

Look at the last term on the right-hand side of (4.39), thanks to (3.1), we have $(a_i(x, T_{2h}(v_n), \nabla T_{2h}(v_n)))_n$ is bounded in $L^{p'_i}(\Omega)$, it follows that: there exists a measurable function $\vartheta_{i,h} \in L^{p'_i(\cdot)}(\Omega)$ such that $a_i(x, T_{2h}(v_n), \nabla T_{2h}(v_n)) \rightharpoonup \vartheta_{i,h}$ weakly in $L^{p'_i(\cdot)}(\Omega)$. Hence

$$\begin{aligned}
\varepsilon_9(n) &= \sum_{i=1}^d \int_{\{|v_n| > k\}} a_i(x, T_{2h}(v_n), \nabla T_{2h}(v_n)) D^i T_k(v) \varphi'(\omega_n) e^{B(|v_n|)} dx \\
&\longrightarrow \varphi'(2k) e^{B(\infty)} \sum_{i=1}^d \int_{\{|v| > k\}} \vartheta_{i,h} D^i T_k(v) \varphi'(0) e^{B(|v|)} dx = 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{4.41}$$

By combining (4.39) – (4.41), we get

$$\begin{aligned}
\mathcal{K}_1^{n,k} &\geq \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\
&\quad \times (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) e^{B(|v_n|)} dx + \varepsilon_{10}(n).
\end{aligned} \tag{4.42}$$

For the second behavior $\mathcal{K}_2^{n,k}$:

By using (4.35), we get

$$|\mathcal{K}_2^{n,k}| \leq \frac{|\varphi(2k)|e^{B(\infty)}}{h} \sum_{i=1}^d \int_{\{h < |v_n| \leq 2h\}} a_i(x, v_n, \nabla v_n) D^i v_n \, dx \rightarrow 0 \quad \text{as } n, h \rightarrow \infty. \quad (4.43)$$

For the third behavior $\mathcal{K}_3^{n,k}$:

We have $\varphi(\omega_n)\text{sign}(v_n) = |\varphi(\omega_n)|$ on the set $\{|v_n| > k\}$, then

$$\begin{aligned} \mathcal{K}_3^{n,k} &= \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v_n)) D^i T_k(v_n) \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} \text{sign}(v_n) \varphi(\omega_n) e^{B(|v_n|)} \, dx \\ &\quad + \sum_{i=1}^d \int_{\{|v_n| > k\}} a_i(x, T_k(v_n), \nabla T_n(v_n)) D^i v_n \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} \, dx \\ &\geq - \left\| (1 + |s|)^\lambda h(|s|) \right\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\ &\quad \times (D^i T_k(v_n) - D^i T_k(v)) |\varphi(\omega_n)| e^{B(|v_n|)} \, dx \\ &\quad - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v_n)) D^i T_k(v) |\varphi(\omega_n)| \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} e^{B(|v_n|)} \, dx \\ &\quad - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v)) (D^i T_k(v_n) - D^i T_k(v)) \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} |\varphi(\omega_n)| e^{B(|v_n|)} \, dx \\ &\quad + \sum_{i=1}^d \int_{\{|v_n| > k\}} h(|v_n|) |D^i v_n|^{p_i(x)} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} \, dx. \end{aligned} \quad (4.44)$$

Concerning the second term on the right-hand side of (4.44), it manifests to see $a_i(x, T_k(v_n), \nabla T_k(v_n)) \rightharpoonup \vartheta_{k,i}$ weakly in $L^{p'_i(\cdot)}(\Omega)$, with $\vartheta_{k,i}$ is a function in $L^{p'_i(\cdot)}(\Omega)$. And thanks to the Lebesgue dominated convergence theorem, we get $D^i T_k(v) \varphi(\omega_n) \rightarrow 0$ strongly in $L^{p_i(\cdot)}(\Omega)$. Therefore

$$\begin{aligned} \varepsilon_{11}(n) &= - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v_n)) D^i T_k(v) |\varphi(\omega_n)| \\ &\quad \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} e^{B(|v_n|)} \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.45)$$

Similary to (4.40), we obtain

$$\begin{aligned} \varepsilon_{12}(n) &= - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v)) (D^i T_k(v_n) - D^i T_k(v)) \\ &\quad \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} |\varphi(\omega_n)| e^{B(|v_n|)} \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.46)$$

By combining (4.44) and (4.45) – (4.46) that

$$\begin{aligned} \mathcal{K}_3^{n,k} &\geq - \left\| (1 + |s|)^\lambda h(|s|) \right\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\ &\quad \times |D^i T_k(v_n) - D^i T_k(v)| \varphi(\omega_n) |e^{B(|v_n|)}| dx \\ &\quad + \sum_{i=1}^d \int_{\{|v_n| > k\}} h(|v_n|) |D^i v_n|^{p_i(x)} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} dx + \varepsilon_{13}(n). \end{aligned} \quad (4.47)$$

For the fourth behavior $\mathcal{K}_4^{n,k}$:

In view of (3.3) and (3.4) we get

$$\begin{aligned} |\mathcal{K}_4^{n,k}| &\leq \left| \int_{\Omega} \mathcal{H}_n(x, (x, v_n, \nabla v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx \right| \\ &\leq e^{B(\infty)} \int_{\Omega} |g_0| |\varphi(\omega_n)| dx + \sum_{i=1}^d \int_{\Omega} h(|v_n|) |D^i v_n|^{p_i(x)} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} dx \\ &\leq e^{B(\infty)} \int_{\Omega} |g_0| |\varphi(\omega_n)| dx + \sum_{i=1}^d \int_{\{|v_n| > k\}} h(|v_n|) |D^i v_n|^{p_i(x)} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} dx \\ &\quad + \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v_n)) D^i T_k(v_n) \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} |\varphi(\omega_n)| e^{B(|v_n|)} dx, \end{aligned} \quad (4.48)$$

Furthermore, we have $|\varphi(\omega_n)| \xrightarrow{*} 0$ weak in $L^\infty(\Omega)$, then

$$\varepsilon_{14}(n) = e^{B(\infty)} \int_{\Omega} |g_0| |\varphi(\omega_n)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.49)$$

According to (4.47), which implies that

$$\begin{aligned} \mathcal{K}_4^{n,k} &\geq - \left\| (1 + |s|)^\lambda h(|s|) \right\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\ &\quad \times (D^i T_k(v_n) - D^i T_k(v)) |\varphi(\omega_n)| e^{B(|v_n|)} dx \\ &\quad - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v_n)) (D^i T_k(v)) \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} |\varphi(\omega_n)| e^{B(|v_n|)} dx \\ &\quad - \sum_{i=1}^d \int_{\Omega} a_i(x, T_k(v_n), \nabla T_k(v)) (D^i T_k(v_n) - D^i T_k(v)) \frac{(1 + |v_n|)^\lambda h(|v_n|)}{b_0} |\varphi(\omega_n)| e^{B(|v_n|)} dx \\ &\quad - \sum_{i=1}^d \int_{\{|v_n| > k\}} h(|v_n|) |D^i v_n|^{p_i(x)} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} dx \\ &\geq - \left\| (1 + |s|)^\lambda h(|s|) \right\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\ &\quad \times (D^i T_k(v_n) - D^i T_k(v)) |\varphi(\omega_n)| e^{B(|v_n|)} dx \\ &\quad - \sum_{i=1}^d \int_{\{|v_n| > k\}} h(|v_n|) |D^i v_n|^{p_i(x)} |\varphi(\omega_n)| S_h(v_n) e^{B(|v_n|)} dx + \varepsilon_{15}(n) \end{aligned} \quad (4.50)$$

For the fifth behavior $\mathcal{K}_5^{n,k}$:

We have

$$\mathcal{K}_5^{n,k} = \int_{\Omega} \phi_{n,i}(x, T_{2h}(v_n)) \cdot (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) S_h(v_n) e^{B(|v_n|)} dx \quad (4.51)$$

Firstly, in view of Lebesgue dominated convergence theorem, we have

$$\phi_i(x, T_{2h}(v_n)) \varphi'(\omega_n) S_h(v_n) e^{B(|v_n|)} \rightarrow \phi_i(x, T_{2h}(v)) \varphi'(0) S_h(v) e^{B(|v|)} \quad \text{strongly in } L^{p'_i(\cdot)}(\Omega),$$

and since $D^i T_k(v_n)$ converges to $D^i T_k(v)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\varepsilon_{17}(n) = \sum_{i=1}^d \int_{\Omega} |\phi_{n,i}(x, T_{2h}(v_n)) (D^i T_k(v_n) - D^i T_k(v)) \varphi'(\omega_n) S_h(v_n) e^{B(|v_n|)}| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.52)$$

For the sixth behavior $\mathcal{K}_6^{n,k}$:

Thanks to (4.37), we get

$$\begin{aligned} \varepsilon_{18}(n, h) &= |\mathcal{K}_6^{n,k}| \\ &\leq \frac{|\varphi(2k)| e^{B(\infty)}}{h} \sum_{i=1}^d \int_{\{h < |v_n| \leq 2h\}} |\phi_{n,i}(x, v_n)| |D^i v_n| dx \rightarrow 0 \quad \text{as } n, h \rightarrow \infty. \end{aligned} \quad (4.53)$$

For the seventh behavior $\mathcal{K}_7^{n,k}$:

We have

$$\mathcal{K}_7^{n,k} = \sum_{i=1}^d \int_{\Omega} \phi_{n,i}(x, T_{2h}(v_n)) D^i T_{2h}(v_n) \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} \text{sign}(v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx, \quad (4.54)$$

Applying the dominated convergence theorem, it follows that:

$$\phi_i(x, T_{2h}(v_n)) \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} \text{sign}(v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} \rightarrow 0 \quad \text{strongly in } L^{p'_i(\cdot)}(\Omega),$$

and since $D^i T_{2h}(v_n)$ converges to $D^i T_{2h}(v)$ weakly in $L^{p_i(\cdot)}(\Omega)$. Hence

$$\begin{aligned} \varepsilon_{19}(n, h) = \mathcal{K}_7^{n,k} &= \sum_{i=1}^d \int_{\Omega} \phi_{n,i}(x, T_{2h}(v_n)) D^i T_{2h}(v_n) \frac{(1 + |v_n|)^{\lambda} h(|v_n|)}{b_0} \\ &\quad \cdot \text{sign}(v_n) \varphi(\omega_n) S_h(v_n) e^{B(|v_n|)} dx \rightarrow 0 \quad \text{as } n, h \rightarrow \infty. \end{aligned} \quad (4.55)$$

For the eighth behavior $\mathcal{K}_8^{n,k}$:

In view of definition of f_n , and since $|\varphi(\omega_n)| \xrightarrow{*} 0$ weak in $L^{\infty}(\Omega)$ as $n \rightarrow \infty$, then

$$\begin{aligned} \varepsilon_{21}(n) &= |\mathcal{K}_8^{n,k}| \\ &\leq e^{B(\infty)} \int_{\Omega} |f_n(x)| |\varphi(\omega_n)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.56)$$

Finally, since $\left(\varphi'(\omega_n) - 2 \left\| (1 + |s|)^{\lambda} h(|s|) \right\|_{L^{\infty}(\mathbb{R})} |\varphi(\omega_n)| \right) e^{B(|v_n|)} \geq \frac{1}{2}$, by combining (4.38) – (4.43), (4.47), (4.47), (4.50), (4.52) – (4.53) and (4.55) – (4.56) we conclude

that

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \\
& \quad \times (D^i T_k(v_n) - D^i T_k(v)) \, dx + \varepsilon_{22}(n, h) \\
& \leq \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) (D^i T_k(v_n) - D^i T_k(v)) \\
& \quad \times \left(\varphi'(\omega_n) - 2 \left\| (1 + |s|)^{\lambda} h(|s|) \right\|_{L^{\infty}(\mathbb{R})} |\varphi(\omega_n)| \right) e^{B(|v_n|)} \, dx + \varepsilon_{23}(n, h) \\
& \leq \mathcal{K}_1^{n,k} + \mathcal{K}_2^{n,k} + \mathcal{K}_3^{n,k} + \mathcal{K}_4^{n,k} - |\mathcal{K}_7^{n,k}| \\
& \leq |\mathcal{K}_5^{n,k}| + |\mathcal{K}_6^{n,k}| + |\mathcal{K}_8^{n,k}| \\
& \leq \varepsilon_{24}(n, h).
\end{aligned} \tag{4.57}$$

Thus, letting n and h tend to infinity in the preceding inequality, and since $T_k(v_n) \rightarrow T_k(v)$ holds in $L^p(\Omega)$ we deduce that:

$$\begin{aligned}
& \sum_{i=1}^d \int_{\Omega} (a_i(x, T_k(v_n), \nabla T_k(v_n)) - a_i(x, T_k(v_n), \nabla T_k(v))) \cdot (D^i T_k(v_n) - D^i T_k(v)) \, dx \\
& \quad + \int_{\Omega} (|T_k(v_n)|^{p-2} T_k(v_n) - |T_k(v)|^{p-2} T_k(v)) (T_k(v_n) - T_k(v)) \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{4.58}$$

By using Lemma 3.3 in [12], we deduce that:

$$T_k(v_n) \longrightarrow T_k(v) \quad \text{strongly in } W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad \nabla v_n \longrightarrow \nabla v \quad \text{a.e. in } \Omega. \tag{4.59}$$

Therefore, for each $i = 1, \dots, d$, $a_i(x, T_n(v_n), \nabla v_n) D^i v_n$ converges almost everywhere to $a_i(x, v, \nabla v) D^i v$ in Ω , By applying Fatou's lemma together with (4.35), it follows that:

$$\begin{aligned}
& \lim_{h \rightarrow \infty} \frac{1}{h} \int_{\{|v| \leq h\}} a_i(x, v, \nabla v) D^i v \, dx \\
& \leq \lim_{h \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{h} \int_{\{|v_n| \leq h\}} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \, dx \\
& \leq \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \int_{\{|v_n| \leq h\}} a_i(x, T_n(v_n), \nabla v_n) D^i v_n \, dx = 0.
\end{aligned} \tag{4.60}$$

Step 6: Equi-integrability of the sequence $\mathcal{H}_n(x, v_n, \nabla v_n)$.

In this section, we will establish that:

$$\mathcal{H}_n(x, v_n, \nabla v_n) \longrightarrow \mathcal{H}(x, v, \nabla v) \quad \text{strongly in } L^1(\Omega). \tag{4.61}$$

In view of (4.59) we have

$$\mathcal{H}_n(x, v_n, \nabla v_n) \longrightarrow \mathcal{H}(x, v, \nabla v) \quad \text{a.e. in } \Omega. \tag{4.62}$$

By applying Vitali's theorem, in order to prove (4.61) it's sufficient to show that the sequence $\mathcal{H}_n(x, v_n, \nabla v_n)$ is uniformly equi-integrable. More precisely, for all

measurable subset $E \subset \Omega$ and for any $h > 0$, we can write:

$$\begin{aligned} \int_E |\mathcal{H}_n(x, v_n, \nabla v_n)| \, dx &\leq \int_E |g_0| \, dx + \sum_{i=1}^d \int_E h(|v_n|) |D^i v_n|^{p_i(x)} \, dx \\ &\leq \int_E |g_0| \, dx + \sum_{i=1}^d \int_E h(|v_n|) |D^i T_h v_n|^{p_i(x)} \, dx + \sum_{i=1}^d \int_{\{|v_n| > h\}} h(|v_n|) |D^i T_h v_n|^{p_i(x)} \, dx. \end{aligned} \quad (4.63)$$

On the one hand, in view of (4.36), we have: for every $\varepsilon > 0$ there exists $h_0 > 0$ such that

$$\sum_{i=1}^d \int_{\{|v_n| > h\}} h(|v_n|) |D^i T_h v_n|^{p_i(x)} \, dx \leq \frac{\varepsilon}{2} \quad \text{for all } h \leq h_0 \quad (4.64)$$

On the other hand, in view of (4.59) there exists a positive constant $\beta(\varepsilon) > 0$ such that: for every measurable set $E \subset \Omega$ satisfying $\text{meas}(E) \leq \beta(\varepsilon)$:

$$\sum_{i=1}^d \int_E h(|v_n|) |D^i T_h v_n|^{p_i(x)} \, dx + \int_E |g_0| \, dx \leq \frac{\varepsilon}{2} \quad (4.65)$$

By combining (4.63), (4.64) and (4.65) we obtain

$$\int_\Omega |\mathcal{H}_n(x, v_n, \nabla v_n)| \, dx \leq \varepsilon \quad (4.66)$$

For all measurable subset $E \subset \Omega$ with $\text{meas}(E) \leq \beta(\varepsilon)$. Thus, the sequence $(\mathcal{H}_n(x, (x, v_n, \nabla v_n)))_n$ is uniformly equi-integrable. Consequently, the strong convergence (4.61) is concluded.

Step 7: Passage to the limit.

Now, we will pass to the limit in the approximate problem (4.1).

Let $\varphi \in K_\psi \cap L^\infty(\Omega)$ and we set $M = k + \|\varphi\|_\infty$. By choosing $v = v_n - \eta T_k(v_n - \varphi)$ as a test function in the approximate problem (4.1) we have

$$\begin{aligned} \sum_{i=1}^d \int_\Omega a_i(x, T_n(v_n), \nabla v_n) \, D^i T_k(v_n - \varphi) \, dx &+ \int_\Omega \mathcal{H}_n(x, v_n, \nabla v_n) T_k(v_n - \varphi) \, dx \\ &\leq \int_\Omega f_n T_k(v_n - \varphi) \, dx + \sum_{i=1}^d \int_\Omega \phi_{i,n}(x, T_n(v_n)) \, D^i T_k(v_n - \varphi) \, dx. \end{aligned} \quad (4.67)$$

On the one hand, in the case of $|v_n| > M$ we have $|v_n - \varphi| \geq |v_n| - \|\varphi\|_\infty \geq k$ then $\{|v_n - \varphi| \leq k\} \subseteq \{|v_n| \leq M\}$. By applying Fatou's lemma, the first term

of the left-hand side of (4.67) can be written as:

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} \int_{\Omega} a_i(x, T_n(v_n), \nabla v_n) D^i T_k(v_n - \varphi) dx \\
&= \liminf_{n \rightarrow \infty} \int_{\Omega} a_i(x, T_M(v_n), \nabla T_M(v_n)) \cdot (D^i T_M(v_n) - D^i \varphi) \chi_{\{|v_n - \varphi| \leq k\}} dx \\
&\geq \int_{\Omega} (a_i(x, T_M(v), \nabla T_M(v)) - a_i(x, T_M(v), \nabla \varphi)) \\
&\quad \times (D^i T_M(v) - D^i \varphi) \chi_{\{|v - \varphi| \leq k\}} dx \\
&\quad + \lim_{n \rightarrow \infty} \int_{\Omega} a_i(x, T_M(v_n), \nabla \varphi) \cdot (D^i T_M(v_n) - D^i \varphi) \chi_{\{|v_n - \varphi| \leq k\}} dx \\
&\geq \int_{\Omega} a_i(x, T_M(v), \nabla T_M(v)) \cdot (D^i T_M(v) - D^i \varphi) \chi_{\{|v - \varphi| \leq k\}} dx \\
&= \int_{\Omega} a_i(x, v, \nabla v) D^i T_k(v - \varphi) dx.
\end{aligned} \tag{4.68}$$

On the other hand, we have $T_k(v_n - \varphi) \xrightarrow{*} T_k(v - \varphi)$ weak in $L^\infty(\Omega)$. thanks to (4.62) we have $\mathcal{H}_n(x, v_n, \nabla v_n) \rightarrow \mathcal{H}(x, v, \nabla v)$ strongly in $L^1(\Omega)$, and having in mind that $f_n \rightarrow f$ strongly in $L^1(\Omega)$, it follows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{H}_n(x, v_n, \nabla v_n) T_k(v_n - \varphi) dx = \int_{\Omega} \mathcal{H}(x, v, \nabla v) T_k(v - \varphi) dx. \tag{4.69}$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n(x) T_k(v_n - \varphi) dx = \int_{\Omega} f(x) T_k(v - \varphi) dx. \tag{4.70}$$

Finally, for the last term on the right-hand side of (4.67) we obtain $\phi_i(x, T_M(v_n)) \rightarrow \phi_i(x, T_M(v))$ strongly in $L^{p'_i(\cdot)}(\Omega)$, we conclude that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{\Omega} \phi_{n,i}(x, T_n(v_n)) D^i T_k(v_n - \varphi) dx \\
&= \lim_{n \rightarrow \infty} \int_{\Omega} \phi_{n,i}(x, T_M(v_n)) (D^i T_M(v_n) - D^i \varphi) \chi_{\{|v_n - \varphi| \leq k\}} dx \\
&= \int_{\Omega} \phi_i(x, T_M(v)) (D^i T_M(v) - D^i \varphi) \chi_{\{|v - \varphi| \leq k\}} dx \\
&= \int_{\Omega} \phi_i(x, v) D^i T_k(v - \varphi) dx.
\end{aligned} \tag{4.71}$$

By combining (4.67) and (4.69) – (4.71) we deduce that

$$\begin{aligned}
& \sum_{i=1}^d \int_{\Omega} a_i(x, v, \nabla v) D^i T_k(v - \varphi) dx + \int_{\Omega} \mathcal{H}(x, v, \nabla v) T_k(v - \varphi) dx \\
&\leq \int_{\Omega} f T_k(v - \varphi) dx + \sum_{i=1}^d \int_{\Omega} \phi_i(x, v) D^i T_k(v - \varphi) dx,
\end{aligned} \tag{4.72}$$

for all $\varphi \in K_\psi \cap L^\infty(\Omega)$, which complete the proof of Theorem 3.2.

Conflict of interest.

All authors declare that they have no conflicts of interest.

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