

A SEMI-FREDHOLM APPROACH TO PERIODIC DYNAMICS IN LINEAR REACTION-DIFFUSION MODELS WITH MULTI-DELAYS

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ABSTRACT. This study explores the existence of periodic solutions in linear reaction-diffusion systems with distributed multi-delays, inspired by models arising from the heat equation and population dynamics. We discuss the Massera's problem to this class of equations. Especially, by merging semi-Fredholm operator perturbation techniques with fixed-point methods, we establish sufficient conditions for the existence of periodic solutions, even without assuming compactness of the semigroup generated by the linear part of the equation. As a concrete application, we investigate selected reaction-diffusion models and complement the theoretical results with numerical simulations for illustration.

Keywords. Periodic solutions, semigroup, Poincaré map, Hille-Yosida operator, semi-Fredholm perturbation, multidelays effects.

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1. INTRODUCTION AND PRELIMINARY RESULTS

This work provides sufficient conditions ensuring the existence of periodic solutions for the following abstract linear reaction-diffusion equations:

$$\frac{d}{dt}x(t) = \Delta x(t) + \sum_{i=0}^n \mathcal{B}_i x(t - \tau_i) + h(t), \quad t \geq 0. \quad (1.1)$$

with initial condition $x_0 = \Phi \in \mathcal{C}_\tau := \mathcal{C}([-\tau, 0], \mathbb{X})$, the space of all continuous functions mapping from $[-\tau, 0]$ to $\mathbb{X}$ such that $|\xi|_\tau = \sup_{-\tau \leq \theta \leq 0} \|\xi(\theta)\|$ and $0 \leq \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = \tau$. The closed linear operator $(\mathcal{D}(\Delta), \Delta)$ mapping from $\mathcal{D}(\Delta) \subset (\mathbb{X}, \|\cdot\|)$ to $(\mathbb{X}, \|\cdot\|)$ such that $\overline{\mathcal{D}(\Delta)} \neq \mathbb{X}$, the operators $\mathcal{B}_i$, $0 \leq i \leq n$, are bounded linear on the Banach space $\mathbb{X}$ and the function $h : \mathbb{R}^+ \rightarrow \mathbb{X}$ is continuous.

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Periodic solutions of Eq. (1.1) play a fundamental role in the study of dynamical systems, providing insights into oscillatory behaviors observed across diverse scientific fields. Classical approaches, such as those developed in [1, 2, 4, 6, 7, 8, 9, 10, 16], usually rely on the ultimate boundedness of solutions together with the compactness of the Poincaré map. However, these techniques often prove inadequate for real-world models in biology, ecology, or physics, where the semigroup generated by the undelayed component of Eq. (1.1) is non-compact. This limitation underlines the need for more flexible analytical tools capable of addressing such challenges. In this work, we introduce a new framework that combines the Chow–Hale fixed-point theorem with key properties of semi-Fredholm operators. This approach bypasses compactness requirements and substantially relaxes the assumptions on $\mathcal{B}_i$, $0 \leq i \leq n$, thereby extending the applicability of periodicity results to a broader class of linear reaction-diffusion systems.

Before delving into the analysis, we begin by introducing the essential definitions and presenting the fundamental theorem on semi-Fredholm operator perturbations, which forms the cornerstone of our approach.

Definition 1.1. A bounded linear operator T is semi-Fredholm in $\mathbb{Y}$ if $\dim \ker(T) < \infty$ and $\mathcal{R}(T)$, the range of T , is closed in $\mathbb{Y}$.

In this study, we will use $\mathcal{SF}_+(\mathbb{Y})$ to represent the set of all semi-Fredholm operators defined from the space $\mathbb{Y}$ to $\mathbb{Y}$.

Now, we announce the following results taken from [3, 13]. Let T be a bounded linear operator in $\mathbb{Y}$. Then, $(\mathbb{X}/\ker(T), |\cdot|)$ is a Banach quotient space such that

$$|\bar{z}| = \text{dist}(z, \ker(T)),$$

where $\bar{z} = z + \ker(T)$ belongs to $\mathbb{X}/\ker(T)$.

Proposition 1.2. [13] *Let T be a bounded linear operator in $\mathbb{Y}$. So, $\mathcal{R}(T)$ is a closed subspace if and only if for $\delta > 0$, the inequality below holds:*

$$|\bar{z}| \leq \delta \|Tz\| \quad \text{for all} \quad z \in \mathbb{Y}.$$

Theorem 1.3. [3] *Let $T \in \mathcal{SF}_+(\mathbb{Y})$ and G be a bounded linear operator in Y . If $|G| < \frac{1}{2\delta(\sqrt{\dim \ker(T)} + 1)}$, where δ is the constant given in Proposition 1.2. Then,*

$$T + G \in \mathcal{SF}_+(\mathbb{Y}).$$

Furthermore, $\dim \ker(T + G) \leq \dim \ker(T)$.

The paper is structured as follows: Section 2 establishes the existence of periodic solutions of Eq.(1.1). In Section 3, we present several concrete reaction-diffusion models that illustrate the abstract framework developed in this work. For each model, we provide numerical simulations that not only validate the theoretical findings but also highlight the richness of the periodic dynamics under the influence of multi-delays. These examples demonstrate the practical relevance of our approach and confirm the robustness of the obtained results.

To develop a rigorous analytical framework for studying the dynamics of Eq. (1.1), we introduce the following assumptions, which are crucial for ensuring the existence and uniqueness of a periodic solution.

(A): There are constants $\nu \in \mathbb{R}$, $K \in [1, +\infty)$ and $p \in \mathbb{N}$ such that $\rho(\Delta) \supset (\nu, +\infty)$ and

$$|(\varsigma I - \Delta)^{-p}| \leq \frac{K}{(\varsigma - \nu)^p}, \quad \nu < \varsigma. \quad (1.2)$$

(B): h is T -periodic.

2. EXISTENCE OF PERIODIC SOLUTIONS

To properly analyze Eq. (1.1) with initial condition $x(0) = x_0$, we must first introduce several key definitions and preliminary results concerning its solutions.

Let $(\mathcal{D}(\Delta_0), \Delta_0)$ be the part of $(\mathcal{D}(\Delta), \Delta)$ defined by:

$$\mathcal{D}(\Delta_0) = \{z \in \mathcal{D}(\Delta) : \Delta z \in \overline{\mathcal{D}(\Delta)}\} \quad \text{and} \quad \Delta_0 z = \Delta z \quad \text{for} \quad z \in \mathcal{D}(\Delta_0).$$

It is established that Δ_0 is the generator of a C_0 -semigroup $\{\mathcal{V}(t), t \geq 0\}$ on $\overline{\mathcal{D}(\Delta)}$ which fulfills:

$$|\mathcal{V}(t)| \leq K e^{\nu t}, \quad t \geq 0,$$

where $K \geq 1$ and $\nu \in \mathbb{R}$.

Remark 2.1. By applying the renorming technique similar to that found in [12], it follows from condition (1.2) that one can define a norm on $\mathbb{X}$, denoted by $|\cdot|_K$, which is equivalent to the original norm $\|\cdot\|$. Specifically, this new norm satisfies the inequalities

$$\|z\| \leq |z|_K \leq K \|z\| \quad \text{for all} \quad z \in \mathbb{X},$$

and ensures the resolvent estimate

$$|(\varsigma I - \Delta)^{-p}| \leq \frac{1}{(\varsigma - \nu)^p} \quad \text{for every } p \in \mathbb{N} \text{ and } \varsigma > \nu.$$

To streamline the analysis, we shall henceforth work under the assumption that $K = 1$, as this entails no loss of generality.

Theorem 2.2. [11] *Under assumption (A). If $x_0 \in \mathcal{C}_\tau$ such that $x(0)$ belongs to $\overline{\mathcal{D}(\Delta)}$, then, $x : (-\tau, +\infty) \rightarrow \mathbb{X}$, the integral solution of Eq. (1.1), is unique and satisfies:*

$$\begin{aligned} x(t) &= \mathcal{V}(t)x(0) \\ &+ \lim_{\varsigma \rightarrow +\infty} \int_0^t \mathcal{V}(t-s) \varsigma (\varsigma I - \Delta)^{-1} \left(\sum_{i=0}^n \mathcal{B}_i x(s - \tau_i) + h(s) \right) ds, \quad t \geq 0. \end{aligned}$$

Throughout this work, we refer to the integral solution of Eq. (1.1) as the solution. Moreover, the solution of Eq. (1.1) such that $h = 0$, denoted for $t \geq 0$ by $\mathcal{W}(t)\varphi = x(\cdot, x_0, \mathcal{B}_i, 0)$. So

$$\mathcal{W}(t)\varphi = \mathcal{V}(t)\varphi + \mathcal{H}(t)\varphi, \quad (2.1)$$

where $\mathcal{H}(t)$, $t \geq 0$ is given by:

$$\mathcal{H}(t)x_0 = \lim_{\mu \rightarrow +\infty} \int_0^t \mathcal{V}(t-s)\mu(\mu I - \Delta)^{-1} \sum_{i=0}^n \mathcal{B}_i x(s - \tau_i, x_0) ds \quad \text{for } t \geq 0.$$

Proposition 2.3. *Under assumption (A). Let $|\mathcal{V}(t)| \leq e^{\nu t}$, $t \geq 0$. Then*

$$|\mathcal{H}(t)| \leq e^{\nu^+ t} \left(e^{\left((n+1) \sup_{0 \leq i \leq n} |\mathcal{B}_i| t \right)} - 1 \right),$$

such that $\nu^+ = \sup \{\nu, 0\}$.

Proof. Let x be the solution of Eq. 1.1 with $h = 0$. Then, for all $i \in \{0; 1; \dots; n\}$, we have

$$\begin{aligned} \|x(t - \tau_i, x_0)\| &\leq e^{\nu^+(t-\tau_i)} \|x_0\| + \int_0^{t-\tau_i} e^{\nu^+(t-s-\tau_i)} \left(\sum_{i=0}^n \|\mathcal{B}_i x(s - \tau_i, x_0)\| \right) ds \\ &\leq e^{\nu^+ t} |x_0|_\tau + \int_0^t e^{\nu^+(t-s)} \left(\sum_{i=0}^n \|\mathcal{B}_i x(s - \tau_i, x_0)\| \right) ds \\ &\leq e^{\nu^+ t} |x_0|_\tau + \sum_{i=0}^n |\mathcal{B}_i| \int_0^t e^{\nu^+(t-s)} \sup_{0 \leq i \leq n} \|x(s - \tau_i, x_0)\| ds. \end{aligned}$$

Thus

$$e^{-\nu^+ t} \sup_{0 \leq i \leq n} \|x(t - \tau_i)\| \leq \|x_0\| + \sum_{i=0}^n |\mathcal{B}_i| \int_0^t e^{-\nu^+ s} \sup_{0 \leq i \leq n} \|x(s - \tau_i, x_0)\| ds.$$

Using Gronwall-Bellman Inequality, we get

$$e^{-\nu^+ t} \sup_{0 \leq i \leq n} \|x(t - \tau_i)\| \leq e^{\int_0^t \sum_{i=0}^n |\mathcal{B}_i| ds} |x_0|_\tau \leq e^{\sum_{i=0}^n |\mathcal{B}_i| t} |x_0|_\tau.$$

So,

$$\sup_{0 \leq i \leq n} \|x(t - \tau_i, x_0)\| \leq e^{\left(\sum_{i=0}^n |\mathcal{B}_i| + \nu^+ \right) t} |x_0|_\tau.$$

Consequently, for $t \geq 0$

$$\begin{aligned}
\|\mathcal{H}(t)x_0\| &\leq \int_0^t e^{\nu^+(t-s)} \sum_{i=0}^n |\mathcal{B}_i| \|x(s - \tau_i, x_0)\| ds \\
&\leq \sum_{i=0}^n |\mathcal{B}_i| \int_0^t e^{\nu^+(t-s)} \sup_{0 \leq i \leq n} \|x(s - \tau_i, x_0)\| ds \\
&\leq \sum_{i=0}^n |\mathcal{B}_i| e^{\nu^+ t} \int_0^t \sum_{i=0}^n |\mathcal{B}_i|^s ds |x_0|_\tau \\
&\leq e^{\nu^+ t} \left(\sum_{i=0}^n |\mathcal{B}_i| t - 1 \right) |x_0|_\tau.
\end{aligned}$$

Which implies that

$$\|\mathcal{H}(t)x_0\| \leq e^{\nu^+ t} \left(e^{\sup_{0 \leq i \leq n} |\mathcal{B}_i| t} - 1 \right) |x_0|_\tau.$$

□

Proposition 2.4. Assume that $I - \mathcal{V}(T) \in \mathcal{SF}_+(\overline{\mathcal{D}(\Delta)})$ such that $d = \dim \ker(I - \mathcal{V}(T))$. Moreover, let $\delta > 0$ such that

$$|\bar{z}| \leq \delta \|(I - \mathcal{V}(T))z\|, \quad z \in \overline{\mathcal{D}(\Delta)}.$$

If

$$\sup_{0 \leq i \leq n} |\mathcal{B}_i| < \frac{1}{(n+1)T} \ln \left(\frac{e^{-\nu^+ T}}{2\delta(1 + \sqrt{d})} + 1 \right), \quad (2.2)$$

then, $I - \mathcal{W}(T)$ is a semi-Fredholm operator in $\overline{\mathcal{D}(\Delta)}$ with $\dim \ker(I - \mathcal{W}(T)) \leq d$.

Proof. The proof is a direct result of estimation (2.2), Proposition 2.3 and Theorem 1.3. □

Theorem 2.5. [3] Define the linear affine map $\mathcal{A}$ by $\mathcal{A}x = \overline{\mathcal{A}}x + y$ for $x \in \mathbb{X}$. Let $(\mathcal{A}^k x_0)_{k \in \mathbb{N}}$ be a bounded set in $\mathbb{X}$, $x_0 \in \mathbb{X}$. Moreover, assume that $I - \overline{\mathcal{A}} \in \mathcal{SF}_+(\mathbb{X})$. Then, $\mathcal{A}$ has a fixed point.

Additionally, for the remainder of this work, we say that an equation satisfies property (P) if it has a T -periodic solutions.

Theorem 2.6. Under assumptions (A) and (B). Assume that $I - \mathcal{V}(T) \in \mathcal{SF}_+(\overline{\mathcal{D}(\Delta)})$. If $\mathcal{B}_i$, $0 \leq i \leq n$, satisfy the estimate:

$$\sup_{0 \leq i \leq n} |\mathcal{B}_i| < \frac{1}{(n+1)T} \ln \left(\frac{e^{-\nu^+ T}}{2\delta(1 + \sqrt{d})} + 1 \right), \quad (2.3)$$

then, the boundedness of solution for Eq. (1.1) over the interval $[0, +\infty)$ implies that Eq. (1.1) verifies the property (P).

Proof. It is sufficient to demonstrate that the Poincaré map $\mathcal{P}_T$, defined by $\mathcal{P}_T(x_0) = x(T, x_0, \mathcal{B}_i, h)$ on $\overline{\mathcal{D}(\Delta)}$ has a fixed point, where $x(T, x_0, \mathcal{B}_i, h)$ denotes the solution of Eq. (1.1) such that $i \in \{0; 1; \dots; n\}$. Since Eq. (1.1) has a unique solution, the Poincaré map $\mathcal{P}_T$ is decomposed as

$$\mathcal{P}_T(x_0) = x(T, x_0, \mathcal{B}_i, 0) + x(T, 0, \mathcal{B}_i, h),$$

where $x(T, x_0, \mathcal{B}_i, 0)$ and $x(T, 0, \mathcal{B}_i, h)$ indicate the solutions of Eq. (1.1) with $h = 0$ and $x_0 = 0$, respectively, then $\mathcal{P}_T$ is given by $\mathcal{P}_T(x_0) = \widehat{\mathcal{P}}_T(x_0) + x_0$, where $\widehat{\mathcal{P}}_T(x_0) = x_T(T, x_0, \mathcal{B}_i, 0)$ and $x_0 = x(T, 0, \mathcal{B}_i, h)$. From (2.1), $\widehat{\mathcal{P}}_T$ is expressed as $\widehat{\mathcal{P}}_T = \mathcal{W}(T) = \mathcal{V}(T) + \mathcal{H}(T)$. In addition, Proposition 2.4 and estimation (2.3) imply that $I - \widehat{\mathcal{P}}_T \in \mathcal{SF}_+(\overline{\mathcal{D}(\Delta)})$. Now, let $x(T, x_0, \mathcal{B}_i, h)$, $0 \leq i \leq n$, be a bounded solution of Eq. (1.1). So,

$$\{\mathcal{P}_T^m x_0 : m \in \mathbb{N}\} = \{x(mT, x_0, \mathcal{B}_i, h) : m \in \mathbb{N}\},$$

this implies the boundedness of $(\mathcal{P}_T^m x_0)_{m \geq 0}$ in $\overline{\mathcal{D}(\Delta)}$. From Theorem 2.5, we deduce that the fixed point set $\mathcal{F}_{\mathcal{P}_T} \neq \emptyset$. Consequently, Eq. (1.1) satisfies the property (P). $\square$

Corollary 2.7. *Under assumptions (A) and (B). Let $|\mathcal{V}(t)| \leq e^{\nu t}$ for $t \geq 0$ and $\nu < 0$. If*

$$\sup_{0 \leq i \leq n} |\mathcal{B}_i| < \frac{1}{(n+1)T} \ln \left(\frac{1 - e^{\nu T}}{2} + 1 \right), \quad (2.4)$$

then, the boundedness of solution for Eq. (1.1) over the interval $[0, +\infty)$ implies that Eq. (1.1) verifies the property (P).

Proof. To establish the desired outcome, we need to determine the explicit values of the parameters δ and d in inequality (2.3). Moreover, the exponential stability of $\{\mathcal{V}(t), t \geq 0\}$ guarantees that the operator $I - \mathcal{V}(T)$ is invertible for $T > 0$, so, $\ker(I - \mathcal{V}(T)) = \overline{\mathcal{D}(\Delta)}$, which implies that $I - \mathcal{V}(T) \in \mathcal{SF}_+(\overline{\mathcal{D}(\Delta)})$ with $d = \dim \ker(I - \mathcal{V}(T)) = 0$. Now, for $z \in \overline{\mathcal{D}(\Delta)}$, we get

$$\|z\| \leq \|z - \mathcal{V}(T)z\| + \|\mathcal{V}(T)z\| \leq \|z - \mathcal{V}(T)z\| + e^{\nu T} \|z\|.$$

and

$$(1 - e^{\nu T}) \|z\| \leq \|z - \mathcal{V}(T)z\|, \quad z \in \overline{\mathcal{D}(\Delta)}.$$

Thus

$$|\bar{z}| = \|z\| \leq \frac{1}{(1 - e^{\nu T})} \|z - \mathcal{V}(T)z\|.$$

Consequently, From Proposition 1.2, we select δ such that

$$\delta \leq \frac{1}{(1 - e^{\nu T})}.$$

Thus, for δ satisfying $\delta \leq \frac{1}{1 - e^{\nu T}}$, the bound in (2.4) ensures the validity of inequality (2.3) and Theorem 2.6 guarantees that Eq. (1.1) verifies the property $(\mathcal{P})$. $\square$

3. APPLICATION

In this section, we consider a class of spatially extended neural field models that incorporate both synaptic transmission delays and diffusion effects, as described by a reaction-diffusion integro-differential equation. This formulation, which has been studied in [14, 15], captures the complex spatiotemporal dynamics of neural activity through the combined effects of local diffusion, delayed nonlocal interactions, and external inputs. The model is introduced as follows:

$$\frac{\partial v(x, t)}{\partial t} - \Delta v(x, t) = -av(x, t) + \sum_{i=1}^n \omega_i \int_{\Omega} J(x, y) v(y, t - \varrho_i) dy + I(x, t), \quad (3.1)$$

which describes the macroscopic dynamics of cortical activity in a continuous population of neurons, where $v(x, t)$ Represents the average membrane potential or firing rate of neurons at position x and time t , a is the decay rate of activity, $J(x, y)$ is the connectivity kernel (strength of synapses from y to x , ω_i Synaptic weights (excitatory if $\omega_i > 0$, inhibitory if $\omega_i < 0$), $I(x, t)$ represents sensory stimuli, thalamic input, or top-down modulation, ϱ_i is the synaptic delays. The index i in the model enumerates different synaptic or transmission delay pathways. Each ϱ_i corresponds to a distinct time delay, and ω_i is the associated weight (how much that delayed influence contributes). So, ϱ_i represents a discrete set of pathways or modes of delayed interactions in the neural field. So, we consider the following equation:

$$\left\{ \begin{array}{l} \frac{d}{dt} v(x, t) = \Delta v(x, t) - av(x, t) \\ \quad + \sum_{i=1}^2 \omega_i \int_0^1 J(x, y) v(y, t - \varrho_i) dy + I(x, t), \quad t \geq 0, \\ v(0, t) = v(1, t) = 0, \quad t \geq 0, \\ v(\theta, x) = \phi_0(\theta, x), \quad \theta \in [-\varrho, 0], \quad x \in [0, 1], \end{array} \right. \quad (3.2)$$

which corresponds to Eq. (3.1) in the case where $i = 2$, such that $a > 0$, $\omega_1 > 0$, $\omega_2 < 0$, $J_0 = \sup_{(x, y) \in [0, 1]^2} |J(x, y)| < +\infty$ and $I(x, t)$ is a continuous T -periodic function. Let

$$\mathcal{E} = \mathbf{C}(\overline{\mathbb{R}}) = \left\{ w \in \mathbf{C}(\mathbb{R}) : \lim_{x \rightarrow \pm\infty} w(x) \text{ exists} \right\}.$$

Moreover, we say that $w \in \mathbf{C}^k(\overline{\mathbb{R}})$ if and only if $w \in \mathbf{C}^k(\mathbb{R})$ and each derivative of order k admits finite limits as $x \rightarrow \pm\infty$. Equipped with the supremum norm

$$\|u\|_\infty = \sup_{y \in \mathbb{R}} |u(y)|,$$

the space $(\mathcal{E}, \|\cdot\|_\infty)$ forms a Banach space.

Let $\mathcal{B} : \mathcal{D}(\Delta) \subset \mathcal{E} \rightarrow \mathcal{E}$ be the operator given by:

$$\begin{cases} \mathcal{D}(\mathcal{B}) &= \{w \in \mathbf{C}^2(\overline{\mathbb{R}}) : \lim_{y \rightarrow \pm\infty} w(y) = 0\}, \\ \mathcal{B}w &= \Delta w - aw. \end{cases}$$

As stated in [2, 5], $\overline{\mathcal{D}(\mathcal{B})} = \{w \in \mathbf{C}(\overline{\mathbb{R}}) : \lim_{y \rightarrow \pm\infty} w(y) = 0\} \neq \mathcal{E}$. Moreover, the part of $\mathcal{B}$ in $\overline{\mathcal{D}(\mathcal{B})}$ generates a C_0 -semigroup denoted by $\{\mathcal{O}(t), t \geq 0\}$ satisfying:

$$|\mathcal{O}(t)| \leq e^{-at}, \quad t \geq 0.$$

On the other hand, let f a continuous function such that: $f(t)(x) = I(x, t)$ and let $\mathcal{D}$ be the bounded linear operator defined on $\mathcal{E}$ by:

$$\mathcal{D}(v)(x, t) = \sum_{i=0}^2 \mathcal{D}_i v(y, t - \varrho_i) = -av(x, t) + \sum_{i=1}^2 \omega_i \int_0^1 J(x, y) v(y, t - \varrho_i) dy.$$

It is clear that

$$\|\mathcal{D}(v)\|_\infty \leq (|a| + J_0(\omega_1 + |\omega_2|)) \|v\|_\infty.$$

To analyze the boundedness of solutions for Equation (3.2), we introduce the following assumption:

(C) there is $l \in (0, 1)$ such that

$$a + J_0(|\omega_1| + |\omega_2|) < (1 - l).$$

Let $R = 1 + \frac{|I|_\infty}{l}$, such that $|I|_\infty = \sup_{0 \leq t \leq T} |I(t)|$.

Proposition 3.1. *Assume that condition (C) holds. Then, for any initial function v_0 satisfying $\|v_0\|_\infty \leq R$, the corresponding solution to Eq. (3.2) remains bounded by R for all $t \in \mathbb{R}^+$.*

Proof. We argue by contradiction. Define

$$t_0 = \inf \{t > 0 : \|v(t)\|_\infty > R\}.$$

By the continuity of v , it follows that

$$\|v(t_0)\|_\infty = R.$$

Moreover, there exists $\lambda > 0$ such that

$$\|v(t)\|_\infty > R \quad \text{for all } t \in (t_0, t_0 + \lambda).$$

Using the variation-of-constants formula applied to Equation (3.2) with initial condition v_0 , we obtain:

$$\begin{aligned}
v(t) &= \mathcal{O}(t)v(0) \\
&+ \lim_{\varsigma \rightarrow +\infty} \int_0^t \mathcal{O}(t-s) \left[-av(s) + \sum_{i=1}^2 \omega_i \int_0^1 J(\cdot, y)v(y, s - \varrho_i) dy + I(s) \right] ds.
\end{aligned}$$

So, for $t \geq 0$, we have

$$\begin{aligned}
\|v(t_0)\|_\infty &\leq \|\mathcal{O}(t_0)\| \|v(0)\|_\infty \\
&+ \int_0^{t_0} |\mathcal{O}(t_0-s)| \left[a|v(s)| + J_0(\omega_1 + |\omega_2|) \sup_{1 \leq i \leq 2} \|v(s - \varrho_i)\|_\infty + |I|_\infty \right] ds.
\end{aligned}$$

As $0 < s < t_0$, it follows that for $1 \leq i \leq 2$, the inequality $-\varrho_i \leq s - \varrho_i \leq t_0 - \varrho_i < t_0$ holds. Consequently, one has that:

$$\begin{aligned}
\|v(t_0)\|_\infty &\leq Re^{-at_0} + (aR + J_0(\omega_1 + |\omega_2|)R + |I|_\infty) \int_0^{t_0} e^{-a(t_0-s)} ds \\
&\leq Re^{-at_0} + (1 - e^{-at_0})(aR + J_0(\omega_1 + |\omega_2|)R + |I|_\infty).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|v(t_0)\|_\infty &\leq Re^{-at_0} + (1 - e^{-at_0})(R - Rl + |I|_\infty) \\
&\leq Re^{-at_0} + (1 - e^{-at_0})(R - l) \\
&\leq R - l(1 - e^{-at_0}) \\
&\leq R.
\end{aligned}$$

This contradicts the definition of t_0 . Consequently, we conclude that

$$\|v(t)\|_\infty \leq R \quad \text{for } t \geq 0.$$

□

Theorem 3.2. Assume that (C) holds. If

$$\max\{a, J_0\omega_1, J_0|\omega_2|\} < \frac{1}{3T} \ln \left(\frac{1 - e^{-T}}{2} + 1 \right), \quad (3.3)$$

then, Eq. (3.2) verifies the property (P).

Proof. The estimations (3.3) indicate that all assumptions of Corollary 2.7 are fulfilled with $n = 2$, $K = \widehat{K} = 1$ and $\nu = -1$. Consequently, Eq. (3.2) verifies the property (P). □

Our goal now is to validate the theoretical findings through numerical simulations. For Eq. (3.2), we select parameters that satisfy all the conditions outlined in Theorem 3.2. Specifically, we set $T = 1$, $a = 0.09$, $\omega_1 = 0.08$, $\omega_2 = -0.04$, $\varrho_1 = 0.1$, $\varrho_2 = 0.2$, $I(t) = \sin(5\pi x + 2\pi t)$. Furthermore, we consider an exponential kernel $J(x, y) = \exp(-5|x - y|)$ and the initial condition $v_0(x, \theta) = 0.1x(x - 1)\sin(2\pi t)$. So, it can be observed that all assumptions of Theorem (3.2) are satisfied. This behavior is visualized in Figure.1 through a 3D representation, and further detailed in Figure.2 with 2D plots corresponding to the fixed spatial position $x = 0.5$.

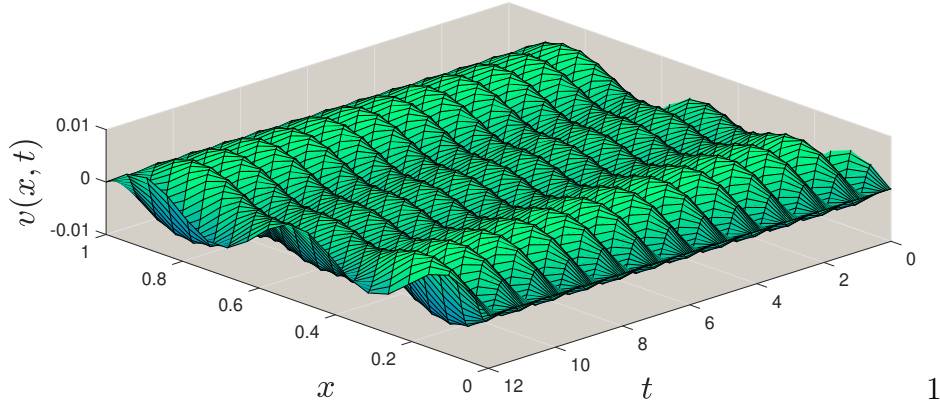


FIGURE 1. The evolution of solution for Eq. (3.2) in three dimensional such that $T = 1$, $a = 0.09$, $I(t) = \sin(5\pi x + 2\pi t)$ and the initial condition $x_0(x, \theta) = 0.1x(x - 1) \sin(2\pi t)$.

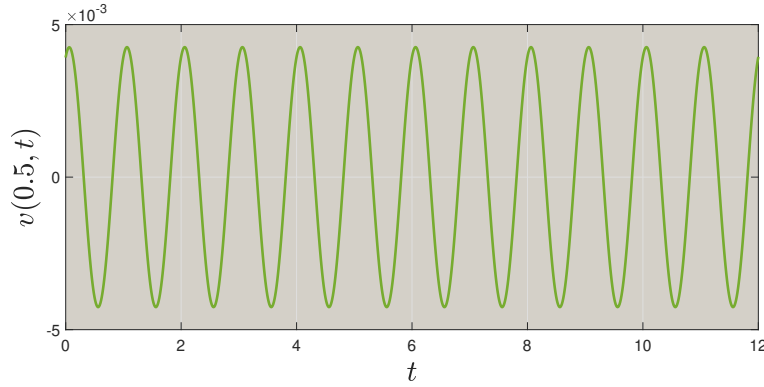


FIGURE 2. The evolution of solution for Eq. (3.2) in two dimensional such that $x = 0.5$.

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