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ON THE STABILITY OF A k-BALANCING FUNCTIONAL EQUATION

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Abstract. We examine a functional equation in Banach spaces that defines k-balancing functions through a recurrence relation. We show these functions are directly linked to k-balancing numbers, prove the Hyers-Ulam stability of the equation, and establish the uniqueness of the solution.

Keywords. k-balancing numbers, k-balancing equation, Hyers-Ulam Stability
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1. Introduction

Balancing numbers are defined by

$$B_{n+2} = 6B_{n+1} - B_n, \quad \text{for } n \geq 2$$

with initials $B_0 = 0$ and $B_1 = 1$. Behera and Panda, see [3], introduced this sequence while they were considering the equation:

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

with $n, r \in \mathbb{Z}^+$. The characteristic equation of this sequence is

$$x^2 - 6x + 1 = 0.$$

A generalized version of the balancing numbers is **k**-balancing numbers defined recursively by $B_{\mathbf{k},0} = 0, B_{\mathbf{k},1} = 1$ and $B_{\mathbf{k},\mathbf{v}} = 6B_{\mathbf{k},\mathbf{v}-1} - B_{\mathbf{k},\mathbf{v}-2}$ for $\mathbf{v} \geq 2$. The sequences formed by k-balancing numbers are governed by linear recurrence relations, analyzable through linear algebra, and are described by rational generating functions—a key tool in algebraic combinatorics. Consequently, the investigation of these numbers creates a unified framework, seamlessly connecting the study of polynomial Diophantine equations and number fields with the principles of modern algebra. For basics of these sequences, see [12], [13], [14] and [15]. For details on the Diophantine equations related to these sequences, see [4], [5], [6], [16] and the references mentioned therein.

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In 1940, Ulam delivered a broad lecture in which he highlighted several imporatant open problems (see [19]). One of the key questions he posed concerned the stability of homomorphisms:

"Under what conditions can a function that approximately satisfies the homomorphism property be approximated by an actual homomorphism?"

In response to Ulam's question, Hyers provided the first significant result on the stability of functional equations, now known as Hyers-Ulam stability, see [7]. Thanks to this paper, the stability theory has received considerable attention and has opened up vast areas of research. See, for example, [1], [2], [8], [9], [10], [11], [17] and [18].

The study of the k-balancing functional equation is closely connected to Geometric Function Theory (GFT) through the analysis of stability, analytic structure, and operator-induced transformations in complex domains. In geometric function theory, analytic and bi-univalent functions are often characterized via functional equations, subordination principles, and convolution operators. The k-balancing functional equation provides a structural framework that can generate or preserve geometric properties such as univalence, starlikeness, and convexity when embedded into analytic operator settings. For instance, recent works on coefficient estimates and operator theory in bi-univalent functions involving q-operators and special functions [20, 21, 22] demonstrate how functional identities influence geometric behavior. Moreover, inclusion properties of Rabotnov-type operators [23] further highlight the role of functional equations in defining analytic subclasses. From a broader perspective, stability concepts developed in generalized metric frameworks [24] support the theoretical foundation for studying the stability of k-balancing equations within analytic function spaces. Hence, the k-balancing functional equation serves as an analytical bridge between stability theory and geometric function theory, particularly in the modern context of q-calculus and operator-based subclasses of bi-univalent functions.

Our aim in this paper is threefold. First, defining the class of **k**-balancing functions and solving the **k**-balancing functional equation. Second, proving the Hyers-Ulam stability for this class in real Banach spaces. Third, establishing uniqueness of the obtained solution.

2. Solution of the k-balancing functional equation

The concept of k-balancing functions is introduced as a natural generalization of k-balancing numbers. We solve the k-balancing functional equation and show its strong connection to the k-balancing numbers. The recurrence relation $B_{k,\vartheta} = 6B_{k,\vartheta-1} - B_{k,\vartheta-2}, \vartheta \in \mathbb{N}$ for k-balancing numbers motivates the following definition.

Definition 2.1. A function $\nu : \mathbb{N} \times \mathbb{R} \to \mathcal{B}$ is called a k-balancing function if the equation:

$$\nu(k, \vartheta) = 6k\nu(k, \vartheta - 1) - \nu(k, \vartheta - 2), \tag{2.1}$$

is satisfied for all $k \in \mathbb{N}$ and $\vartheta \in \mathbb{R}$.

We call this equation the k-balancing functional equation.

Example 2.2. (1) Define $\nu : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by:

$$\nu(k,\vartheta) = \begin{cases} 0, & \vartheta = 0, \\ B_{k,\vartheta}, & \vartheta \in \mathbb{N}, \\ 0, & \vartheta \notin \mathbb{N}. \end{cases}$$

Then for $\vartheta \in \mathbb{R}$, we get:

$$\nu(k,\vartheta) = B_{k,\vartheta} = 6kB_{k,\vartheta-1} - B_{k,\vartheta-2} = 6k\nu(k,\vartheta-1) - \nu(k,\vartheta-2).$$

Thus, ν is a k-balancing function. This shows that k-balancing functions can be considered as a natural extension of k-balancing numbers.
(2) Let $\nu : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be defined by $\nu(k,\vartheta) = \frac{1}{k}\rho^{\vartheta}$. Then

$$\begin{aligned} 6k\nu(k,\vartheta-1) - \nu(k,\vartheta-2) &= 6\rho^{\vartheta-1} - \frac{1}{k}\rho^{\vartheta-2} \\ &= \frac{1}{k}\rho^{\vartheta-2}(6k\rho - 1). \end{aligned}$$

The characteristic equation of the k-balancing numbers gives $6k\rho - 1 = \rho^2$. Then

$$6k\nu(k,\vartheta-1) - \nu(k,\vartheta-2) = \frac{1}{k}\rho^{\vartheta} = \nu(k,\vartheta).$$

Hence, ν is a k-balancing function.

Theorem (2.3) provides the solution of the k-balancing equation and establishes the intimate connection between k-balancing numbers and k-balancing functions.

Theorem 2.3. *A function $\nu : \mathbb{N} \times \mathbb{R} \to \mathcal{B}$ is a k-balancing function if and only if there exists a function $\mu : \mathbb{N} \times [-1,1) \to \mathcal{B}$ such that*

$$\nu(k,\vartheta) = \begin{cases} B_{k,[\vartheta]+1}\mu(k,\vartheta-[\vartheta]) - B_{k,[\vartheta]}\mu(k,\vartheta-[\vartheta]-1), & \text{if } \vartheta \geq 0, \\ -B_{k,-[\vartheta]-1}\mu(k,\vartheta-[\vartheta]) + B_{k,-[\vartheta]}\mu(k,\vartheta-[\vartheta]-1), & \text{if } \vartheta < 0. \end{cases}$$

Proof. Using the relations $\rho + \sigma = 6k$ and $\rho\sigma = 1$, Eq. (2.1) can be written as

$$\begin{aligned} \nu(k,\vartheta) &= (\rho+\sigma)\nu(k,\vartheta-1) - \rho\sigma\nu(k,\vartheta-2) \\ &= \rho\nu(k,\vartheta-1) + \sigma\nu(k,\vartheta-1) - \rho\sigma\nu(k,\vartheta-2). \end{aligned}$$

Consequently,

$$\nu(k,\vartheta) - \rho\nu(k,\vartheta-1) = \sigma[\nu(k,\vartheta-1) - \rho\nu(k,\vartheta-2)],$$

and

$$\nu(k,r) - \sigma\nu(k,\vartheta-1) = \rho[\nu(k,\vartheta-1) - \sigma\nu(k,\vartheta-2)].$$

Inductively, one can show that

$$\nu(k,\vartheta) - \rho\nu(k,\vartheta-1) = \sigma^n[\nu(k,\vartheta-n) - \rho\nu(k,\vartheta-n-1)], \tag{2.2}$$

and

$$\nu(k,\vartheta) - \sigma\nu(k,\vartheta-1) = \rho^n[\nu(k,\vartheta-n) - \sigma\nu(k,\vartheta-n-1)]. \tag{2.3}$$

Hence, these equations hold for all $n \in \mathbb{Z}$. Multiplying Eq. (2.2) by σ and Eq. (2.3) by $-\rho$ and summing, we get

$$\nu(k,v) = \frac{\sigma^{n+1}-\rho^{n+1}}{\sigma-\rho}\nu(k,\vartheta-n) - \frac{\sigma^{n}-\rho^{n}}{\sigma-\rho}\nu(k,\vartheta-n-1).$$

Plugging Binet's formulas into the above, we obtain

$$\nu(k,\vartheta) = B_{k,n+1}\nu(k,\vartheta-n) - B_{k,n}\nu(k,\vartheta-n-1). \tag{2.4}$$

We split the investigation as follows:
Case I: If $\vartheta \geq 0$, let $n = [\vartheta]$. Then

$$\nu(k,\vartheta) = B_{k,[\vartheta]+1}\nu(k,\vartheta-[r]) - B_{k,[\vartheta]}\nu(k,\vartheta-[\vartheta]-1). \tag{2.5}$$

Case II: If $\vartheta < 0$, let $n = [\vartheta] = -|[\vartheta]|$. Then, using the previous equation,

$$\nu(k,\vartheta) = \frac{\sigma^{-|[\vartheta]|+1}-\rho^{-|[\vartheta]|+1}}{\sigma-\rho}\nu(k,\vartheta-[\vartheta]) - \frac{\sigma^{-|[\vartheta]|}-\rho^{-|[\vartheta]|}}{\sigma-\rho}\nu(k,\vartheta-[\vartheta]-1).$$

Since $\rho\sigma = 1$, then

$$\nu(k,v) = \frac{\sigma^{|[\vartheta]|-1}-\rho^{|[\vartheta]|-1}}{\sigma-\rho}\nu(k,\vartheta-[\vartheta]) + \frac{\sigma^{|[\vartheta]|}-\rho^{|[\vartheta]|}}{\sigma-\rho}\nu(k,\vartheta-[\vartheta]-1).$$

Therefore,

$$\nu(k,\vartheta) = -B_{k,|[\vartheta]|-1}\nu(k,\vartheta-[\vartheta]) + B_{k,|[\vartheta]|}\nu(k,\vartheta-[\vartheta]-1). \tag{2.6}$$

We have $0 \leq \vartheta-[\vartheta] < 1$ and $-1 \leq \vartheta-[\vartheta]-1 < 0$. Define $\mu : \mathbb{N}\times[-1,1) \to \mathcal{B}$ by $\mu = \nu|_{\mathbb{N}\times[-1,1)}$. Then, by Eq.(2.5) and Eq.(2.6), $\nu(k,\vartheta)$ takes the required form. Conversely, let $\mu : \mathbb{N}\times[-1,1) \to \mathcal{B}$ be any function. Our aim is to prove that μ is a k-balancing function. This will be achieved through the following cases:
Case I: If $\vartheta < 0$, then

$$\nu(k,\vartheta) = -B_{k,-[\vartheta]-1}\mu(k,\vartheta-[\vartheta]) + B_{k,-[\vartheta]}\mu(k,\vartheta-[\vartheta]-1).$$

$$\begin{aligned}\nu(k,\vartheta-1) &= -B_{k,-[\vartheta-1]-1}\mu(k,\vartheta-1-[\vartheta-1]) + B_{k,-[\vartheta-1]}\mu(k,\vartheta-1-[\vartheta-1]-1)\\ &= -B_{k,-[\vartheta]}\mu(k,\vartheta-[\vartheta]) + B_{k,-[\vartheta]+1}\mu(k,\vartheta-[\vartheta]-1).\end{aligned}$$

And

$$\begin{aligned}\nu(k,\vartheta-2) &= -B_{k,-[\vartheta-2]-1}\mu(k,\vartheta-2-[\vartheta-2]) + B_{k,-[\vartheta-2]}\mu(k,\vartheta-2-[\vartheta-2]-1)\\ &= -B_{k,-[\vartheta]+1}\mu(k,\vartheta-[\vartheta]) + B_{k,-[\vartheta]+2}\mu(k,\vartheta-[\vartheta]-1).\end{aligned}$$

Since $B_{k,-[\vartheta]+1} = 6kB_{k,-[\vartheta]} - B_{k,-[\vartheta]-1}$ and $B_{k,-[\vartheta]+2} = 6kB_{k,-[\vartheta]+1} - B_{k,-[\vartheta]}$, then

$$\begin{aligned} 6k\nu(k,\vartheta-1)-\nu(k,\vartheta-2) &= -B_{k,-[\vartheta]-1}\mu(k,\vartheta-[\vartheta])+B_{k,-[\vartheta]}\mu(k,\vartheta-[\vartheta]-1) \\ &= \nu(k,\vartheta). \end{aligned}$$

Case II: If $0 \le \vartheta < 1$, then $[\vartheta]=0$, $[\vartheta-1]=-1$, and $[\vartheta-2]=-2$. We obtain:

$$6k\nu(k,\vartheta-1)-\nu(k,\vartheta-2) = 6k\mu(k,\vartheta-1)+\nu(k,\vartheta)-6k\nu(k,\vartheta-1) = \mu(k,\vartheta) = \nu(k,\vartheta).$$

Case III: If $1 \le \vartheta < 2$, then $[\vartheta]=1$, $[\vartheta-1]=0$, and $[\vartheta-2]=-1$. One can drive:

$$6k\nu(k,\vartheta-1)-\nu(k,\vartheta-2) = 6k\mu(k,\vartheta-1)-\nu(k,\vartheta-2) = \nu(k,\vartheta).$$

Case IV: If $\vartheta \ge 2$, then:

$$\begin{aligned} 6k\nu(k,\vartheta-1)-\nu(k,\vartheta-2) &= [6kB_{k,[\vartheta]}-B_{k,[\vartheta]-1}]\mu(k,\vartheta-[\vartheta]) \\ &- [6kB_{k,[\vartheta]-1}-B_{k,[\vartheta]-2}]\mu(k,\vartheta-[\vartheta]-1) \\ &= \nu(k,\vartheta). \end{aligned}$$

This finishes the proof. □

3. Hyers-Ulam stability of the k-balancing equation

The Hyers-Ulam stability of k-balancing functions is established in the following result.

Theorem 3.1. *Let $(\mathcal{B}, \|.\|)$ be a real Banach space. If, for some $\epsilon \ge 0$, a function $\nu : \mathbb{N}\times\mathbb{R} \to \mathcal{B}$ satisfies:*

$$\|\nu(k,\vartheta)-6k\nu(k,\vartheta-1)-\nu(k,\vartheta-2)\| \le \epsilon, \tag{3.1}$$

for all $\vartheta \in \mathbb{R}$, then a unique k-balancing function Λ exists such that:

$$\|\nu(k,\vartheta)-\Lambda(k,\vartheta)\| \le \frac{\epsilon}{2(3k-1)}.$$

Proof. We rewrite Eq. (3.1) in the form

$$\|\nu(k,\vartheta)-\rho\nu(k,\vartheta-1)-\sigma[\nu(k,\vartheta-1)-\rho\nu(k,\vartheta-2)]\| \le \epsilon. \tag{3.2}$$

Replacing ϑ by $\vartheta-t$ and multiplying by σ^t yields

$$\|\sigma^t[\nu(k,\vartheta-t)-\rho\nu(k,\vartheta-t-1)]-\sigma^{t+1}[\nu(k,\vartheta-t-1)-\rho\nu(k,\vartheta-t-2)]\| \le \sigma^t\epsilon.$$

Summing over t from 0 to $n-1$ gives

$$\begin{aligned} &\sum_{t=0}^{n-1}\sigma^t[\nu(k,\vartheta-t)-\rho\nu(k,\vartheta-t-1)]-\sigma^{t+1}[\nu(k,\vartheta-t-1)-\rho\nu(k,\vartheta-t-2)] \\ &\qquad = \nu(k,\vartheta)-\rho\nu(k,\vartheta-1)-\sigma^n[\nu(k,\vartheta-n)-\rho\nu(k,\vartheta-n-1)]. \end{aligned}$$

Therefore,

$$\|\nu(k,\vartheta)-\rho\nu(k,\vartheta-1)-\sigma^n[\nu(k,\vartheta-n)-\rho\nu(k,\vartheta-n-1)]\| \le \sum_{t=0}^{n-1}\sigma^t\epsilon.$$

Since $\sigma < 1$, then $\{\sigma^t[\nu(k,\vartheta-t)-\rho\nu(k,\vartheta-t-1)]\}$ is a Cauchy sequence. Now, the function

$$\eta_1 := \lim_{t\to\infty} \sigma^t[\nu(k,\vartheta-t)-\rho\nu(k,\vartheta-t-1)] \tag{3.3}$$

is well defined. In addition,

$$\begin{aligned}
6k\eta_1(k,\vartheta-1)-\eta_1(k,\vartheta-2) &= 6k\lim_{t\to\infty}\sigma^t[\nu(k,\vartheta-t-1)-\rho\nu(k,\vartheta-t-2)]\\
&- \lim_{t\to\infty}\sigma^t[\nu(k,\vartheta-t-2)-\rho\nu(k,\vartheta-t-3)]\\
&= 6k\sigma^{-1}\eta_1(k,\vartheta)-\sigma^{-2}\eta_1(k,\vartheta)\\
&= \sigma^{-2}(6k\sigma^{-1})\eta_1(k,\vartheta)\\
&= \eta_1(k,\vartheta).
\end{aligned}$$

In the last equality, we used that $\sigma^2-6k\sigma+1=0$. Hence, η_1 is a k-balancing function. Taking $n\to\infty$ in Eq. (3.3), we obtain

$$\sum_{t=0}^{\infty}\|\nu(k,\vartheta)-\rho\nu(k,\vartheta-1)-\eta_1(k,\vartheta)\| \le \sum_{t=0}^{\infty}\sigma^t\epsilon = \frac{\epsilon}{1-\sigma}. \tag{3.4}$$

We have

$$\frac{1}{1-\sigma} = \frac{1}{1-(3k-\sqrt{9k^2-1})} = \frac{(1-3k)-\sqrt{9k^2-1}}{(1-3k)^2-(9k^2-1)}.$$

Thus, by Eq. (3.1), we get

$$\frac{1}{1-\sigma} = \frac{1}{2}\left(1+\frac{\sqrt{9k^2-1}}{3k-1}\right). \tag{3.5}$$

$$\|\nu(k,\vartheta)-\sigma\nu(k,\vartheta-1)-\rho[\nu(k,\vartheta-1)-\sigma\nu(k,\vartheta-2)]\| \le \epsilon.$$

Replacing ϑ with $\vartheta+t$ and multiplying by ρ^{-t}, we obtain

$$\|\rho^{-t}[\nu(k,\vartheta+t)-\sigma\nu(k,\vartheta+t-1)]-\rho^{-t+1}[\nu(k,\vartheta+t-1)-\sigma\nu(k,\vartheta+t-2)]\| \le \rho^{-t}\epsilon. \tag{3.6}$$

Since $\rho > 1$, then $\{\rho^{-t}[\nu(k,\vartheta+t)-\sigma\nu(k,\vartheta+t-1)]\}$ is a Cauchy sequence. We have

$$\begin{aligned}
&\sum_{t=1}^{n}\rho^{-t}[\nu(k,\vartheta+t)-\sigma\nu(k,\vartheta+t-1)]-\rho^{-t+1}[\nu(k,\vartheta+t-1)-\sigma\nu(k,\vartheta+t-2)]\\
&\qquad = \rho^{-n}[\nu(k,\vartheta+n)-\sigma\nu(k,\vartheta+n-1)]-[\nu(k,\vartheta)-\sigma\nu(k,\vartheta-1)].
\end{aligned}$$

Then,

$$\|\rho^{-n}[\nu(k,\vartheta+n)-\sigma\nu(k,\vartheta+n-1)]-[\nu(k,\vartheta)-\sigma\nu(k,\vartheta-1)]\| \le \sum_{t=1}^{n}\rho^{-t}\epsilon. \tag{3.7}$$

Let

$$\eta_2 := \lim_{t\to\infty} \rho^{-t}[\nu(k,\vartheta+t) - \sigma\nu(k,\vartheta+t-1)].$$

$$\begin{aligned} 6k\eta_2(k,\vartheta-1) - \eta_2(k,\vartheta-2) &= 6k \lim_{t\to\infty} \rho^{-t}[\nu(k,\vartheta+t-1) - \sigma\nu(k,\vartheta+t-2)] \\ &- \lim_{t\to\infty} \rho^{-t}[\nu(k,\vartheta+t-2) - \sigma\nu(k,\vartheta+t-3)] \\ &= 6k\rho^{-1}\eta_2(k,\vartheta) - \rho^{-2}\eta_2(k,\vartheta) \\ &= \rho^{-2}(6k\rho^{-1})\eta_2(k,\vartheta) \\ &= \eta_2(k,\vartheta). \end{aligned}$$

Hence, η_2 is a k-balancing function. Taking $n \to \infty$ in Eq. (3.6), we obtain

$$\sum_{t=1}^{\infty} \|\nu(k,\vartheta) - \sigma\nu(k,\vartheta-1) - \eta_2(k,\vartheta)\| \le \sum_{t=1}^{\infty} \rho^{-t}\epsilon = \frac{\epsilon}{\rho-1}. \tag{3.8}$$

Now,

$$\frac{1}{\rho-1} = \frac{1}{(3k-1)+\sqrt{9k^2-1}} = \frac{(3k-1)-\sqrt{9k^2-1}}{(3k-1)^2-(9k^2-1)}.$$

Thus,

$$\frac{1}{\rho-1} = -\frac{1}{2}\left(1 - \frac{\sqrt{9k^2-1}}{3k-1}\right). \tag{3.9}$$

Define

$$\Lambda(k,\vartheta) := \frac{\sigma}{\sigma-\rho}\eta_1(k,\vartheta) - \frac{\rho}{\sigma-\rho}\eta_2(k,\vartheta). \tag{3.10}$$

Then,

$$\begin{aligned} \|\nu(k,\vartheta) - \Lambda(k,\vartheta)\| &= \left\|\nu(k,\vartheta) - \frac{\sigma}{\sigma-\rho}\eta_1(k,\vartheta) + \frac{\rho}{\sigma-\rho}\eta_2(k,\vartheta)\right\| \\ &\le \frac{1}{\rho-\sigma}\,\|\sigma[\nu(k,\vartheta) - \rho\nu(k,\vartheta-1) - \eta_1(k,\vartheta)]\| \\ &+ \frac{1}{\rho-\sigma}\,\|\rho[\eta_2(k,\vartheta) - \nu(k,\vartheta) - \sigma\nu(k,\vartheta-1)]\| \end{aligned}$$

Using the relations (3.4), (3.5), (3.8) and (3.9), we obtain

$$\|\nu(k,\vartheta) - \Lambda(k,\vartheta)\| \le \frac{\epsilon}{2(3k-1)}.$$

□

4. Uniqueness of the solution

We prove that the function Λ is unique. The following result is crucial for the proof.

Lemma 4.1. *Let $(\mathcal{X}, \|.\|)$ be a real normed space. Let $v_1, v_2 \in \mathcal{X}$. Suppose that, for some $c \geq 0$, we have*

$$\|B_{k,\vartheta+1}v_1 + B_{k,\vartheta}v_2\| \leq c$$

for every $\vartheta \in \mathbb{N}$. Then $\rho v_1 + v_2 = 0$.

Proof. We have

$$\begin{aligned} B_{k,\vartheta}\|\rho v_1 + v_2\| &= \|\rho B_{k,\vartheta}v_1 + B_{k,\vartheta}v_2 + B_{k,\vartheta+1}v_1 - B_{k,\vartheta+1}v_1\| \\ &\leq \|B_{k,\vartheta+1}v_1 + B_{k,\vartheta}v_2\| + |B_{k,\vartheta+1} - \rho B_{k,\vartheta}|\|v_1\| \\ &\leq c + |\frac{\rho^{\vartheta+1} - \sigma^{\vartheta+1}}{\rho - \sigma} - \rho\frac{\rho^{\vartheta} - \sigma^{\vartheta}}{\rho - \sigma}|\|v_1\| \\ &= c + |\sigma|^n|\|v_1\| \end{aligned}$$

Since $|\sigma| < 1$ and $B_{k,\vartheta} \to \infty$ as $\vartheta \to \infty$, we deduce that $\rho v_1 + v_2 = 0$. □

Theorem 4.2. *The solution of the Hyers-Ulam problem for k-balancing functions is unique.*

Proof. Suppose that there exist k-balancing functions Λ_1 and Λ_2 that satisfy

$$\|\nu(k,\vartheta) - \Lambda_i(k,\vartheta)\| \leq \frac{\epsilon}{2(3k-1)}.$$

for $i \in \{1, 2\}$. Then, for $i \in \{1, 2\}$, there exist functions $\mu_i : \mathbb{N} \times [-1, 1) \to \mathcal{B}$ such that

$$\nu(k,\vartheta) = \begin{cases} B_{k,[\vartheta]+1}\mu_i(k,\vartheta-[\vartheta]) - B_{k,[\vartheta]}\mu_i(k,\vartheta-[\vartheta]-1), & \text{if } \vartheta \geq 0, \\ -B_{k,-[\vartheta]-1}\mu_i(k,\vartheta-[\vartheta]) + B_{k,-[\vartheta]}\mu_i(k,\vartheta-[\vartheta]-1), & \text{if } \vartheta < 0. \end{cases} \tag{4.1}$$

Fixing $t \in [0, 1)$, we get

$$\begin{aligned} \|\Lambda_1(k,\vartheta+t) - \Lambda_2(k,\vartheta+t)\| &\leq \|\Lambda_1(k,\vartheta+t) - \nu(k,\vartheta+t)\| \\ &+ \|\nu(k,\vartheta+t) - \Lambda_2(k,\vartheta+t)\| \\ &\leq \frac{\epsilon}{(3k-1)}, \end{aligned}$$

for all $k \in \mathbb{N}$ and $\vartheta \in \mathbb{Z}$. Set $\mu_{k,t} = \mu_1(k,t) - \mu_2(k,t)$. Then, for all $k, \vartheta \in \mathbb{N}$, we have

$$\begin{aligned} \|B_{k,\vartheta+1}\mu_{k,t} - B_{k,\vartheta}\mu_{k,t-1}\| &\leq \|\Lambda_1(k,\vartheta+t) - \Lambda_2(k,\vartheta+t)\| \\ &\leq \frac{\epsilon}{(3k-1)}. \end{aligned}$$

And,

$$\begin{aligned} \|B_{k,\vartheta-1}\mu_{k,t} - B_{k,\vartheta}\mu_{k,t-1}\| &\leq \|\Lambda_1(k,-\vartheta+t) - \Lambda_2(k,-\vartheta+t)\| \\ &\leq \frac{\epsilon}{(3k-1)}. \end{aligned}$$

Lemma (4.1) entails that

$$\begin{aligned} \rho\mu_{k,t} - \mu_{k,t-1} &= 0 \\ -\rho\mu_{k,t-1} + \mu_{k,t} &= 0. \end{aligned}$$

Solving the equations and using $-\rho^2 + 1 \neq 0$, we arrive that

$$\mu_{k,t} = \mu_{k,t-1} = 0.$$

Equivalently,

$$\mu_1(k,t) - \mu_2(k,t) = \mu_1(k,t-1) - \mu_2(k,t-1) = 0.$$

This holds for arbitrary $t \in [0,1)$. By Eq.(4.1), we deduce that $\Lambda_1(k,\vartheta) = \Lambda_2(k,\vartheta)$ for all $\vartheta \in \mathbb{R}$. □

5. Conclusion

The k-balancing functional equation is solved. The Hyers-Ulam problem is solved, in real Banach spaces, for this equation. The uniqueness of the solution is also proved. In future, the same strategy may be applied to study other equations related to different sequences of integers.

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