

A UNIFIED ANALYSIS OF STABILIZED MIXED FINITE ELEMENT METHODS FOR THE GENERALIZED STOKES PROBLEM

AKRAM BOUKABACHE¹

ABSTRACT. In this paper, we present a unified analytical framework for stabilized finite element methods applied to the generalized Stokes problem using $P_1 - P_0$ and $Q_1 - Q_0$ element pairs. Stability is attained by including discrete pressure stabilization terms within the variational formulation. Specifically, we investigate the global jump stabilization method, the local jump stabilization method, and their reduced variants. The global approach penalizes pressure jumps across all interior element interfaces, whereas the local and reduced techniques confine these penalties to macro-element partitions through localized jump operators. We establish a unified theoretical framework that encompasses all stabilization variants within a single analytical setting. This framework enables systematic and rigorous proofs of stability and convergence.

Keywords. Generalized Stokes problem, finite elements, mixed methods, stabilized methods.

2020 Mathematics Subject Classification. Primary 76D07, 65N30; Secondary 65N15.

1. INTRODUCTION

The Stokes problem describes a wide range of fluid flows in which viscous forces dominate over inertial forces. This transpires across a diverse array of applications in both nature and technology (see, e.g., [24, 19]). The Darcy-Stokes-Brinkman problem, or Brinkman problem, is an extension of the Stokes problem that takes into account Darcy-type resistance in very porous media [8]. The generalized Stokes problem, as it is commonly known, is a particular type of the Navier-Stokes equations with time dependence, where nonlinear terms are handled explicitly within the framework of a semi-implicit temporal discretization scheme. The generalized Stokes problem incorporates both Darcy and Stokes equations for porous media and viscous fluid flow into a single equation. If the viscosity disappears, the model becomes the Darcy problem, and if the permeability is set to zero, it becomes the Stokes problem. Thus, this is important when studying the coupling between Stokes and Darcy equations. This problem can be used

Date: Received: Jan 23, 2026; Revised: Feb 19, 2026; Accepted: March 1, 2026.

© The Author(s) 2025. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit <https://creativecommons.org/licenses/by-nc-nd/4.0/>.

to model a variety of fluid flows, including groundwater flow; see, for instance, [13, 17].

The numerical approximation of viscous flow problems has been the subject of extensive investigation in the literature; see, for instance, [10, 4, 12, 5, 21]. The Stokes and generalized Stokes equations appear simple, but their numerical approximation has been found to be a bit difficult. The well-known Babuška-Brezzi condition [6, 1] lies at the heart of the problem. In simple terms, the condition states that a stable and convergent discrete formulation requires careful selection of the discrete spaces. Low-order mixed approximation methods are very popular in engineering practice, since they have simple and convenient implementations. Nonetheless, many of them, including the mixed approaches known as $Q_1 - Q_1$, $P_1 - P_1$, $Q_1 - Q_0$ and $P_1 - P_0$, do not fulfill the Babuška-Brezzi criteria. The instability of these pairs motivated several researchers to find ways to overcome this drawback. The idea of so-called stabilization was first introduced in [7] for the $P_1 - P_1$ equal-order pair. Following that, Hughes and Franca [16] developed a stabilized technique for solving Stokes equations. To ensure what is known as universal stability for discontinuous pressure estimates, additional pressure jump elements can be included into the usual Galerkin discrete approach. The jump terms must, however, handle pressure jumps along every shared boundary between elements in order to provide stability. At the beginning of the 1990s, Silvester and Kechkar [23] established a more rigorous stabilization framework for mixed methods with discontinuous pressure approximations. Their approach, called local jump stabilization, restricts the global jump operator to a macro-element subdivision of the problem domain. Additionally, it was demonstrated in [18] that local jump stabilization may regain the optimal convergence behavior for the $P_1 - P_0$ and $Q_1 - Q_0$ approaches. A fundamental aspect of this stabilization technique is that it can be implemented using a conventional macroelement, allowing the resulting stabilized formulation to be incorporated within element-by-element iterative solution approaches. The stabilized $P_1 - P_0$ and $Q_1 - Q_0$ approaches, presented and analyzed in [23, 18] have been applied in many applications, see, e.g., [10, 20].

Recently, Chibani and Kechkar proposed in [10] some modifications of the local jump stabilization approach for stabilizing the $Q_1 - Q_0$ element through the so-called reduced local jump stabilization methods. These strategies simplify the discrete formulation by reducing the number of local jump terms to just two jumps—or even a single jump—for each 2×2 macro-element. From a theoretical perspective, the global stabilization method differs significantly from the three locally stabilized variants. Likewise, the locally stabilized method and its two reduced versions are based on distinct theoretical principles, making direct comparisons challenging. Thus far, only numerical comparisons have been conducted in [10] to evaluate these jump stabilization techniques on some test problems. To address this challenge, adopting a unified approach for analyzing stabilized methods applied to generalized Stokes equations proves highly advantageous. Such an approach not only provides a cohesive theoretical framework but also enables meaningful comparisons between different stabilization techniques. Furthermore, this unified perspective is particularly well-suited for infiltration problems, such

as those involving coupled (Navier-)Stokes and Darcy equations. These problems often require a consistent framework to analyze both equations simultaneously, despite their differing stabilization terms and physical characteristics [9, 11]. In this paper, we propose a general unified methodology for analyzing the stability and accuracy of both globally stabilized and locally stabilized (including reduced variants) discrete formulations for $Q_1 - P_0$ and $P_1 - P_0$ elements within the framework of the generalized Stokes problem. We present a detailed examination of these methods and demonstrate their integration into this unified framework.

The remainder of this paper is structured as follows. In the next section, we present the considered continuous problem and the weak formulations that are used in the subsequent analysis. In Section 3, the finite element discretizations and the various jump stabilized formulations are reviewed. Then, Stability and convergence analysis are obtained in Sections 4 and 5, respectively. Finally, some closing observations are made in Section 6.

2. GENERALIZED STOKES PROBLEM

Consider an open, bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with a boundary $\partial\Omega$ that is polygonal when $d = 2$ or polyhedral when $d = 3$. We study the generalized Stokes problem, which consists in finding a velocity $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and a pressure $p : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \alpha \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where α and μ are two positive parameters that cannot vanish simultaneously, denote the inverse of the permeability and the viscosity coefficient, respectively, and $\mathbf{f}$ is a body force.

We begin by introducing the notation and function spaces needed for the weak formulation of (2.1). We denote by $H^m(\Omega) = W^{m,2}(\Omega)$ the usual Sobolev space equipped with the norm $\|\cdot\|_{m,\Omega}$ and the seminorm $|\cdot|_{m,\Omega}$. In particular, $H^0(\Omega) = L^2(\Omega)$ equipped with the standard inner product $(\cdot, \cdot)_{0,\Omega}$ and the norm $\|\cdot\|_{0,\Omega}$. Furthermore, we define $\mathbf{H}^m(\Omega)$ to consist of functions $\mathbf{u} = (u_1, \dots, u_d)$, where each u_i belongs to $H^m(\Omega)$. The norm and seminorm on $\mathbf{H}^m(\Omega)$ are as follows

$$\|\mathbf{w}\|_{m,\Omega} = \left(\sum_{i=1}^d \|w^{(i)}\|_{m,\Omega}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad |\mathbf{w}|_{m,\Omega} = \left(\sum_{i=1}^d |w^{(i)}|_{m,\Omega}^2 \right)^{\frac{1}{2}}.$$

We further introduce the spaces

$$\mathbf{V} := \mathbf{H}_0^1(\Omega) = \{ \mathbf{w} \in \mathbf{H}^1(\Omega) : \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \},$$

and

$$P := L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int_{\Omega} p dx = 0 \right\}.$$

For all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, the identity [15]

$$(\nabla \mathbf{v}, \nabla \mathbf{u})_{0,\Omega} = (\operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{u})_{0,\Omega} + (\nabla \times \mathbf{v}, \nabla \times \mathbf{u})_{0,\Omega},$$

holds, as the homogeneous Dirichlet boundary condition eliminates all boundary terms arising in the integration-by-parts formulas. Consequently, setting $\mathbf{v} = \mathbf{u}$ yields

$$\|\operatorname{div} \mathbf{u}\|_{0,\Omega} \leq \|\nabla \mathbf{u}\|_{0,\Omega}, \quad (2.2)$$

a bound that relies on the assumption $\mathbf{u} \in \mathbf{V}$.

The problem (2.1) admits the following weak formulation:

$$\begin{aligned} &\text{Find } (\mathbf{u}, p) \in \mathbf{V} \times P \text{ satisfying} \\ &\alpha (\mathbf{u}, \mathbf{v})_{0,\Omega} + \mu (\nabla \mathbf{u}, \nabla \mathbf{v})_{0,\Omega} - (p, \operatorname{div} \mathbf{v})_{0,\Omega} = (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathbf{V}, \\ &\quad - (q, \operatorname{div} \mathbf{u})_{0,\Omega} = 0 \quad \forall q \in P. \end{aligned} \quad (2.3)$$

The second equation of (2.3) is multiplied by -1 to obtain a symmetric formulation. In addition, we assume that $\mathbf{f}$ is in $\mathbf{L}^2(\Omega)$ to ensure that (2.3) is meaningful.

The existence of a unique solution $(\mathbf{u}, p)$ in (2.3) is ensured by the Babuška-Brezzi theory [6, 1].

The problem (2.3) can be rewritten in the equivalent form:

$$\begin{aligned} &\text{Find } (\mathbf{u}, p) \in \mathbf{V} \times P \text{ such that} \\ &Q((\mathbf{u}, p); (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times P, \end{aligned} \quad (2.4)$$

where

$$Q((\mathbf{u}, p); (\mathbf{v}, q)) = \alpha (\mathbf{u}, \mathbf{v})_{0,\Omega} + \mu (\nabla \mathbf{u}, \nabla \mathbf{v})_{0,\Omega} - (p, \operatorname{div} \mathbf{v})_{0,\Omega} - (q, \operatorname{div} \mathbf{u})_{0,\Omega}.$$

3. FINITE ELEMENT FORMULATION

The domain Ω is partitioned into triangles or quadrilaterals, forming a partitioning $\mathcal{T}_h$ that satisfies the standard regularity conditions (see [15, 18]). Specifically, there exist constants $\varrho > 1$ and $0 < \psi < 1$,

$$h_K \leq \varrho \sigma_K \quad \text{and} \quad |\cos \theta_{i,K}| \leq \psi, \quad \forall K \in \mathcal{T}_h \quad (3.1)$$

where h_K denotes the diameter of K , σ_K is the radius of the largest circle contained in K and $\theta_{i,K}$ ($i = 1, \dots, 4$) are the angles of K when the partitioning consists of quadrilaterals. The discretization parameter h is defined as $h = \max_{K \in \mathcal{T}_h} h_K$.

The finite element method of interest in this paper consists of seeking approximations $(\mathbf{u}_h, p_h)$ of $(\mathbf{u}, p)$, where $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$. Here, $\mathbf{V}_h$ and Q_h are the classical finite element spaces defined by

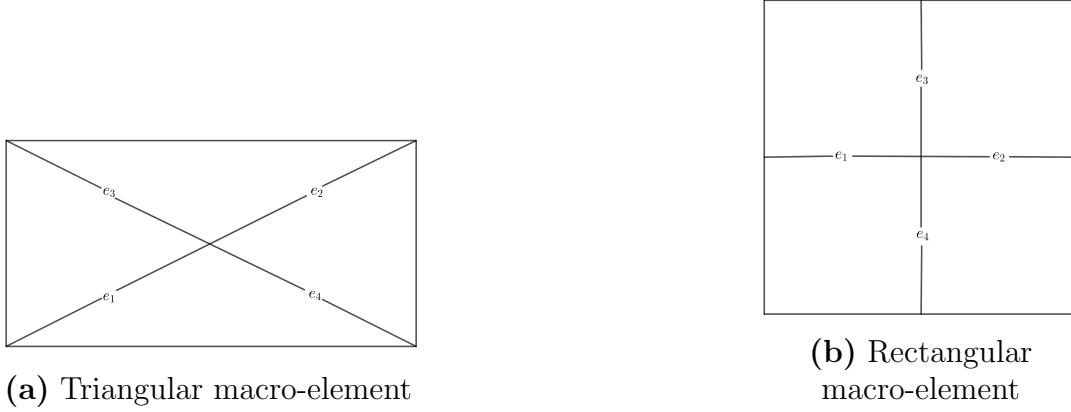
$$\mathbf{V}_h = \left\{ u \in \mathbf{V} \cap \mathbf{C}^0 : \mathbf{u}|_K \in [R_1(K)]^d, \forall K \in \mathcal{T}_h \right\}, \quad (3.2)$$

and

$$P_h = \left\{ p \in P : p|_K \in P_0(K), \forall K \in \mathcal{T}_h \right\}, \quad (3.3)$$

where R_1 represents (P_1) on triangles or (Q_1) on quadrilaterals.

As it was pointed out above, the combination $\mathbf{V}_h \times P_h$ is unstable in the standard Babuška-Brezzi sense. For simplicity of presentation, we will restrict our attention to rectangular axis-parallel meshes of Ω when K is quadrilateral. However, using standard isoparametric transformations, the findings can be adapted to accommodate more general quadrilateral meshes.

FIGURE 1. A 2×2 macro-element.

Based on the previously defined finite element spaces, we now recall the global jump stabilization formulation

$$s_h(p_h, q_h) = \beta \sum_{K \in \mathcal{T}_h} \int_{\partial K \setminus \partial \Omega} h_{\partial K} \llbracket p_h \rrbracket \llbracket q_h \rrbracket ds, \quad (3.4)$$

with $\beta > 0$ is the global stabilization coefficient and $\llbracket \cdot \rrbracket$ denotes the jump operator along interior edges or faces, excluding those on $\partial \Omega$.

To develop a locally stabilized formulation of the generalized Stokes problem, we define a macro-element subdivision $\mathcal{M}_h$ as described below (see [18]): A macro-element subdivision $\mathcal{M}_h$ can be constructed for any given mesh $\mathcal{T}_h$ such that each macro-element M_K consists of a connected group of neighboring elements from $\mathcal{T}_h$. Each element K is entirely contained within a single macro-element, ensuring that macro-elements remain disjoint. For each M_K , let $\mathcal{E}_{M_K}$ represent the collection of inter-element edges e located strictly within the interior of M_K . Additionally, for every edge $e \in \mathcal{E}_{M_K}$, its length is denoted by h_e .

We assume that for the triangular $P_1 - P_0$ pair the macro-element satisfies the following geometrical assumption: there should be only one edge of a element which is an external edge of the macro-element to which this element belongs.

For instance, to illustrate the concept of macro-element, the $Q_1 - P_0$ and $P_1 - P_0$ methods can be stabilized by constructing the mesh $\mathcal{T}_h$ where the elements are organized into non-overlapping 2×2 macro-elements (see Figure 1).

With these additional definitions, the choice of (3.4) can be simplified into a local jump stabilization formulation [18]

$$s_h(p_h, q_h) = \beta \sum_{M_K \in \mathcal{M}_h} \sum_{e \in \mathcal{E}_{M_K}} h_e \int_e \llbracket p_h \rrbracket_e \llbracket q_h \rrbracket_e ds \quad (3.5)$$

where $\beta > 0$ is the local stabilization coefficient and $\llbracket \cdot \rrbracket_e$ represents the jump operator along inter-element edges entirely contained within each macro-element.

Furthermore, Chibani and Kechkar in [10] showed that the local jump method can be reduced to the local (two or one)-jump formulations for the $Q_1 - Q_0$ mixed element. Thanks to the macro-element geometrical assumption, we can

stabilized the mixed $P_1 - P_0$ method by the reduced local jump stabilization technique discussed in [10]. To this end, we account for pressure jumps in one direction (parallel jumps) across edges located entirely inside each macro-element in the case of two-jump formulations. For one-jump formulations, we consider a single jump along one of the four interior boundaries of the M -th macro-element.

Thus, we assume that $\mathcal{E}_{M_K}$ is partitioned into $\mathcal{E}_{M_K} = \mathcal{E}_{M_K}^H \cup \mathcal{E}_{M_K}^V$ such that $\mathcal{E}_{M_K}^H = \{e_1, e_2\}$ and $\mathcal{E}_{M_K}^V = \{e_3, e_4\}$, see Figure 1. Furthermore, we use the notation $\mathcal{E}_{M_K}^X$ where X might be either H or V .

The reduced local two-jump representation is written as

$$s_h(p_h, q_h) = \beta \sum_{M_K \in \mathcal{M}_h} \sum_{e \in \mathcal{E}_{M_K}^X} h_e \int_e \llbracket p_h \rrbracket_e \llbracket q_h \rrbracket_e ds, \quad (3.6)$$

and the reduced local one-jump formulations by

$$s_h(p_h, q_h) = \beta \sum_{M_K \in \mathcal{M}_h} h_e \int_e \llbracket p_h \rrbracket_e \llbracket q_h \rrbracket_e ds, \quad (3.7)$$

where $e \in \mathcal{E}_{M_K}$.

The stabilized discrete formulation of the problem (2.3) may be expressed as follows:

$$\begin{aligned} & \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h \text{ such that} \\ & \alpha(\mathbf{u}_h, \mathbf{v}_h)_{0,\Omega} + \mu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{0,\Omega} - (p_h, \operatorname{div} \mathbf{v}_h)_{0,\Omega} = (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ & - (q_h, \operatorname{div} \mathbf{u}_h)_{0,\Omega} - s_h(p_h, q_h) = 0 \quad \forall q_h \in P_h, \end{aligned} \quad (3.8)$$

where s_h is one of the previous stabilization terms presented before.

Here again, we can rewrite (3.8) as:

$$\begin{aligned} & \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h \text{ such that} \\ & Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h, \end{aligned} \quad (3.9)$$

where

$$Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = Q((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) - s_h(p_h, q_h). \quad (3.10)$$

Assuming that the pressure belongs to $H^1(\Omega)$, the penalization term is consistent; thus, we obtain the following result.

Lemma 3.1. *If $(\mathbf{u}, p)$ is the weak solution of (2.1) with $(\mathbf{u}, p) \in \mathbf{V} \times H^1(\Omega)$ then*

$$Q_h((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)) = 0 \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h. \quad (3.11)$$

Proof. By the definition of Q_h , we have

$$\begin{aligned} Q_h((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)) &= Q_h((\mathbf{u}, p); (\mathbf{v}_h, q_h)) - Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) \\ &= Q((\mathbf{u}, p); (\mathbf{v}_h, q_h)) - s_h(p, q_h) - Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)). \end{aligned}$$

From (2.4) and (3.9), it follows that

$$Q_h((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)) = -s_h(p, q_h).$$

Since $p \in H^1(\Omega)$, its trace is single-valued across every interior interface; that is, the values from adjacent elements coincide on each interior face. Hence, the inter-element jump $\llbracket p \rrbracket$ vanishes, and it follows that $s_h(p, q_h) = 0$ for all $q_h \in P_h$. $\square$

4. STABILITY

We begin by equipping the space $\mathbf{V}_h \times P_h$ with the following norm

$$\|(\mathbf{u}_h, p_h)\|_s^2 = \|\mathbf{u}_h\|_{1,\Omega}^2 + \|p_h\|_{0,\Omega}^2 + s_h(p_h, p_h).$$

Throughout this analysis, c denotes a positive constant, possibly with a subscript, independent of h but possibly depending on α and μ .

Let π_h denote the Scott–Zhang interpolation operator for both the velocity and pressure spaces, i.e., $\pi_h : \mathbf{L}_2(\Omega) \rightarrow \mathbf{V}_h$ and $\pi_h : L_2(\Omega) \rightarrow P_h$. This operator satisfies the following standard properties ([22, 3, 14]):

(i): Stability:

$$\|\pi_h v\|_{m,\Omega} \leq c \|v\|_{l,\Omega}, \quad \forall v \in H^l(\Omega) \quad (4.1)$$

for $0 \leq m \leq \min(1, l)$.

(ii): Interpolation:

$$\|v - \pi_h v\|_{m,\Omega} \leq c h^{l-m} \|v\|_{l,\Omega}, \quad \forall v \in H^l(\Omega) \quad (4.2)$$

for $0 \leq m \leq l$.

(iii): Orthogonality:

$$(v - \pi_h v, q_h)_{0,\Omega} = 0, \quad \forall q_h \in P_h. \quad (4.3)$$

Finally, we have the trace inequality [25]:

$$\|v\|_{0,\partial K}^2 \leq c \left(h_K^{-1} \|v\|_{0,K}^2 + h_K \|v\|_{1,K}^2 \right), \quad \forall v \in H^1(K). \quad (4.4)$$

We can now establish that the stabilized discrete formulation (3.9) is well-posed.

Theorem 4.1. *The formulation (3.9) satisfies the following conditions:*

(C1): For all $(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h$ we have

$$|Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h))| \leq \delta_1 \|(\mathbf{u}_h, p_h)\|_s \|(\mathbf{v}_h, q_h)\|_s \quad (4.5)$$

where δ_1 is a positive constant independent of h .

(C2): For all $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h$, there exists a pair $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h$ such that

$$Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) \geq \delta_2 \|(\mathbf{u}_h, p_h)\|_s \|(\mathbf{v}_h, q_h)\|_s \quad (4.6)$$

where δ_2 is a positive constant independent of h .

Proof. Using Cauchy-Schwarz inequality followed by (2.2), we obtain

$$\begin{aligned} |Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h))| &\leq c_1 \|\mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} + \|p_h\|_{0,\Omega} \|\mathbf{v}_h\|_{1,\Omega} + \|q_h\|_{0,\Omega} \|\mathbf{u}_h\|_{1,\Omega} \\ &\quad + (s_h(p_h, p_h))^{1/2} (s_h(q_h, q_h))^{1/2}. \end{aligned}$$

Noting that $\sqrt{a} \leq \sqrt{a+b+c}$ for $a, b, c > 0$, we obtain

$$|Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h))| \leq \delta_1 \|(\mathbf{u}_h, p_h)\|_s \|(\mathbf{v}_h, q_h)\|_s.$$

That is, (4.5) is satisfied.

Setting $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, -p_h)$ in (3.10) yields

$$Q_h((\mathbf{u}_h, p_h); (\mathbf{u}_h, -p_h)) \geq c_2 \|\mathbf{u}_h\|_{1,\Omega}^2 + s_h(p_h, p_h). \quad (4.7)$$

Let $p_h \in P_h \subset P$ be arbitrary and ξ_1 a positive constant to be determined below. Since $\xi_1 p_h \in P$, it is well known that by the surjectivity of the divergence operator [15], there is a function $\mathbf{v}_p \in \mathbf{V}$ such that

$$\operatorname{div} \mathbf{v}_p = \xi_1 p_h \quad \text{and} \quad \|\mathbf{v}_p\|_{1,\Omega} \leq c_3 \|p_h\|_{0,\Omega}. \quad (4.8)$$

Now, choosing $(\mathbf{v}_h, q_h) = (-\pi_h \mathbf{v}_p, 0)$ in (3.10) and use the fact that

$$\xi_1 \|p_h\|_{0,\Omega}^2 - (p_h, \operatorname{div} \mathbf{v}_p) = 0,$$

by the properties of $\mathbf{v}_p$ yields

$$\begin{aligned} Q_h((\mathbf{u}_h, p_h); (-\pi_h \mathbf{v}_p, 0)) &= -\alpha (\mathbf{u}_h, \pi_h \mathbf{v}_p)_{0,\Omega} - \mu (\nabla \mathbf{u}_h, \nabla (\pi_h \mathbf{v}_p))_{0,\Omega} + \xi_1 \|p_h\|_{0,\Omega}^2 \\ &\quad + (p_h, \operatorname{div} (\pi_h \mathbf{v}_p - \mathbf{v}_p))_{0,\Omega}. \end{aligned}$$

We estimate each term. Using Young's inequality with $\varepsilon > 0$ together with the stability of π_h (4.1) and (4.8) gives

$$\left| \alpha (\mathbf{u}_h, \pi_h \mathbf{v}_p)_{0,\Omega} \right| \leq \frac{1}{\varepsilon} \|\mathbf{u}_h\|_{0,\Omega}^2 + c_4 \varepsilon \|p_h\|_{0,\Omega}^2, \quad (4.9)$$

$$\left| \mu (\nabla \mathbf{u}_h, \nabla (\pi_h \mathbf{v}_p))_{0,\Omega} \right| \leq \frac{1}{\varepsilon} \|\nabla \mathbf{u}_h\|_{0,\Omega}^2 + c_5 \varepsilon \|p_h\|_{0,\Omega}^2, \quad (4.10)$$

where the constants c_4 and c_5 are proportional to α^2 and μ^2 , respectively.

Consequently,

$$-\alpha (\mathbf{u}_h, \pi_h \mathbf{v}_p)_{0,\Omega} \geq -\frac{1}{\varepsilon} \|\mathbf{u}_h\|_{0,\Omega}^2 - c_4 \varepsilon \|p_h\|_{0,\Omega}^2, \quad (4.11)$$

$$-\mu (\nabla \mathbf{u}_h, \nabla (\pi_h \mathbf{v}_p))_{0,\Omega} \geq -\frac{1}{\varepsilon} \|\nabla \mathbf{u}_h\|_{0,\Omega}^2 - c_5 \varepsilon \|p_h\|_{0,\Omega}^2. \quad (4.12)$$

Finally, using Cauchy-Schwarz inequality, (4.2) and (4.8) we get

$$\left| (p_h, \operatorname{div} (\pi_h \mathbf{v}_p - \mathbf{v}_p))_{0,\Omega} \right| \leq c_6 \|p_h\|_{0,\Omega} \|\mathbf{v}_p\|_{1,\Omega} \leq c_7 \|p_h\|_{0,\Omega}^2, \quad (4.13)$$

and hence

$$(p_h, \operatorname{div} (\pi_h \mathbf{v}_p - \mathbf{v}_p))_{0,\Omega} \geq -c_7 \|p_h\|_{0,\Omega}^2. \quad (4.14)$$

Then combining (4.11), (4.12) and (4.14) yields

$$Q_h((\mathbf{u}_h, p_h); (-\pi_h \mathbf{v}_p, 0)) \geq -\frac{1}{\varepsilon} \|\mathbf{u}_h\|_{1,\Omega}^2 + (\xi_1 - (c_8 \varepsilon + c_7)) \|p_h\|_{0,\Omega}^2, \quad (4.15)$$

with choosing $\xi_1 > c_8 \varepsilon + c_7$, where $c_8 = \max(c_4, c_5)$.

To conclude we take $(\mathbf{v}_h, q_h) = (\xi_2 \mathbf{u}_h - \pi_h \mathbf{v}_p, -\xi_2 p_h)$ with ξ_2 sufficiently large. which yields by (4.7) and (4.15)

$$\begin{aligned} Q_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) &\geq \left(\xi_2 c_2 - \frac{1}{\varepsilon} \right) \|\mathbf{u}_h\|_{1,\Omega}^2 + (\xi_1 - (c_8 \varepsilon + c_7)) \|p_h\|_{0,\Omega}^2 + \xi_2 s_h(p_h, p_h) \\ &\geq c_9 \|(\mathbf{u}_h, p_h)\|_s^2, \end{aligned} \quad (4.16)$$

where $c_9 = \min(\xi_2 c_2 - \frac{1}{\varepsilon}, \xi_1 - (c_8 \varepsilon + c_7), \xi_2)$.

In addition, we have

$$\begin{aligned}
\|(\mathbf{v}_h, q_h)\|_s^2 &= \|\xi_2 \mathbf{u}_h - \pi_h \mathbf{v}_p\|_{1,\Omega}^2 + \xi_2^2 \|p_h\|_{0,\Omega}^2 + \xi_2^2 s_h(p_h, p_h) \\
&\leq 2\xi_2 \|\mathbf{u}_h\|_{1,\Omega}^2 + (2c_{10} + \xi_2^2) \|p_h\|_{0,\Omega}^2 + \xi_2^2 s_h(p_h, p_h) \\
&\leq c_{11}^2 \left(\|\mathbf{u}_h\|_{1,\Omega}^2 + \|p_h\|_{0,\Omega}^2 + s_h(p_h, p_h) \right) \\
&= c_{11}^2 \|(\mathbf{u}_h, p_h)\|_s^2.
\end{aligned} \tag{4.17}$$

The claim now follows by combining (4.16) and (4.17) with $\delta_2 = \frac{c_9}{c_{11}}$. $\square$

Remark 4.2. The constants δ_1 and δ_2 in Theorem 4.1 are independent of the mesh size h , but they do depend on the physical parameters μ and α . In particular, δ_2 deteriorates as $\mu \rightarrow 0$, corresponding to the Darcy limit of the generalized Stokes problem. This behavior stems from the choice of the energy norm $\|\cdot\|_s$, which is natural for the Stokes-dominated regime, as it includes the full H^1 -norm of the velocity; in the Darcy limit, uniform stability would require a different triple norm incorporating control of the divergence of the velocity, for instance, through the L^2 -norm of $\nabla \cdot \mathbf{u}$.

Remark 4.3. The local stabilization term can also be replaced by alternative reduced forms by minimizing the number of local jump terms within the discretized formulation. For instance, this can involve reducing the terms to three jumps or to two non-parallel jumps within each 2×2 macro-element.

5. A PRIORI ERROR ANALYSIS

The stability result established earlier, together with the fact that the stabilized formulation is consistent, yields an a priori error estimate. To begin, we present the following property.

Lemma 5.1. *For $p \in H^1(\Omega)$. Then, we have*

$$s_h(\pi_h p - p, \pi_h p - p) \leq ch^2 \|p\|_{1,\Omega}^2. \tag{5.1}$$

Proof. Since the global jump operator takes into account all the inter-element boundaries, whereas the local jump operators discards those not strictly within any macro-element from $\mathcal{M}_h$, we have

$$s_h(\pi_h p - p, \pi_h p - p) \leq \beta \sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} \llbracket \pi_h p - p \rrbracket^2 ds.$$

Using the inequality $(a - b)^2 \leq 2a^2 + 2b^2$, which holds for all real numbers a and b , we obtain:

$$s_h(\pi_h p - p, \pi_h p - p) \leq c_1 \sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |\pi_h p - p|^2 ds.$$

Further, we have, using (4.4) and (4.2),

$$\begin{aligned}
\|p - \pi_h p\|_{0,\partial K}^2 &\leq c_2 \left(h_K^{-1} \|p - \pi_h p\|_{0,K}^2 + h_K \|p - \pi_h p\|_{1,K}^2 \right) \\
&\leq c_3 h_K \|p\|_{1,K}^2.
\end{aligned}$$

By summation, we obtain:

$$s_h(\pi_h p - p, \pi_h p - p) \leq ch^2 \|p\|_{1,\Omega}^2.$$

□

Lemma 5.2. *Let $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$. Then, we have*

$$\|(\mathbf{u} - \pi_h \mathbf{u}, p - \pi_h p)\|_s \leq ch \left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \right). \quad (5.2)$$

Proof. By writing the terms in $\|(\mathbf{u} - \pi_h \mathbf{u}, p - \pi_h p)\|_s^2$, we get

$$\|(\mathbf{u} - \pi_h \mathbf{u}, p - \pi_h p)\|_s^2 = \|\mathbf{u} - \pi_h \mathbf{u}\|_{1,\Omega}^2 + \|p - \pi_h p\|_{0,\Omega}^2 + s_h(p - \pi_h p, p - \pi_h p).$$

We estimate the three terms as follows

$$\|\mathbf{u} - \pi_h \mathbf{u}\|_{1,\Omega}^2 \leq c_1 h^2 \|\mathbf{u}\|_{2,\Omega}^2,$$

and

$$\|p - \pi_h p\|_{0,\Omega}^2 \leq c_2 h^2 \|p\|_{1,\Omega}^2,$$

where we used the standard interpolation error estimates (4.2). Further, (5.1) implies

$$|s_h(p - \pi_h p, p - \pi_h p)| \leq c_5 h^2 \|p\|_{1,\Omega}^2.$$

Collecting the above bounds provides the required inequality (5.2). □

Theorem 5.3. *Let $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h$ be the solution of (2.4) and (3.9), respectively. Then, we have*

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \leq ch \left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \right). \quad (5.3)$$

Proof. First, we note that

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \leq \|(\mathbf{u} - \pi_h \mathbf{u}, p - \pi_h p)\|_s + \|(\pi_h \mathbf{u} - \mathbf{u}_h, \pi_h p - p_h)\|_s.$$

By Theorem 4.1, we have

$$\|(\pi_h \mathbf{u} - \mathbf{u}_h, \pi_h p - p_h)\|_s \leq \frac{1}{\delta_2} \sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h} \frac{Q_h((\pi_h \mathbf{u} - \mathbf{u}_h, \pi_h p - p_h); (\mathbf{v}_h, q_h))}{\|(\mathbf{v}_h, q_h)\|_s}$$

and, by Lemma 3.1,

$$\|(\pi_h \mathbf{u} - \mathbf{u}_h, \pi_h p - p_h)\|_s \leq \frac{1}{\delta_2} \sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times P_h} \frac{Q_h((\pi_h \mathbf{u} - \mathbf{u}, \pi_h p - p); (\mathbf{v}_h, q_h))}{\|(\mathbf{v}_h, q_h)\|_s}. \quad (5.4)$$

Expanding the terms in $Q_h((\pi_h \mathbf{u} - \mathbf{u}, \pi_h p - p); (\mathbf{v}_h, q_h))$ gives

$$\begin{aligned} Q_h((\pi_h \mathbf{u} - \mathbf{u}, \pi_h p - p); (\mathbf{v}_h, q_h)) &= \alpha (\pi_h \mathbf{u} - \mathbf{u}, \mathbf{v}_h)_{0,\Omega} + \mu (\nabla (\pi_h \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h)_{0,\Omega} \\ &\quad - (\pi_h p - p, \operatorname{div} \mathbf{v}_h)_{0,\Omega} - (q_h, \operatorname{div} (\pi_h \mathbf{u} - \mathbf{u}))_{0,\Omega} \\ &\quad - s_h(\pi_h p - p, q_h), \end{aligned} \quad (5.5)$$

The result is obtained through the use of interpolation estimates to the right-hand side. The first four terms are handled by applying Cauchy–Schwartz inequality followed by the interpolation operator properties (4.2)

$$\alpha (\pi_h \mathbf{u} - \mathbf{u}, \mathbf{v}_h)_{0,\Omega} \leq c_1 \|\pi_h \mathbf{u} - \mathbf{u}\|_{1,\Omega} \|\mathbf{v}_h\|_{0,\Omega} \leq c_2 h \|\mathbf{u}\|_{2,\Omega} \|(\mathbf{v}_h, q_h)\|_s, \quad (5.6)$$

$$\mu (\nabla (\pi_h \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h)_{0,\Omega} \leq \mu \|\nabla (\pi_h \mathbf{u} - \mathbf{u})\|_{0,\Omega} \|\nabla \mathbf{v}_h\|_{0,\Omega} \leq c_3 h \|\mathbf{u}\|_{2,\Omega} \|(\mathbf{v}_h, q_h)\|_s, \quad (5.7)$$

$$(\pi_h p - p, \operatorname{div} \mathbf{v}_h) \leq \|\pi_h p - p\|_{0,\Omega} \|\operatorname{div} \mathbf{v}_h\|_{0,\Omega} \leq c_4 h \|p_h\|_{1,\Omega} \|(\mathbf{v}_h, q_h)\|_s, \quad (5.8)$$

and

$$(q_h, \operatorname{div} (\pi_h \mathbf{u} - \mathbf{u}))_{0,\Omega} \leq \|q_h\|_{0,\Omega} \|\nabla (\pi_h \mathbf{u} - \mathbf{u})\|_{0,\Omega} \leq c_5 h \|\mathbf{u}\|_{2,\Omega} \|(\mathbf{v}_h, q_h)\|_s. \quad (5.9)$$

For the jump term, we use the same proof as in Lemma 5.1. We obtain

$$|s_h (\pi_h p - p, q_h)| \leq c_6 \left(\sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |\pi_h p - p|^2 ds \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |q_h|^2 ds \right)^{1/2}.$$

The terms are bounded as follows.

$$\begin{aligned} \left(\sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |\pi_h p - p|^2 ds \right)^{1/2} &\leq c_7 h \|p_h\|_{0,\Omega}, \\ \left(\sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |q_h|^2 ds \right)^{1/2} &\leq c_8 \|q_h\|_{0,\Omega}, \end{aligned}$$

implying that

$$|s_h (p_h, q_h)| \leq c_9 h \|p_h\|_{0,\Omega} \|(\mathbf{v}_h, q_h)\|_s.$$

So,

$$|s_h (p_h, q_h)| \leq c_{10} h \|p_h\|_{1,\Omega} \|(\mathbf{v}_h, q_h)\|_s. \quad (5.10)$$

Combining (5.6)-(5.10), we get

$$\begin{aligned} Q_h ((\pi_h \mathbf{u} - \mathbf{u}, \pi_h p - p); (\mathbf{v}_h, q_h)) &\leq c_{11} h \|\mathbf{u}\|_{2,\Omega} \|(\mathbf{v}_h, q_h)\|_s + c_{12} h \|p_h\|_{1,\Omega} \|(\mathbf{v}_h, q_h)\|_s \\ &\leq c_{13} h \left(\|\mathbf{u}\|_{2,\Omega} + \|p_h\|_{1,\Omega} \right) \|(\mathbf{v}_h, q_h)\|_s, \end{aligned}$$

which yields

$$\|(\pi_h \mathbf{u} - \mathbf{u}_h, \pi_h p - p_h)\|_s \leq c_{13} h \left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \right).$$

The Theorem follows by Lemma 5.2. $\square$

In the following, we will prove L^2 -error estimate for the velocities using the Aubin-Nitsche duality argument. Let $(\varphi, r) \in \mathbf{V} \times P$ denote the solution of the dual equation

$$Q((\mathbf{v}, q); (\varphi, r)) = (\eta, \mathbf{v})_{0,\Omega} \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times P. \quad (5.11)$$

We further assume that the dual solution satisfies the required smoothness condition

$$\|\varphi\|_{2,\Omega} + \|r\|_{1,\Omega} \leq c \|\eta\|_{0,\Omega}, \quad (5.12)$$

if the boundary is sufficiently smooth, see [15].

Theorem 5.4. *Let $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h$ be the solution of (2.4) and (3.9), respectively. Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \leq ch^2 \left(\|\mathbf{u}\|_{2,\Omega} + \|p_h\|_{0,\Omega} \right). \quad (5.13)$$

Proof. Choosing $(\mathbf{v}, q) = (\mathbf{u} - \mathbf{u}_h, p - p_h)$ and $\eta = \mathbf{u} - \mathbf{u}_h$ in (5.11) and proceed using Galerkin orthogonality to get

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 &= Q((\mathbf{u} - \mathbf{u}_h, p - p_h); (\varphi, r)) \\ &= Q((\mathbf{u} - \mathbf{u}_h, p - p_h); (\varphi - \pi_h \varphi, r - \pi_h r)) + s_h(p - p_h, \pi_h r) \\ &= \alpha(\mathbf{u} - \mathbf{u}_h, \varphi - \pi_h \varphi)_{0,\Omega} + \mu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla(\varphi - \pi_h \varphi))_{0,\Omega} \\ &\quad - (p - p_h, \operatorname{div}(\varphi - \pi_h \varphi))_{0,\Omega} - (r - \pi_h r, \operatorname{div}(\mathbf{u} - \mathbf{u}_h))_{0,\Omega} \\ &\quad + s_h(p - p_h, \pi_h r). \end{aligned} \quad (5.14)$$

We now estimate the right-hand side terms of the above equation. By applying Cauchy–Schwarz inequality followed by the interpolation operator properties we get

$$\alpha(\mathbf{u} - \mathbf{u}_h, \varphi - \pi_h \varphi)_{0,\Omega} \leq c_1 h \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \|\varphi\|_{2,\Omega}, \quad (5.15)$$

$$\mu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla(\varphi - \pi_h \varphi))_{0,\Omega} \leq c_2 h \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \|\varphi\|_{2,\Omega}, \quad (5.16)$$

$$(p - p_h, \operatorname{div}(\varphi - \pi_h \varphi))_{0,\Omega} \leq c_3 h \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \|\varphi\|_{2,\Omega}, \quad (5.17)$$

$$(r - \pi_h r, \operatorname{div}(\mathbf{u} - \mathbf{u}_h))_{0,\Omega} \leq c_4 h \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \|r\|_{1,\Omega}. \quad (5.18)$$

To estimate the jump term in (5.14), observe that $r \in H^1$ implies $s_h(p - p_h, r) = 0$. Consequently, $s_h(p - p_h, \pi_h r) = s_h(p - p_h, \pi_h r - r)$. Applying the triangle inequality and the Schwarz inequality (while leveraging the regularity condition (3.1)) yields

$$\begin{aligned} s_h(p - p_h, \pi_h r) &\leq s_h(p - p_h, p - p_h)^{1/2} s_h(\pi_h r - r, \pi_h r - r)^{1/2} \\ &\leq c_5 \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \left(\sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |\pi_h r - r|^2 ds \right)^{1/2}, \end{aligned}$$

and by using (4.4) and (4.2) we have

$$\left(\sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |\pi_h r - r|^2 ds \right)^{1/2} \leq c_6 h \|r\|_{1,\Omega},$$

implying that

$$s_h(p - p_h, \pi_h r) \leq c_7 h \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s \|r\|_{1,\Omega}. \quad (5.19)$$

Combining (5.15)–(5.19), we get

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 \leq c_8 h \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_s (\|\varphi\|_{2,\Omega} + \|r\|_{1,\Omega}).$$

We conclude by combining (5.3) with the regularity condition on the dual problem stated in (5.12). $\square$

6. CONCLUSION

In this paper, we have investigated various stabilization terms introduced in [18, 16, 10] within a unified framework for the generalized Stokes problem. Additionally, we have analyzed the reduced local jump stabilization methods for the $P_1 - P_0$ mixed element method. Stability and error estimates for all these

methods were derived within a common theoretical framework, demonstrating the versatility and robustness of the unified approach. This unified perspective is especially advantageous for infiltration problems, such as those involving coupled (Navier-)Stokes and Darcy equations. In these problems, it is critical to analyze both equations within a consistent framework, even though they involve different stabilization terms and physical characteristics. The proposed methodology offers a robust and cohesive foundation for effectively addressing these challenges. Ongoing work aims to further explore the effectiveness of the reduced local jump stabilization methods for the $P_1 - P_0$ mixed element method through comprehensive numerical benchmarks. Our objective is to refine the unified model, enhance its applicability, and develop more efficient numerical schemes to improve computational performance. These efforts will contribute to advancing the robustness and efficiency of stabilized finite element methods for complex multiphysics problems.

REFERENCES

1. Babuška, I. (1971). Error-bounds for finite element method. *Numerische Mathematik*, 16(4), 322–333. <https://doi.org/10.1007/BF02165003>
2. Babuška, I., Osborn, J., & Pitkäranta, J. (1980). Analysis of mixed methods using mesh dependent norms. *Mathematics of Computation*, 35(152), 1039–1062. <https://doi.org/10.2307/2006374>
3. Becker, R., & Braack, M. (2001). A finite element pressure gradient stabilization for the Stokes equations based on local projections. *Calcolo*, 38(4), 173–199. <https://doi.org/10.1007/s10092-001-8180-4>
4. Boukabache, A. (2022). A locally stabilized collocated finite volume method for the stationary Stokes problem. *Mathematical Methods in the Applied Sciences*, 45(17), 11714–11735. <https://doi.org/10.1002/mma.8476>
5. Boukabache, A. (2025). On the stability of collocated finite volume method for the generalized Stokes problem. *Vestnik of Udmurt University. Mathematics. Mechanics. Computer Science*, 35(2), 169–187. <https://doi.org/10.35634/vm250201>
6. Brezzi, F. (1974). On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *RAIRO Analyse Numérique*, 8(R2), 129–151. <https://doi.org/10.1051/m2an/197408R201291>
7. Brezzi, F., & Pitkäranta, J. (1984). On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch (Ed.), *Efficient solutions of elliptic systems: Notes on numerical fluid mechanics* (Vol. 10, pp. 11–19). Vieweg+Teubner Verlag. https://doi.org/10.1007/978-3-663-14169-3_2
8. Brinkman, H. C. (1949). A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Applied Scientific Research*, 1(1), 27–34. <https://doi.org/10.1007/BF02120313>
9. Burman, E., & Hansbo, P. (2007). A unified stabilized method for Stokes’ and Darcy’s equations. *Journal of Computational and Applied Mathematics*, 198(1), 35–51. <https://doi.org/10.1016/j.cam.2005.11.022>
10. Chibani, A., & Kechkar, N. (2020). Minimal locally stabilized $Q_1 - Q_0$ schemes for the generalized Stokes problem. *Journal of the Korean Mathematical Society*, 57(5), 1239–1266. <https://doi.org/10.4134/JKMS.j190628>
11. D’Angelo, C., & Zunino, P. (2012). Numerical approximation with Nitsche’s coupling of transient Stokes’/Darcy’s flow problems applied to hemodynamics. *Applied Numerical Mathematics*, 62(4), 378–395. <https://doi.org/10.1016/j.apnum.2011.03.001>

12. Dang, H., Peng, H., Zhai, Q., & Zhang, R. (2025). A stabilizer free weak Galerkin finite element method for Brinkman equations. *Journal of Computational Mathematics*, 43(1), 1–17. <https://doi.org/10.4208/jcm.2307-m2022-0264>
13. Das, D. (2002). Hydrodynamic modelling for groundwater flow through permeable reactive barriers. *Hydrological Processes*, 16(17), 3393–3418. <https://doi.org/10.1002/hyp.1107>
14. Ern, A., & Guermond, J.-L. (2004). *Theory and Practice of Finite Elements*. Springer, New York. <https://doi.org/10.1007/978-1-4757-4355-5>
15. Girault, V., & Raviart, P.-A. (2012). *Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms*. Springer Berlin, Heidelberg. <https://doi.org/10.1007/978-3-642-61623-5>
16. Hughes, T. J. R., & Franca, L. P. (1987). A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions. *Computer Methods in Applied Mechanics and Engineering*, 65(1), 85–96. [https://doi.org/10.1016/0045-7825\(87\)90184-8](https://doi.org/10.1016/0045-7825(87)90184-8)
17. Kaya, T., & Goldak, J. (2007). Three-dimensional numerical analysis of heat and mass transfer in heat pipes. *Heat and Mass Transfer*, 43(8), 775–785. <https://doi.org/10.1007/s00231-006-0166-y>
18. Kechkar, N., & Silvester, D. (1992). Analysis of locally stabilized mixed finite element methods for the Stokes problem. *Mathematics of Computation*, 58(197), 1–10. <https://doi.org/10.2307/2153016>
19. Kirby, B. (2010). *Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices*. Cambridge University Press, New York. <http://www.cambridge.org/9780521119030>
20. Liu, X., Song, Q., Gao, Y., & Chen, Z. (2024). The lowest-order stabilized virtual element method for the Stokes problem. *Communications in Computational Physics*, 36(1), 221–247. <https://doi.org/10.4208/cicp.0A-2023-0233>
21. Ouya, J., & Ouedraogo, A. (2024). A well-defined Godunov-type scheme in a Navier–Stokes model with a friction term. *Gulf Journal of Mathematics*, 16(2), 213–231. <https://doi.org/10.56947/gjom.v16i2.1882>
22. Scott, L. R., & Zhang, S. (1990). Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Mathematics of Computation*, 54(190), 483–493. <https://doi.org/10.1090/S0025-5718-1990-1011446-7>
23. Silvester, D., & Kechkar, N. (1990). Stabilised bilinear-constant velocity-pressure finite elements for the conjugate gradient solution of the Stokes problem. *Computer Methods in Applied Mechanics and Engineering*, 79(1), 71–86. [https://doi.org/10.1016/0045-7825\(90\)90095-4](https://doi.org/10.1016/0045-7825(90)90095-4)
24. Stadler, G., Gurnis, M., Burstedde, C., Wilcox, L., Alisic, L., & Ghattas, O. (2010). The dynamics of plate tectonics and mantle flow: From local to global scales. *Science*, 329(5995), 1033–1038. <https://doi.org/10.1126/science.1191223>
25. Thomée, V. (2007). *Galerkin Finite Element Methods for Parabolic Problems*. Springer Berlin, Heidelberg. <https://doi.org/10.1007/3-540-33122-0>

¹ DEPARTMENT OF SCIENCES, TEACHER EDUCATION COLLEGE OF SETIF-MESSAOUD ZEGHAR, P.O. BOX 556, EL-EULMA, 19600, SETIF, ALGERIA.

Email address: akramboukabache@gmail.com (a.boukabache@ens-setif.dz)