Gulf Journal of Mathematics
Vol 22, Issue 2 (2026) 1-9
ISSN: 2309-4966
https://doi.org/10.56947/gjom.v22i2.4203

CONSTRUCTION OF DUAL BASES FOR FUSS-CATALAN MODULES

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Abstract. We introduce a systematic method to derive the dual bases associated with certain cell modules of the Fuss–Catalan algebras with respect to the canonical bilinear form. Our approach is based on a detailed and explicit analysis of the corresponding Gram matrices, together with the computation of their inverses. By exploiting the cellular structure of the algebra, we obtain concrete formulas for the dual basis elements, expressed directly in terms of diagrammatic data. This construction provides an effective computational framework inside the cell modules and clarifies the role played by the bilinear form in their representation theory. The resulting dual basis proves to be a powerful tool for structural investigations of Fuss–Catalan algebras. In particular, it plays a fundamental role in the explicit determination of central primitive idempotents and facilitates the computation of characters associated with simple modules. Moreover, the method developed here highlights the close relationship between Gram matrix degeneracy, parameter specialisation, and the appearance of reducible cell modules. These results offer new insights into the internal structure of Fuss–Catalan algebras and provide a foundation for further applications to related diagram algebras and their representation theory.

Keywords. Representation theory, cell modules, dual basis, Gram Matrix
2020 Mathematics Subject Classification. Primary 16D10; Secondary 16D80

1. Introduction

Fuss–Catalan algebras form an important family of diagram algebras that generalize the Temperley–Lieb algebra and were originally introduced in connection with subfactor theory and planar algebras. Since their appearance in the work of Bisch and Jones [1], these algebras have attracted considerable attention due to their rich combinatorial structure and their relevance to several areas of mathematics and mathematical physics. In particular, Fuss–Catalan algebras arise naturally in the study of intermediate subfactors, statistical mechanical models, and exactly solvable lattice systems, as well as in connections with quantum

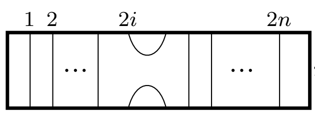
Date: Received: Feb 2, 2026; Revised: Mar 2, 2026; Accepted: Mar 16.

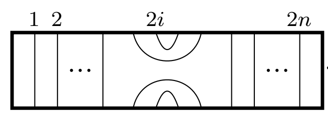
groups and tensor categories (see, for example, [1, 2, 3]). From the representation-theoretic point of view, Fuss–Catalan algebras belong to the broader class of cellular algebras in the framework of Graham and Lehrer [4]. This cellular structure provides a systematic way to construct and study their representations via cell modules, which admit natural diagrammatic bases reflecting the underlying combinatorics of the algebra. Understanding the structure of these cell modules is therefore a key step toward describing the representation theory of Fuss–Catalan algebras in detail [5]. Many important representation-theoretic objects, such as characters, trace functions, and central primitive idempotents, depend crucially on the bilinear forms defined on cell modules and on the ability to work effectively with dual bases. While diagram bases are well suited for combinatorial arguments, the lack of an explicit dual basis often makes concrete calculations difficult, especially in higher rank or multi-colored settings inherent to Fuss–Catalan algebras. In this paper, we address this issue by constructing a dual basis for a class of cell modules over the Fuss–Catalan algebras. Our approach relies on elementary linear algebraic techniques, centered around the analysis of Gram matrices associated with the natural bilinear forms. By explicitly describing the inverse Gram matrix, we obtain a dual basis compatible with the diagrammatic structure of the modules. The construction presented here is intended as a foundational tool for further applications. In particular, the availability of an explicit dual basis opens the way to the computation of characters and to the explicit determination of central primitive idempotents for Fuss–Catalan algebras. Such results are expected to be useful not only for representation theory, but also for applications in subfactor theory, where idempotents correspond to projections onto irreducible sectors, and in statistical mechanics, where they are related to transfer matrices and partition functions. These directions will be explored in future work.

In Section 2, for the reader's convenience, we recall the definition of cellular algebras and their corresponding cell modules, following the framework introduced by Graham and Lehrer [4]. Section 3 is devoted to the construction of a dual basis for a family of cell modules over the Fuss-Catalan algebras with prescribed labels. Inspired by the approach of Y. Li and D. Zhao [8] in their study of the centers of symmetric cellular algebras, our construction makes use of the duality condition and an explicit description of the associated Gram matrices.

2. Definitions

Fuss–Catalan algebras, denoted $\mathrm{FC}_n(a, b)$, are a family of diagram algebras that can be defined for each positive integer n and non-zero parameters $a, b \in \mathbb{C}$ using planar diagrams consisting of $2n$ nodes on the top and $2n$ nodes on the bottom, connected by noncrossing strands. In contrast to the Temperley–Lieb case, Fuss–Catalan diagrams may include additional structures, such as colors or layers, which correspond combinatorially to Fuss–Catalan numbers. Let B_n be the set of these diagrams, so, the algebra $\mathrm{FC}_n(a, b)$ is spanned as a vector space by B_n, and its multiplication is defined diagrammatically, given two diagrams, one places the first diagram on top of the second and connects the bottom nodes of the top diagram to the top nodes of the bottom diagram. Any closed loops that arise in this process are removed, and each removed loop contributes a scalar factor, either a or b, depending on its type. This multiplication is associative and turns the set of diagrams into a finite-dimensional algebra over the field of complex numbers. Diagrammatically, the product is simply the composition of strands, with loops evaluated according to the parameters a and b, which allows for explicit computation and provides an intuitive visual framework for studying the algebra's structure. For a diagram $U \in \mathrm{FC}_n(a,b)$ a through line is a strand that connects a node in the top of U to a node in the bottom of U. The label of U is the color corresponding to each through line. The Fuss-Catalan algebra $\mathrm{FC}_n(a,b)$ is generated by the diagrams 1, ${}_1E_i$ and ${}_2E_i$, where $1 \leq i \leq n-1$, and

${}_1E_i =$ 

, ${}_2E_i =$ 

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The coloring for a basis diagram in $\mathrm{FC}_n(a,b)$ has the pattern $(abba)^{\frac{n}{2}}$ if n is even, and $(abba)^{\frac{n-1}{2}}ab$ if n is odd. [1].

Let Γ denote the labels set associated with diagrams in B_n. Further, we define the initial part of U as an upper half diagram of U, and for each $\gamma \in \Gamma$, let $\mathcal{D}_n(\gamma)$ be the set consisting of the different initial parts arising from diagrams in B_n with label γ. Using this notation, every diagram in B_n can be expressed as $B_n = \{W^{\gamma}_{M,N} \mid \gamma \in \Gamma,\ M, N \in \mathcal{D}_n(\gamma)\}$, where the diagram $W^{\gamma}_{M,N}$ is constructed by placing M on top of N^*, and N^* denotes the reflection of N across a horizontal axis, i.e., turning the diagram upside down. This construction ensures that each diagram in B_n is uniquely determined by its label γ and the corresponding initial parts M and N. In [5, Theorem 4.5], it was shown that the algebra $\mathrm{FC}_n(a,b)$, equipped with the cell datum $(\Gamma, \mathcal{D}_n(\gamma), B_n, *)$, satisfies the axioms of a cellular algebra as defined by Graham and Lehrer [4]. This means, in particular, that $\mathrm{FC}_n(a,b)$ admits a cellular basis indexed by pairs of initial diagrams, and that the multiplication and involution $*$ are compatible with the cellular structure. Consequently, one can define cell modules for $\mathrm{FC}_n(a,b)$ corresponding to each label $\gamma \in \Gamma$, providing a combinatorial and algebraic framework that is crucial for studying representations, Gram matrices, and dual bases of the algebra.

Definition 2.1 ([4, Definition 2.1])**.** Let A be a cellular algebra, described by a cell datum $(\Gamma, \mathcal{D}, B, *)$. For each element γ in the indexing set Γ, we can construct a left A-module $\Delta(\gamma)$, called a cell module. As an R-module, $\Delta(\gamma)$ is free and has a basis containing of elements W_M for all $M \in \mathcal{D}(\gamma)$. The action of A on these basis elements is determined by

$$aW_M = \sum_{M' \in \mathcal{D}(\gamma)} \alpha_a(M', M) W_{M'} \qquad (a \in A,\ M \in \mathcal{D}(\gamma))$$

where $\alpha_a(M', M) \in R$. This module called the *cell module* of A labeled by γ.

Given $U \in \mathrm{FC}_n(a,b)$ and an element $U' \in \mathrm{FC}_n(a,b)$ with initial part N, we define the action of U on N by taking the initial part of the product UU'. This gives a well-defined action.

Definition 2.2 ([5, Definition 4.6])**.** For each $\gamma \in \Gamma$, there is a corresponding cell module $\Delta_n(\gamma)$ whose basis is the set $\mathcal{D}_n(\gamma)$. The algebra $\mathrm{FC}_n(a,b)$ acts on this module in the following way, for any element $U \in \mathrm{FC}_n(a,b)$ and any basis element $N \in \mathcal{D}_n(\gamma)$, the action of U on N is defined by

$$U \cdot N = \begin{cases} UN, & \text{if } \ \ell(UN) = \ell(N), \\ 0, & \text{otherwise.} \end{cases}$$

where $\ell(N)$ and $\ell(UN)$ is the number of through lines in N and UN respectively.

3. Dual basis and Gram matrix

In [8], the dual basis method was used to investigate the structure of the center of graded symmetric cellular algebras. Their approach relies on studying the Higman ideal of the algebra, which provides a structured way to identify central elements via the interaction of a basis with its dual. In this section, we extend this idea to the algebras $\mathrm{FC}_n(a,b)$. Specifically, We provide an explicit construction of a dual basis for a large class of cell modules, namely $\Delta_n(\gamma_n)$, for $n \geq 6$, and

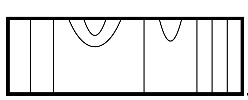
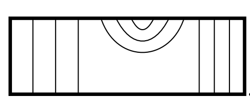
$$\gamma_n = \begin{cases} abbbba(abba)^{\frac{n-6}{2}} & \text{if n even} \\ abbbba(abba)^{\frac{n-7}{2}}ab & \text{if n odd .} \end{cases} \tag{3.1}$$

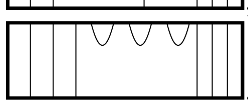
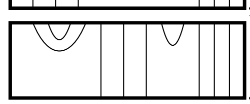
These cell modules are essential in understanding the representation theory of $\mathrm{FC}_n(a,b)$, as the dual basis allows us to explicitly compute inner products and Gram matrices, and to study properties such as idempotents, centers, and trace forms. This construction provides a concrete tool to explore how the algebra acts on its modules and gives a pathway to generalize results from classical Temperley–Lieb algebras to the more intricate Fuss–Catalan setting.

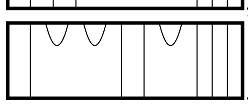
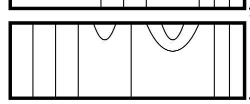
Our motivation to choose the label γ_n is that this family forms a natural and inductively stable class within $\mathrm{FC}_n(a,b)$, and it is the first sequence for which the structure of the Gram matrix exhibits a uniform recursive behaviour that can be analyzed explicitly. In particular, these modules capture the essential features of reducibility and determinant structure that are central to our approach.

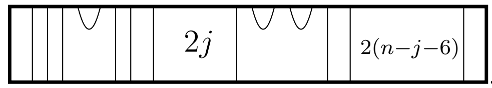
Definition 3.1. We define the tensor product of two diagrams S_1 and S_2 by putting S_2 to the right of S_1 and denoted by $S_1 \otimes S_2$.

Definition 3.2. For all $i = 0, 1, \cdots, 5$, we shall define the diagrams $S_{n,i} = f_i \otimes \mathrm{id}_{n-6}$, where id_{n-6} is the identity diagram for $\mathrm{FC}_{n-6}(a,b)$, and

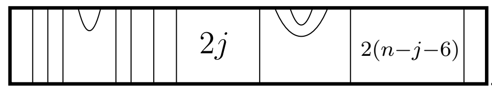
$f_0 =$ , $f_1 =$ ,

$f_2 =$ , $f_3 =$ ,

$f_4 =$ , $f_5 =$ , moreover, two diagrams $P_{n,j}$ and $Q_{n,j}$ are defined by

$P_{n,j} =$ 

,

$Q_{n,j} =$ 

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for all $0 \leq j \leq n-6$.

From [7, Proposition 3.1 and Proposition 3.5], we get

Proposition 3.3. *Suppose that $n \geq 6$ and let γ_n be defined as in (3.1). Thus the cell module $\Delta_n(\gamma_n)$ has dimension $2(n-2)$ and is spanned by the set of diagrams $\{S_{n,i}, P_{n,j}, Q_{n,j} | i = 0, 1, \cdots, 5, j = 0, 1, \cdots, n-6\}$ where $S_{n,i}, P_{n,j}$, and $Q_{n,j}$ are as defined in Definition 3.2.*

It follows from [4, Theorem 3.8] that, in a cellular algebra, cell modules are irreducible provided their associated bilinear forms are non-degenerate.

Definition 3.4 ([6, Definition 5.5])**.** We define a bilinear form $\langle -, - \rangle : \Delta_n(\gamma) \times \Delta_n(\gamma) \to \mathbb{C}$ on basis diagrams as follows. Given basis elements N_1 and N_2, rotate N_2 by 180° and place it on top of N_1. If this concatenation preserves the number of through lines, then $\langle N_1, N_2 \rangle = a^{k_1} b^{k_2}$, where k_1 and k_2 denote the numbers of closed a- and b-labeled circles formed in the diagram. If, on the other hand, the number of through lines decreases, we set $\langle N_1, N_2 \rangle = 0$. The form is extended bilinearly to all of $\Delta_n(\gamma)$.

Definition 3.5. For the cell module $\Delta_n(\gamma)$ with ordered basis $(N_1, N_2, \ldots, N_r)$, the Gram matrix $G_n(\gamma)$ records the values of the bilinear form on basis elements. Its (i,j)-entry is given by $(G_n(\gamma))_{i,j} = \langle N_i, N_j \rangle, \quad i, j = 1, 2, \ldots, r.$

Proposition 3.6 ([7, Proposition 3.9])**.** *Let γ_n be defined as in equation (3.1). Consider the cell module $\Delta_n(\gamma_n)$ with the ordered basis*

$$\{S_{n,i}, P_{n,j}, Q_{n,j} | i = 0, 1, \cdots, 5, j = 0, 1, \cdots, n-6\}.$$

The Gram matrix of $\Delta_n(\gamma_n)$ with respect to this basis, denoted $G(\gamma_n)$, has the following structure

$$\text{(i)} \quad G(\gamma_n) = a \begin{bmatrix} ab & 1 & a & 1 & a & 0 & 0 & 0 \\ 1 & ab & b & 0 & 0 & 1 & 0 & 0 \\ a & b & ab & 0 & 0 & a & 0 & 0 \\ 1 & 0 & 0 & ab & b & 0 & 0 & 0 \\ a & 0 & 0 & b & ab & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & ab & a & 1 \\ 0 & 0 & 0 & 0 & 0 & a & ab & b \\ 0 & 0 & 0 & 0 & 0 & 1 & b & ab \end{bmatrix} \quad \text{if } n = 6.$$

$$\text{(ii)} \quad G(\gamma_n) = \begin{pmatrix} G(\gamma_{n-1}) & H_s^T \\ H_s & D_t \end{pmatrix} \, if n > 6,$$

where $H_s = \begin{pmatrix} 0 & \dots & 0 & as \\ 0 & \dots & 0 & a \end{pmatrix}_{2\times(2n-6)}$, $D_t = \begin{pmatrix} a^2b & at \\ at & a^2b \end{pmatrix}$, *and* $(s,t) = (a,b)$ *if* n *even while* $(s,t) = (b,a)$ *if* n *odd.*

Our method to constructing a dual basis for $\Delta_n(\gamma_n)$ relies on computing the inverse of the Gram matrix $G_n(\gamma_n)$. Since this is generally a difficult task, we instead rewrite $G_n(\gamma_n)$ in a block-diagonal form using an appropriate change of basis. This allows us to compute its inverse recursively, simplifying the overall process.

Proposition 3.7. *Suppose* $n > 6$, $s \neq 0, t \notin \{0, \pm 1\}$ *and* γ_n *is given in (3.1). Consider the cell module* $\Delta_n(\gamma_n)$ *equipped with the ordered basis* $B_n = \{S_{n,i}, P_{n,j}, Q_{n,j} | i = 0, 1, \cdots, 5, j = 0, 1, \cdots, n-6\}$. *Then*

1) The set $J_n = \{S_{n,0}, \cdots, S_{n,5}, P_{n,0}, Q_{n,0}, \cdots, P_{n,n-7}, Q_{n,n-7}, u_1, u_2\}$ *is a basis for* $\Delta_n(\gamma_n)$ *where* $u_1 = \frac{1}{t^2-1} P_{n,n-7} - \frac{t}{t^2-1} Q_{n,n-7} + P_{n,n-6}$ *and* $u_2 = \frac{1}{s(t^2-1)} P_{n,n-7} - \frac{t}{s(t^2-1)} Q_{n,n-7} + Q_{n,n-6}$.

2) The Gram matrix associated with J_n *can be written as follows*

$$G_J(\gamma_n) = \begin{pmatrix} G(\gamma_{n-1}) & \mathbf{0} \\ \mathbf{0} & K \end{pmatrix},$$

such that $K = \frac{at(t^2-2)}{t^2-1} \begin{bmatrix} s & 1 \\ 1 & \frac{1 - s^2(t^2-1)}{s(t^2-2)} \end{bmatrix}$, *and* $(s,t) = (a,b)$ *if* n *even whereas* $(s,t) = (b,a)$ *if* n *is odd.*

Proof. 1) The sets J_n and B_n contain an equal number of diagrams, and the change of basis matrix from B_n to J_n has determinant 1. This implies that J_n is linearly independent. In addition, the number of elements in J_n matches the dimension of $\Delta_n(\gamma_n)$, we conclude that J_n forms a complete basis for this module.
2) It follows, from Proposition 3.6, that

$$G(\gamma_n) = \begin{pmatrix} G(\gamma_{n-1}) & H_s^T \\ H_s & D_t \end{pmatrix} = \left(\begin{array}{cc|c} G(\gamma_{n-2}) & H_t^T & \\ H_t & D_s & H_s^T \\ \hline \multicolumn{2}{c|}{H_s} & D_t \end{array} \right).$$

Replacing the block matrices H_s, H_t, D_s, H_s^T and D_t, we get

$$G(\gamma_n) = \begin{array}{c|c|c|c} \langle -,- \rangle & S_{n,0} \ \ S_{n,1} \ \cdots \ Q_{n,n-8} & P_{n,n-7} \ \ Q_{n,n-7} & P_{n,n-6} \ \ Q_{n,n-6} \\ \hline \begin{matrix} S_{n,0} \\ S_{n,1} \\ \vdots \\ Q_{n,n-8} \end{matrix} & G(\gamma_{n-2}) & H_t^T & 0 \\ \hline \begin{matrix} P_{n,n-7} \\ Q_{n,n-7} \end{matrix} & \begin{matrix} 0 & 0 & \cdots & 0 & at \\ 0 & 0 & \cdots & 0 & a \end{matrix} & \begin{matrix} a^2b & as \\ as & a^2b \end{matrix} & \begin{matrix} 0 & 0 \\ as & a \end{matrix} \\ \hline \begin{matrix} P_{n,n-6} \\ Q_{n,n-6} \end{matrix} & 0 & \begin{matrix} 0 & as \\ 0 & a \end{matrix} & \begin{matrix} a^2b & at \\ at & a^2b \end{matrix} \end{array}$$

To compute the entries of $G_J(\gamma_n)$, it suffices to calculate the inner products $\langle u_1, x\rangle$ and $\langle u_2, x\rangle$ for all $x \in J_n$. All the remaining entries of $G_J(\gamma_n)$ are the same as the corresponding entries in $G(\gamma_n)$. That is,

$$G_J(\gamma_n) = \begin{pmatrix} G(\gamma_{n-1}) & X^T \\ X & Y \end{pmatrix}$$

where, for $i = 1, 2$, and $x_j \in \{S_{n,0}, \cdots, S_{n,5}, P_{n,0}, Q_{n,0}, \cdots, P_{n,n-7}, Q_{n,n-7}\}$

$$X_{ij} = \langle u_i, x_j\rangle,$$

and for $i, j = 1, 2$,

$$Y_{ij} = \langle u_i, u_j\rangle.$$

One can check that

$\langle -, -\rangle$	$S_{n,0}$ $S_{n,1}$ $\cdots$ $Q_{n,n-8}$	$P_{n,n-7}$ $Q_{n,n-7}$	u_1 u_2
u_1	0	0 0	K
u_2		0 0	

Therefore, we obtain $X = 0$, and $Y = K$ as claimed. □

Let V be a vector space with basis $\{v_0, v_1, \cdots, v_r\}$. A set $\{v'_0, v'_1, \cdots, v'_r\}$ is called a dual basis of V if it also spans V and satisfies the property $\langle v_i, v'_j\rangle = \delta_{ij}$, where $\delta_{ii} = 1$ when $i = j$ and 0 otherwise (see [9, sec. 1]). We now turn our attention to the space $\Delta_n(\gamma_n)$ and the construction of a dual basis within it. Let J_n denote a fixed basis of this space. Following the method outlined in [9, sec. 2], it is possible to express each element of the dual basis $\{v_0, v_1, \cdots, v_{2n-5}\}$ as a linear combination of diagrams from J_n:

$$v_i = \sum_{k=0}^{2n-5} c_{ik} x_k, \qquad (x_k \in J_n).$$

The coefficients c_{ik} are determined by the duality condition, which requires that each dual basis element v_i pairs to 1 with the corresponding basis element x_i, and to 0 with all others:

$$\langle x_j, v_i\rangle = \delta_{ij}, \quad 0 \le i, j \le 2n-5.$$

Substituting the linear combination for v_i into this condition gives a system of linear equations:

$$\delta_{ij} = \sum_{k=0}^{2n-5} c_{ik} \langle x_j, x_k\rangle.$$

Solving this system for the coefficients c_{ik} yields the dual basis.

$$\begin{bmatrix} \langle x_0, x_0\rangle & \langle x_0, x_1\rangle & \cdots & \langle x_0, x_{2n-5}\rangle \\ \langle x_1, x_0\rangle & \langle x_1, x_1\rangle & \cdots & \langle x_1, x_{2n-5}\rangle \\ \vdots & \vdots & & \vdots \\ \langle x_i, x_0\rangle & \langle x_i, x_1\rangle & \cdots & \langle x_i, x_{2n-5}\rangle \\ \vdots & \vdots & & \vdots \\ \langle x_{2n-5}, x_0\rangle & \langle x_{2n-5}, x_1\rangle & \cdots & \langle x_{2n-5}, x_{2n-5}\rangle \end{bmatrix} \begin{bmatrix} c_{i0} \\ c_{i1} \\ \vdots \\ c_{ii} \\ \vdots \\ c_{i,2n-5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

In this way, the abstract notion of duality is concretely realized in terms of the original basis diagrams. The system can be expressed in matrix form as $G_J(\gamma_n)\ C = E$, and therefore,

$$C = G_J^{-1}(\gamma_n)E \tag{3.2}$$

We now present the main theorem, giving a recursive procedure for computing the matrix of coefficients.

Theorem 3.8. *Keeping the notation as above, assume that $n > 6$. The cell module $\Delta_n(\gamma_n)$, with basis J_n, admits a dual basis $\{v_i\}$ defined by*

$$v_i = \sum_{k=0}^{2n-5} c_{ik}x_k, \qquad (x_k \in J_n).$$

For each fixed i, the coefficients c_{ik} are given by the entries of the corresponding matrix

$$C = \begin{bmatrix} L_{n-1}^T\ G_J^{-1}(\gamma_{n-1})\ L_{n-1} & \mathbf{0} \\ \mathbf{0} & K^{-1} \end{bmatrix} E$$

Proof. Recalling equation (3.2), we have $C = G_J^{-1}(\gamma_n)E$. Furthermore, using Proposition 3.7, it follows that

$$G_J^{-1}(\gamma_n) = \begin{bmatrix} G^{-1}(\gamma_{n-1}) & \mathbf{0} \\ \mathbf{0} & K^{-1} \end{bmatrix}.$$

The change of basis from B_n to J_n is given by the matrix L_n, thus $G_J(\gamma_n) = L_nG(\gamma_n)L_n^T$, so we have $G^{-1}(\gamma_{n-1}) = L_{n-1}^T\ G_J^{-1}(\gamma_{n-1})\ L_{n-1}$. This completes the proof. □

From Proposition 3.7, it follows that the change of basis matrix L_n from B_n to J_n is explicitly given by

$$L_n = \begin{bmatrix} I_{d_{n-2}} & \mathbf{0} \\ Z_n & I_2 \end{bmatrix},$$

where $I_{d_{n-2}}$ denotes the identity matrix of size d_{n-2}, with $d_{n-2} = \dim G(\gamma_{n-2})$, and $Z_n = \frac{1}{t^2-1}\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & -t \\ 0 & 0 & \cdots & 0 & \frac{1}{s} & \frac{-t}{s} \end{bmatrix}_{2\times d_{n-2}}$.

Observing that L_n is readily determined, and

$$G_J^{-1}(\gamma_n) = \begin{bmatrix} L_{n-1}^T\ G_J^{-1}(\gamma_{n-1})\ L_{n-1} & 0 \\ 0 & K^{-1} \end{bmatrix},$$

we obtain a recursive description of the dual basis of $\Delta_n(\gamma_n)$.

Acknowledgement. The author would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad-Iraq for its support in the present work.

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