

WELL-POSEDNESS AND NUMERICAL APPROXIMATION FOR NONLINEAR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO DERIVATIVE

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ABSTRACT. This paper considers a class of nonlinear hybrid fractional differential equations with the Caputo derivative. The hybrid feature means that the fractional derivative is taken for the corrected state $z(t) - f(t, z(t))$, which can model memory effects together with a feedback-type term. We first convert the initial value problem into an equivalent Volterra integral equation. Using this form, we define an operator on $C([0, T], \mathbb{R})$ and prove existence and uniqueness of solutions by the Banach fixed point theorem under simple Lipschitz conditions. We also establish well-posedness by showing continuous dependence on the initial data and deriving an explicit a priori bound. Finally, we develop a three-step Adams–Bashforth product-integration scheme for numerical approximation and provide numerical experiments to support the theoretical results.

Keywords. hybrid fractional differential equation, well-posedness, numerical approximation, Adams–Bashforth method, Caputo derivative

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1. INTRODUCTION

Fractional differential equations are often used to model processes with memory [3, 15–18], where the present state depends on past values. Related studies in hybrid Caputo-type stability results, neutral evolution models with delay and nonlocal conditions, pantograph-type Caputo problems, solvability for Caputo–Hadamard integro-differential equations, and numerical methods for fractional delay integro-differential algebraic equations [1, 6, 7, 9, 14]. Such effects appear in many applications, for example in diffusion in complex media, viscoelasticity, electrical circuits with constant-phase elements, and control systems. Among the various definitions of fractional derivatives, the Caputo derivative is popular in applications because it allows standard initial conditions stated in terms of classical values at the initial time.

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In this paper, we study a nonlinear *hybrid* fractional initial value problem with the Caputo derivative. The hybrid feature means that the fractional derivative acts on the difference $z(t) - f(t, z(t))$ rather than on $z(t)$ itself. More precisely, for $0 < \nu < 1$ and $T > 0$, we consider

$${}^C D_{0+}^\nu (z(t) - f(t, z(t))) = g(t, z(t)), \quad t \in [0, T], \quad (1.1)$$

together with the initial condition

$$z(0) - f(0, z(0)) = \xi, \quad (1.2)$$

where $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are given nonlinear functions and $\xi \in \mathbb{R}$ is a prescribed constant. Problems of the form (1.1)–(1.2) arise when the evolving quantity z is coupled with a measured, corrected, or effective state

$$u(t) = z(t) - f(t, z(t)),$$

and the memory law is imposed on u instead of z . This structure is natural in several situations:

(i) *Feedback and correction mechanisms.* In many control and estimation models, the controller does not act on the true state $z(t)$ directly, but on a corrected signal such as $u(t) = z(t) - f(t, z(t))$. Here $f(t, z)$ can represent a nonlinear feedback, a compensation term, or a correction based on a sensor model. Fractional dynamics then enters as a compact way to describe long-memory effects in the actuator, the plant, or the closed-loop response.

(ii) *Implicit constitutive relations and effective variables.* Hybrid forms may also appear when z is linked to another physical variable through an implicit relation. For instance, in viscoelasticity and rheology, an effective stress/strain variable can include instantaneous and nonlinear contributions, and the memory operator acts on the effective quantity rather than the raw state. The term $f(t, z(t))$ models instantaneous nonlinear response, while the Caputo derivative describes hereditary effects.

(iii) *Modeling uncertainties and nonlinear measurements.* In data-driven settings, one often measures a nonlinear transformation of the state. If $f(t, z)$ captures a measurement bias or a nonlinear sensor map, then $u = z - f(t, z)$ is a corrected estimate. The hybrid equation (1.1) can be interpreted as an evolution law written directly in terms of that corrected variable, which is convenient for analysis and numerical simulation.

Equation (1.1) combines three key features: (a) nonlocal memory in time (Caputo derivative), (b) nonlinearity in the state, and (c) a hybrid/implicit structure due to $f(t, z(t))$. From the analysis point of view, the hybrid term makes the equation more involved than standard fractional equations because the unknown appears both inside and outside a nonlinear function. From the modeling point of view, the hybrid term provides flexibility to incorporate instantaneous corrections or feedback without losing the memory description.

The main difficulty of (1.1)–(1.2) is that a direct treatment in differential form is not always convenient. To overcome this, we first rewrite the problem as an equivalent Volterra-type integral equation. This integral form is well suited for

fixed point arguments and also provides a natural starting point for numerical methods.

The main contributions of this paper are summarized as follows:

- We derive an equivalent integral equation for the hybrid Caputo problem (1.1)–(1.2).
- Under simple Lipschitz conditions on f and g , we prove existence and uniqueness of solutions on $[0, T]$ by the Banach fixed point theorem.
- We establish well-posedness in the sense of Hadamard by proving continuous dependence on the initial data, and we provide an explicit a priori bound for solutions.
- We propose a three-step Adams–Bashforth type product-integration method to approximate the solution and present numerical simulations.

The remainder of this paper is organized as follows. Section 2 recalls preliminaries on the Caputo derivative and the associated fractional integral. Section 3 derives an equivalent Volterra integral formulation and establishes existence, uniqueness, and well-posedness. Section 4 presents the proposed product-integration scheme and its implementation. Section 4.7 reports numerical experiments validating the method.

2. PRELIMINARIES

In this section we recall basic definitions and tools from fractional calculus and Volterra integral equations that will be used in the sequel. Throughout the paper, let $T > 0$ and $0 < \nu < 1$ be fixed. We denote by $C([0, T], \mathbb{R})$ the Banach space of continuous functions on $[0, T]$ endowed with the sup-norm

$$\|z\|_\infty := \sup_{t \in [0, T]} |z(t)|.$$

We also use $AC([0, T], \mathbb{R})$ for the space of absolutely continuous functions on $[0, T]$. For background and further properties, we refer to [4, 10, 19, 20].

Definition 2.1 (Riemann–Liouville fractional integral). Fix $\nu > 0$ and take $\varphi \in L^1([0, T], \mathbb{R})$. We define the left Riemann–Liouville fractional integral of order ν by

$$(I_{0+}^\nu \varphi)(t) := \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \varphi(s) ds, \quad (2.1)$$

for $t \in [0, T]$. The symbol $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 (Caputo fractional derivative). Let $u \in AC([0, T], \mathbb{R})$ and $0 < \nu < 1$. The Caputo fractional derivative of order ν is defined by

$$({}^C D_{0+}^\nu u)(t) := \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} u'(s) ds, \quad t \in [0, T]. \quad (2.2)$$

The following identities connect the Caputo derivative with the Riemann–Liouville integral.

Lemma 2.3 (Left inverse property). *Let $u \in AC([0, T], \mathbb{R})$ and $0 < \nu < 1$. Then*

$$I_{0+}^\nu {}^C D_{0+}^\nu u(t) = u(t) - u(0), \quad t \in [0, T]. \quad (2.3)$$

Lemma 2.4 (Right inverse property). *Let $\varphi \in C([0, T], \mathbb{R})$ and $0 < \nu < 1$. Then*

$${}^C D_{0+}^\nu (I_{0+}^\nu \varphi)(t) = \varphi(t), \quad t \in [0, T]. \quad (2.4)$$

Lemma 2.5. *For $0 < \nu < 1$ and $t \in [0, T]$, we have*

$$\int_0^t (t-s)^{\nu-1} ds = \frac{t^\nu}{\nu}, \quad \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds = \frac{t^\nu}{\Gamma(\nu+1)}. \quad (2.5)$$

In particular,

$$\sup_{t \in [0, T]} \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds = \frac{T^\nu}{\Gamma(\nu+1)}. \quad (2.6)$$

We will use the Banach contraction principle [11].

Theorem 2.6 (Banach fixed point theorem). *Let $(X, \|\cdot\|)$ be a Banach space and let $M : X \rightarrow X$ be a contraction, i.e., there exists $0 < q < 1$ such that*

$$\|Mx - My\| \leq q\|x - y\| \quad \text{for all } x, y \in X.$$

Then M has a unique fixed point $x^* \in X$, and the sequence $x_{n+1} = Mx_n$ converges to x^* for any initial $x_0 \in X$.

Remark 2.7. In Section 3, we transform the hybrid Caputo problem into an equivalent Volterra-type integral equation [4, 10]. Then we define an operator on $C([0, T], \mathbb{R})$ and show that it becomes a contraction under suitable Lipschitz assumptions.

3. MAIN RESULTS

In this section we present the well-posedness theory for the hybrid Caputo problem. We first rewrite the differential equation as an equivalent Volterra integral equation, and then we apply the Banach fixed point theorem.

Let $T > 0$ and $0 < \nu < 1$. We consider the nonlinear hybrid fractional initial value problem

$${}^C D_{0+}^\nu (z(t) - f(t, z(t))) = g(t, z(t)), \quad t \in [0, T], \quad (3.1)$$

with the initial condition

$$z(0) - f(0, z(0)) = \xi, \quad (3.2)$$

where $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions and $\xi \in \mathbb{R}$.

We use the following assumptions.

- (H1) f and g are continuous on $[0, T] \times \mathbb{R}$.
- (H2) Assume that, for each fixed $t \in [0, T]$, the mappings $x \mapsto f(t, x)$ and $x \mapsto g(t, x)$ are globally Lipschitz on $\mathbb{R}$, with Lipschitz bounds independent of t . Hence, there exist $L_f, L_g \geq 0$ such that

$$|f(t, x) - f(t, y)| \leq L_f |x - y|, \quad |g(t, x) - g(t, y)| \leq L_g |x - y| \quad (3.3)$$

for all $t \in [0, T]$ and $x, y \in \mathbb{R}$.

- (H3) The functions $t \mapsto f(t, 0)$ and $t \mapsto g(t, 0)$ are bounded on $[0, T]$.

The equivalence between Caputo-type problems and Volterra integral equations is standard; see, for example, [2, 4, 8, 10].

Proposition 3.1. *Let $z \in C([0, T], \mathbb{R})$ be such that $u(t) := z(t) - f(t, z(t)) \in AC([0, T], \mathbb{R})$. Then z satisfies (3.1)–(3.2) if and only if z satisfies*

$$z(t) = f(t, z(t)) + \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds, \quad t \in [0, T]. \quad (3.4)$$

Proof of Proposition 3.1. We prove the two implications separately.

($\Rightarrow$). Assume that z satisfies (3.1)–(3.2). Set

$$u(t) := z(t) - f(t, z(t)).$$

Then $u \in AC([0, T], \mathbb{R})$ by assumption and (3.1) reads

$${}^C D_{0+}^\nu u(t) = g(t, z(t)), \quad t \in [0, T].$$

Apply the Riemann–Liouville fractional integral I_{0+}^ν to both sides and use Lemma 2.3:

$$I_{0+}^\nu {}^C D_{0+}^\nu u(t) = u(t) - u(0) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds.$$

By the initial condition (3.2), we have $u(0) = z(0) - f(0, z(0)) = \xi$. Hence,

$$u(t) = \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds,$$

that is,

$$z(t) - f(t, z(t)) = \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds,$$

which is equivalent to (3.4).

($\Leftarrow$). Conversely, assume that z satisfies (3.4). Define

$$u(t) := z(t) - f(t, z(t)).$$

Then (3.4) implies

$$u(t) = \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds = \xi + (I_{0+}^\nu (g(\cdot, z(\cdot))))(t).$$

Taking $t = 0$ gives $u(0) = \xi$, i.e. (3.2) holds. Now apply ${}^C D_{0+}^\nu$ to both sides. Using Lemma 2.4, we get

$${}^C D_{0+}^\nu u(t) = {}^C D_{0+}^\nu \left(\xi + I_{0+}^\nu (g(\cdot, z(\cdot))) \right)(t) = g(t, z(t)), \quad t \in [0, T],$$

which is exactly (3.1). Therefore (3.1)–(3.2) holds.

This completes the proof. $\square$

3.1. Existence and uniqueness. Define the operator $M : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$ by

$$(Mz)(t) := f(t, z(t)) + \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds, \quad t \in [0, T]. \quad (3.5)$$

Theorem 3.2. Assume (H1)–(H2) and

$$L_f + \frac{L_g T^\nu}{\Gamma(\nu+1)} < 1. \quad (3.6)$$

Then (3.1)–(3.2) admits a unique solution $z \in C([0, T], \mathbb{R})$.

Proof of Theorem 3.2. Let $z, y \in C([0, T], \mathbb{R})$. Using (3.5), (3.3), and Lemma 2.5, for each $t \in [0, T]$ we estimate

$$\begin{aligned} |(Mz)(t) - (My)(t)| &\leq |f(t, z(t)) - f(t, y(t))| + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |g(s, z(s)) - g(s, y(s))| ds \\ &\leq L_f |z(t) - y(t)| + \frac{L_g}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |z(s) - y(s)| ds \\ &\leq \left(L_f + \frac{L_g t^\nu}{\Gamma(\nu+1)} \right) \|z - y\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0, T]$ yields

$$\|Mz - My\|_\infty \leq \left(L_f + \frac{L_g T^\nu}{\Gamma(\nu+1)} \right) \|z - y\|_\infty.$$

By (3.6), M is a contraction on $C([0, T], \mathbb{R})$. Therefore, by Theorem 2.6, M has a unique fixed point $z^* \in C([0, T], \mathbb{R})$. By Proposition 3.1, this fixed point is the unique solution of (3.1)–(3.2). $\square$

3.2. Well-posedness. We use the classical (Hadamard) meaning of well-posedness: existence, uniqueness, and continuous dependence on the given data.

Definition 3.3 (Well-posedness). Problem (3.1)–(3.2) is said to be well-posed in $C([0, T], \mathbb{R})$ if it has a unique solution and the solution depends continuously on the initial value ξ .

Theorem 3.4 (Continuous dependence on ξ). Assume (H1)–(H2) and (3.6). Let $\xi_1, \xi_2 \in \mathbb{R}$ and let z_1, z_2 be the corresponding unique solutions of (3.1)–(3.2). Then

$$\|z_1 - z_2\|_\infty \leq \frac{1}{1-q} |\xi_1 - \xi_2|, \quad q := L_f + \frac{L_g T^\nu}{\Gamma(\nu+1)} \in (0, 1). \quad (3.7)$$

Proof of Theorem 3.4. By Proposition 3.1, for $i = 1, 2$ the solution z_i satisfies the integral equation

$$z_i(t) = f(t, z_i(t)) + \xi_i + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z_i(s)) ds, \quad t \in [0, T]. \quad (3.8)$$

Subtracting (3.8) for $i = 1$ and $i = 2$ yields, for each $t \in [0, T]$,

$$z_1(t) - z_2(t) = (f(t, z_1(t)) - f(t, z_2(t))) + (\xi_1 - \xi_2) + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} (g(s, z_1(s)) - g(s, z_2(s))) ds.$$

Taking absolute values and using the Lipschitz conditions (3.3), we obtain

$$\begin{aligned} |z_1(t) - z_2(t)| &\leq L_f |z_1(t) - z_2(t)| + |\xi_1 - \xi_2| + \frac{L_g}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |z_1(s) - z_2(s)| ds \\ &\leq L_f \|z_1 - z_2\|_\infty + |\xi_1 - \xi_2| + \frac{L_g}{\Gamma(\nu)} \|z_1 - z_2\|_\infty \int_0^t (t-s)^{\nu-1} ds. \end{aligned}$$

Since $\int_0^t (t-s)^{\nu-1} ds = \frac{t^\nu}{\nu}$ and $\Gamma(\nu+1) = \nu\Gamma(\nu)$, we have

$$\frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds = \frac{t^\nu}{\Gamma(\nu+1)} \leq \frac{T^\nu}{\Gamma(\nu+1)}.$$

Therefore, for all $t \in [0, T]$,

$$|z_1(t) - z_2(t)| \leq \left(L_f + \frac{L_g T^\nu}{\Gamma(\nu+1)} \right) \|z_1 - z_2\|_\infty + |\xi_1 - \xi_2| = q \|z_1 - z_2\|_\infty + |\xi_1 - \xi_2|.$$

Taking the supremum over $t \in [0, T]$ gives

$$\|z_1 - z_2\|_\infty \leq q \|z_1 - z_2\|_\infty + |\xi_1 - \xi_2|.$$

Since $q \in (0, 1)$ by (3.6), we rearrange to obtain

$$(1 - q) \|z_1 - z_2\|_\infty \leq |\xi_1 - \xi_2|,$$

which implies (3.7). This completes the proof. $\square$

Corollary 3.5. *Assume that the hypotheses of Theorem 3.2 hold. Then the problem (3.1)–(3.2) is well posed in $C([0, T], \mathbb{R})$ according to Definition 3.3.*

3.3. A priori bound. We now provide a simple bound for solutions.

Lemma 3.6. *Assume (H1)–(H3) and $L_f < 1$. Let $z \in C([0, T], \mathbb{R})$ be a solution of (3.1)–(3.2). Set*

$$F_0 := \sup_{t \in [0, T]} |f(t, 0)|, \quad G_0 := \sup_{t \in [0, T]} |g(t, 0)|.$$

If

$$\frac{L_g T^\nu}{(1 - L_f)\Gamma(\nu+1)} < 1, \tag{3.9}$$

then

$$\|z\|_\infty \leq \frac{\frac{F_0 + |\xi|}{1 - L_f} + \frac{G_0 T^\nu}{(1 - L_f)\Gamma(\nu+1)}}{1 - \frac{L_g T^\nu}{(1 - L_f)\Gamma(\nu+1)}}. \tag{3.10}$$

Proof of Lemma 3.6. Since z solves (3.1)–(3.2), by Proposition 3.1 it satisfies the integral equation

$$z(t) = f(t, z(t)) + \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds, \quad t \in [0, T]. \tag{3.11}$$

Taking absolute values and using the triangle inequality, we obtain for each $t \in [0, T]$,

$$|z(t)| \leq |f(t, z(t))| + |\xi| + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |g(s, z(s))| ds.$$

Add and subtract $f(t, 0)$ and $g(s, 0)$, then apply the Lipschitz condition (3.3):

$$\begin{aligned} |f(t, z(t))| &\leq |f(t, 0)| + |f(t, z(t)) - f(t, 0)| \leq F_0 + L_f |z(t)|, \\ |g(s, z(s))| &\leq |g(s, 0)| + |g(s, z(s)) - g(s, 0)| \leq G_0 + L_g |z(s)|. \end{aligned}$$

Substituting these estimates into (3.11) gives

$$\begin{aligned} |z(t)| &\leq F_0 + L_f |z(t)| + |\xi| + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} (G_0 + L_g |z(s)|) ds \\ &= F_0 + |\xi| + L_f |z(t)| + \frac{G_0}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds + \frac{L_g}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |z(s)| ds. \end{aligned}$$

Rearranging and using $L_f < 1$ we obtain

$$|z(t)| \leq \frac{F_0 + |\xi|}{1 - L_f} + \frac{G_0}{(1 - L_f)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds + \frac{L_g}{(1 - L_f)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |z(s)| ds. \quad (3.12)$$

Now use the identity

$$\int_0^t (t-s)^{\nu-1} ds = \frac{t^\nu}{\nu} \quad \text{and} \quad \Gamma(\nu+1) = \nu\Gamma(\nu),$$

to get

$$\frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds = \frac{t^\nu}{\Gamma(\nu+1)} \leq \frac{T^\nu}{\Gamma(\nu+1)}.$$

Also,

$$\frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |z(s)| ds \leq \frac{1}{\Gamma(\nu)} \|z\|_\infty \int_0^t (t-s)^{\nu-1} ds \leq \frac{T^\nu}{\Gamma(\nu+1)} \|z\|_\infty.$$

Taking the supremum of (3.12) over $t \in [0, T]$ yields

$$\|z\|_\infty \leq \frac{F_0 + |\xi|}{1 - L_f} + \frac{G_0 T^\nu}{(1 - L_f)\Gamma(\nu+1)} + \frac{L_g T^\nu}{(1 - L_f)\Gamma(\nu+1)} \|z\|_\infty. \quad (3.13)$$

Set

$$\alpha := \frac{L_g T^\nu}{(1 - L_f)\Gamma(\nu+1)}.$$

By (3.9), we have $\alpha < 1$. Hence, moving the last term in (3.13) to the left gives

$$(1 - \alpha) \|z\|_\infty \leq \frac{F_0 + |\xi|}{1 - L_f} + \frac{G_0 T^\nu}{(1 - L_f)\Gamma(\nu+1)}.$$

Dividing by $1 - \alpha > 0$ we obtain exactly (3.10). $\square$

4. NUMERICAL METHOD

In this section we propose a practical numerical method for approximating the solution of the hybrid Caputo problem (3.1)–(3.2). We work with the integral form (3.4) and approximate the weakly singular Volterra integral by a product-integration rule. Such approaches are common for Volterra equations and fractional models; see, for example, [5, 12, 13]. The multistep idea is inspired by classical Adams–Bashforth methods.

4.1. Grid and notation. Let $N \in \mathbb{N}$ and set $h := T/N$. Define a uniform grid

$$t_n := nh, \quad n = 0, 1, \dots, N.$$

Let $z_n \approx z(t_n)$ be the numerical approximation at t_n , and define

$$G_n := g(t_n, z_n), \quad n = 0, 1, \dots, N.$$

Recall the equivalent integral equation (Proposition 3.1)

$$z(t) = f(t, z(t)) + \xi + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s, z(s)) ds, \quad t \in [0, T]. \quad (4.1)$$

For each $n \geq 0$ we denote the fractional integral term by

$$\mathcal{I}(t_{n+1}) := \frac{1}{\Gamma(\nu)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\nu-1} g(s, z(s)) ds.$$

4.2. Local quadratic approximation. Fix $n \geq 2$. We decompose $\mathcal{I}(t_{n+1})$ as

$$\int_0^{t_{n+1}} = \sum_{j=0}^n \int_{t_j}^{t_{j+1}}.$$

On each subinterval $[t_j, t_{j+1}]$ we keep the kernel $(t_{n+1} - s)^{\nu-1}$ exact and approximate $g(s, z(s))$ by a quadratic interpolant using the three most recent values G_j, G_{j-1}, G_{j-2} (three-step Adams–Bashforth type).

For $j \geq 2$ write $s = t_j + h\theta$ with $\theta \in [0, 1]$ and define

$$P_j(\theta) := G_j L_0(\theta) + G_{j-1} L_1(\theta) + G_{j-2} L_2(\theta),$$

where the Lagrange basis functions corresponding to nodes $\theta = 0, -1, -2$ are

$$L_0(\theta) = \frac{(\theta+1)(\theta+2)}{2}, \quad L_1(\theta) = -\theta(\theta+2), \quad L_2(\theta) = \frac{\theta(\theta+1)}{2}.$$

Set $k := n+1-j \geq 1$. Then $t_{n+1} - s = h(k - \theta)$ and

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} g(s, z(s)) ds \approx h^\nu \int_0^1 (k - \theta)^{\nu-1} P_j(\theta) d\theta. \quad (4.2)$$

4.3. Weights and explicit formulas. Define the weights, for $k \geq 1$ and $r = 0, 1, 2$,

$$I_r(k) := \int_0^1 (k - \theta)^{\nu-1} L_r(\theta) d\theta. \quad (4.3)$$

Then (4.2) becomes

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} g(s, z(s)) ds \approx h^\nu \left(I_0(k) G_j + I_1(k) G_{j-1} + I_2(k) G_{j-2} \right), \quad k = n+1-j. \quad (4.4)$$

To compute $I_r(k)$ explicitly, introduce

$$J_p(k) := \int_0^1 (k - \theta)^{\nu-1} \theta^p d\theta, \quad p = 0, 1, 2.$$

A direct calculation yields

$$J_0(k) = \frac{k^\nu - (k-1)^\nu}{\nu}, \quad (4.5)$$

$$J_1(k) = \frac{k(k^\nu - (k-1)^\nu)}{\nu} - \frac{k^{\nu+1} - (k-1)^{\nu+1}}{\nu+1}, \quad (4.6)$$

$$J_2(k) = \frac{k^2(k^\nu - (k-1)^\nu)}{\nu} - \frac{2k(k^{\nu+1} - (k-1)^{\nu+1})}{\nu+1} + \frac{k^{\nu+2} - (k-1)^{\nu+2}}{\nu+2}. \quad (4.7)$$

Since

$$L_0(\theta) = \frac{\theta^2 + 3\theta + 2}{2}, \quad L_1(\theta) = -(\theta^2 + 2\theta), \quad L_2(\theta) = \frac{\theta^2 + \theta}{2},$$

we obtain

$$I_0(k) = \frac{1}{2} (J_2(k) + 3J_1(k) + 2J_0(k)), \quad (4.8)$$

$$I_1(k) = -(J_2(k) + 2J_1(k)), \quad (4.9)$$

$$I_2(k) = \frac{1}{2} (J_2(k) + J_1(k)). \quad (4.10)$$

These formulas can be used to precompute the weights $I_r(k)$ for $k = 1, 2, \dots, N$.

4.4. Fully discrete scheme. Let $n \geq 2$. Combining (4.4) and summing over $j = 2, \dots, n$, we define

$$\widehat{\mathcal{I}}_{n+1} := \frac{h^\nu}{\Gamma(\nu)} \sum_{j=2}^n \left(I_0(n+1-j) G_j + I_1(n+1-j) G_{j-1} + I_2(n+1-j) G_{j-2} \right) + \widehat{S}_{n+1}, \quad (4.11)$$

where $\widehat{S}_{n+1}$ collects the starting contributions from $j = 0, 1$. For simplicity we use lower-order rules for these first intervals:

- on $[t_0, t_1]$, use $g(s, z(s)) \approx G_0$;
- on $[t_1, t_2]$, use linear interpolation based on G_1 and G_0 .

(Other starting procedures are possible; see [5]).

Starting contributions $\widehat{S}_{n+1}$. For later use and implementation, we write $\widehat{S}_{n+1}$ explicitly. Let $k := n+1$ on $[t_0, t_1]$ and $k := n$ on $[t_1, t_2]$.

- Constant rule on $[t_0, t_1]$:

$$\int_{t_0}^{t_1} (t_{n+1} - s)^{\nu-1} g(s, z(s)) ds \approx h^\nu J_0(n+1) G_0.$$

- Linear interpolation on $[t_1, t_2]$: with $s = t_1 + h\theta$, $g(s, z(s)) \approx (1 - \theta)G_1 + \theta G_0$, we get

$$\int_{t_1}^{t_2} (t_{n+1} - s)^{\nu-1} g(s, z(s)) ds \approx h^\nu \left[(J_1(n) + J_0(n))G_1 - J_1(n)G_0 \right].$$

Therefore,

$$\widehat{S}_{n+1} = \frac{h^\nu}{\Gamma(\nu)} \left\{ J_0(n+1) G_0 + (J_1(n) + J_0(n))G_1 - J_1(n) G_0 \right\}. \quad (4.12)$$

Using (4.1) at t_{n+1} , the discrete scheme is

$$z_{n+1} = f(t_{n+1}, z_{n+1}) + \xi + \widehat{\mathcal{I}}_{n+1}, \quad n = 2, 3, \dots, N-1. \quad (4.13)$$

In general, (4.13) is implicit because z_{n+1} appears inside f .

4.5. Fixed-point solver at each time step. Assume f is Lipschitz in z with constant $L_f < 1$. Then (4.13) can be solved by a simple fixed-point iteration:

$$z_{n+1}^{(m+1)} := f(t_{n+1}, z_{n+1}^{(m)}) + \xi + \widehat{\mathcal{I}}_{n+1}, \quad m = 0, 1, 2, \dots, \quad (4.14)$$

starting from $z_{n+1}^{(0)} := z_n$. We stop when

$$|z_{n+1}^{(m+1)} - z_{n+1}^{(m)}| \leq \text{tol}$$

and set $z_{n+1} := z_{n+1}^{(m+1)}$.

Algorithm

- (1) Solve $z_0 - f(0, z_0) = \xi$ (e.g., by fixed-point iteration).
- (2) Compute z_1 and z_2 using (4.1) with lower-order quadrature for the fractional integral and the same fixed-point solver for the hybrid term.
- (3) For $n = 2, 3, \dots, N-1$:
 - (a) Compute $G_j = g(t_j, z_j)$ for $j = 0, 1, \dots, n$.
 - (b) Compute $\widehat{\mathcal{I}}_{n+1}$ using (4.11)–(4.10) and (4.12).
 - (c) Solve (4.13) by the iteration (4.14) and set z_{n+1} .

Remark 4.1. A direct implementation of (4.11) requires a sum over all previous time levels, so the cost is $O(N^2)$. This is acceptable for moderate N . For large N , one may use fast convolution ideas such as convolution quadrature to reduce the cost.

Remark 4.2. If $t \mapsto g(t, z(t))$ is sufficiently smooth, the quadratic approximation in (4.2) yields higher accuracy than first-order product-integration rules. A detailed convergence analysis is not included here; we illustrate the performance numerically in Section 4.7.

Remark 4.3 (Initial singularity). The convergence estimate is derived under additional smoothness assumptions on $\phi(t) = g(t, z(t))$ and sufficiently accurate initialization values. For general Caputo fractional differential equations, solutions may exhibit weak singular behavior near $t = 0$ (e.g., containing t^ν -type terms), which can lead to order reduction on uniform meshes for polynomial-based product-integration schemes. In such cases, graded meshes or correction terms are commonly used to recover high-order accuracy.

4.6. Convergence analysis. Additional regularity. Let z be the unique solution of (3.1)–(3.2), and define

$$\phi(t) := g(t, z(t)), \quad t \in [0, T].$$

In addition to (H1)–(H2), we assume:

(H4) $\phi \in C^3([0, T], \mathbb{R})$ and $M_3 := \max_{t \in [0, T]} |\phi^{(3)}(t)| < \infty$.

(H5) g is globally Lipschitz in z with constant L_g (as in (H2)).

Accurate starting values. To retain the third-order accuracy of the three-step rule, we assume that the starting values satisfy

$$|z(t_j) - z_j| \leq C_\star h^3, \quad j = 0, 1, 2, \quad (4.15)$$

for some constant $C_\star$ independent of h . This can be achieved, for instance, by replacing the low-order initialization on $[t_0, t_2]$ with any suitable third-order initialization for weakly singular Volterra integrals (see, e.g., [4, 5]).

Error notation. Let $\{z_n\}$ be generated by (4.13). Set

$$e_n := z(t_n) - z_n, \quad E_n := \max_{0 \leq j \leq n} |e_j|.$$

Lemma 4.4 (Interpolation error bound). *Let $\phi \in C^3([0, T])$ and $j \geq 2$. Let P_j be the quadratic interpolant of ϕ built on the nodes t_j, t_{j-1}, t_{j-2} . Then for all $s \in [t_j, t_{j+1}]$,*

$$|\phi(s) - P_j(s)| \leq M_3 h^3. \quad (4.16)$$

Proof of Lemma 4.4. Fix $j \geq 2$ and $s \in [t_j, t_{j+1}]$. By the standard Lagrange interpolation remainder formula for quadratic interpolation, there exists a point ξ_s lying between the smallest and largest interpolation node and s , hence $\xi_s \in [t_{j-2}, t_{j+1}]$, such that

$$\phi(s) - P_j(s) = \frac{\phi^{(3)}(\xi_s)}{3!} (s - t_j)(s - t_{j-1})(s - t_{j-2}).$$

Taking absolute values and using $|\phi^{(3)}(\xi_s)| \leq \max_{t \in [0, T]} |\phi^{(3)}(t)| =: M_3$, we obtain

$$|\phi(s) - P_j(s)| \leq \frac{M_3}{6} |s - t_j| |s - t_{j-1}| |s - t_{j-2}|.$$

Since $s \in [t_j, t_{j+1}]$ and $t_n = nh$, we have

$$|s - t_j| \leq h, \quad |s - t_{j-1}| \leq |s - t_j| + |t_j - t_{j-1}| \leq h + h = 2h,$$

and similarly

$$|s - t_{j-2}| \leq |s - t_j| + |t_j - t_{j-2}| \leq h + 2h = 3h.$$

Therefore

$$|s - t_j| |s - t_{j-1}| |s - t_{j-2}| \leq (h)(2h)(3h) = 6h^3.$$

Substituting this into the remainder estimate yields

$$|\phi(s) - P_j(s)| \leq \frac{M_3}{6} \cdot 6h^3 = M_3 h^3,$$

which proves (4.16). $\square$

Lemma 4.5 (Consistency of the product-integration). *Assume (H4). Define the exact fractional integral term*

$$\mathcal{I}(t_{n+1}) = \frac{1}{\Gamma(\nu)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\nu-1} \phi(s) ds.$$

Let $\widehat{\mathcal{I}}_{n+1}^*$ be the numerical approximation (4.11) constructed using the exact nodal values $\phi(t_j)$ and a initialization of order at least 3 on $[t_0, t_2]$. Then there exists a constant $C_{\text{loc}} > 0$, independent of h and n , such that

$$|\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}^*| \leq C_{\text{loc}} h^3, \quad n = 2, 3, \dots, N-1. \quad (4.17)$$

Proof of Lemma 4.5. Fix $n \in \{2, \dots, N-1\}$. Split the integral into subintervals:

$$\mathcal{I}(t_{n+1}) = \frac{1}{\Gamma(\nu)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} \phi(s) ds.$$

The product-integration approximation $\widehat{\mathcal{I}}_{n+1}^*$ can be viewed in the same form, where on each interval $[t_j, t_{j+1}]$ one replaces ϕ by a polynomial Π_j (constructed from *exact* nodal values):

$$\widehat{\mathcal{I}}_{n+1}^* = \frac{1}{\Gamma(\nu)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} \Pi_j(s) ds.$$

Hence,

$$\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}^* = \frac{1}{\Gamma(\nu)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} (\phi(s) - \Pi_j(s)) ds.$$

Taking absolute values and splitting the sum into the initialization part ($j = 0, 1$) and the three-step part ($j \geq 2$) yields

$$\begin{aligned} |\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}^*| &\leq \frac{1}{\Gamma(\nu)} \sum_{j=2}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} |\phi(s) - \Pi_j(s)| ds \\ &\quad + \frac{1}{\Gamma(\nu)} \sum_{j=0}^1 \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} |\phi(s) - \Pi_j(s)| ds \\ &=: \text{Err}_{\geq 2} + \text{Err}_{\text{start}}. \end{aligned}$$

Step 1: estimate on $j \geq 2$. For $j \geq 2$, $\Pi_j = P_j$ is the quadratic interpolant of ϕ on the nodes t_j, t_{j-1}, t_{j-2} . By Lemma 4.4, for all $s \in [t_j, t_{j+1}]$,

$$|\phi(s) - P_j(s)| \leq M_3 h^3.$$

Therefore,

$$\begin{aligned}
\text{Err}_{\geq 2} &\leq \frac{M_3 h^3}{\Gamma(\nu)} \sum_{j=2}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - s)^{\nu-1} ds \\
&= \frac{M_3 h^3}{\Gamma(\nu)} \int_{t_2}^{t_{n+1}} (t_{n+1} - s)^{\nu-1} ds \\
&= \frac{M_3 h^3}{\Gamma(\nu)} \cdot \frac{(t_{n+1} - t_2)^\nu}{\nu} \leq \frac{M_3 h^3}{\Gamma(\nu)} \cdot \frac{T^\nu}{\nu} = \frac{M_3 T^\nu}{\Gamma(\nu+1)} h^3,
\end{aligned}$$

where we used $\Gamma(\nu+1) = \nu\Gamma(\nu)$ and $t_{n+1} - t_2 \leq T$.

Step 2: initialization contribution on $[t_0, t_2]$. By the assumption “initialization of order at least 3 on $[t_0, t_2]$ ”, there exists a constant $C_{\text{start}} > 0$, independent of h and n , such that the quadrature/polynomial choice on $[t_0, t_1] \cup [t_1, t_2]$ satisfies

$$\text{Err}_{\text{start}} \leq C_{\text{start}} h^3.$$

Combining the two steps, we obtain

$$|\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}^*| \leq \left(C_{\text{start}} + \frac{M_3 T^\nu}{\Gamma(\nu+1)} \right) h^3 =: C_{\text{loc}} h^3,$$

where C_{loc} is independent of h and n . This proves (4.17). $\square$

Lemma 4.6 (Stability of the discrete fractional integral operator). *Assume (H2). Let $\{u_j\}_{j \geq 0}$ and $\{v_j\}_{j \geq 0}$ be two sequences and set $G_j^u := g(t_j, u_j)$, $G_j^v := g(t_j, v_j)$. For $n \geq 2$, denote by $\widehat{\mathcal{I}}_{n+1}[u]$ and $\widehat{\mathcal{I}}_{n+1}[v]$ the discrete fractional integral terms (4.11) computed from $\{G_j^u\}$ and $\{G_j^v\}$, respectively. Define*

$$\Delta_j := \max_{0 \leq m \leq j} |u_m - v_m|, \quad j \geq 0.$$

Then there exists a constant $C_\nu > 0$ depending only on ν (and on the fixed initialization rule on $[t_0, t_2]$) such that, for all $n \geq 2$,

$$|\widehat{\mathcal{I}}_{n+1}[u] - \widehat{\mathcal{I}}_{n+1}[v]| \leq \frac{C_\nu L_g}{\Gamma(\nu)} h^\nu \sum_{j=0}^n (n+1-j)^{\nu-1} \Delta_j. \quad (4.18)$$

Proof of Lemma 4.6. Fix $n \geq 2$. Recall that $\widehat{\mathcal{I}}_{n+1}$ is a linear combination of the nodal values G_j with weights depending on $(n+1-j)$; hence the difference $\widehat{\mathcal{I}}_{n+1}[u] - \widehat{\mathcal{I}}_{n+1}[v]$ is obtained by replacing each G_j by $(G_j^u - G_j^v)$.

Step 1: Lipschitz control of nodal differences. By (H2), g is Lipschitz in its second argument with constant L_g , hence

$$|G_j^u - G_j^v| = |g(t_j, u_j) - g(t_j, v_j)| \leq L_g |u_j - v_j| \leq L_g \Delta_j, \quad j \geq 0.$$

Step 2: Contribution from the three-step part $j \geq 2$. For $j \geq 2$, on the interval $[t_j, t_{j+1}]$ the scheme replaces g by the quadratic combination

$$P_j(\theta) = G_j L_0(\theta) + G_{j-1} L_1(\theta) + G_{j-2} L_2(\theta), \quad \theta \in [0, 1],$$

and the local contribution to the discrete integral at time t_{n+1} reads

$$\frac{h^\nu}{\Gamma(\nu)} \int_0^1 (k - \theta)^{\nu-1} P_j(\theta) d\theta, \quad k := n + 1 - j \geq 1.$$

Therefore, the *difference* between the local contributions (built from u and v) is

$$\begin{aligned} D_{j,n} &:= \frac{h^\nu}{\Gamma(\nu)} \left| \int_0^1 (k - \theta)^{\nu-1} (P_j^u(\theta) - P_j^v(\theta)) d\theta \right| \\ &\leq \frac{h^\nu}{\Gamma(\nu)} \int_0^1 (k - \theta)^{\nu-1} |P_j^u(\theta) - P_j^v(\theta)| d\theta. \end{aligned}$$

Using the representation of P_j and the triangle inequality,

$$|P_j^u(\theta) - P_j^v(\theta)| \leq |G_j^u - G_j^v| |L_0(\theta)| + |G_{j-1}^u - G_{j-1}^v| |L_1(\theta)| + |G_{j-2}^u - G_{j-2}^v| |L_2(\theta)|.$$

Set

$$\Lambda := \max_{r=0,1,2} \max_{\theta \in [0,1]} |L_r(\theta)|.$$

A direct check gives $\Lambda \leq 3$ for the Lagrange basis corresponding to nodes $\theta = 0, -1, -2$. Hence

$$|P_j^u(\theta) - P_j^v(\theta)| \leq \Lambda (|G_j^u - G_j^v| + |G_{j-1}^u - G_{j-1}^v| + |G_{j-2}^u - G_{j-2}^v|).$$

Combining with Step 1 and using $\Delta_{j-1} \leq \Delta_j$, $\Delta_{j-2} \leq \Delta_j$, we obtain, for all $\theta \in [0, 1]$,

$$|P_j^u(\theta) - P_j^v(\theta)| \leq 3\Lambda L_g \Delta_j.$$

Therefore

$$D_{j,n} \leq \frac{h^\nu}{\Gamma(\nu)} 3\Lambda L_g \Delta_j \int_0^1 (k - \theta)^{\nu-1} d\theta = \frac{h^\nu}{\Gamma(\nu)} 3\Lambda L_g \Delta_j J_0(k),$$

where $J_0(k) = \int_0^1 (k - \theta)^{\nu-1} d\theta = \frac{k^\nu - (k-1)^\nu}{\nu}$.

Step 3: Bounding $J_0(k)$ by $k^{\nu-1}$. For $k \geq 1$ and $0 < \nu < 1$, by the mean value theorem applied to $x \mapsto x^\nu$ on $[k-1, k]$, there exists $\xi_k \in (k-1, k)$ such that

$$k^\nu - (k-1)^\nu = \nu \xi_k^{\nu-1} \leq \nu (k-1)^{\nu-1}.$$

Hence, for $k \geq 2$,

$$J_0(k) = \frac{k^\nu - (k-1)^\nu}{\nu} \leq (k-1)^{\nu-1} \leq 2^{1-\nu} k^{\nu-1}.$$

For $k = 1$, we have $J_0(1) = \int_0^1 (1 - \theta)^{\nu-1} d\theta = \frac{1}{\nu}$, and thus $J_0(1) \leq \frac{1}{\nu} 1^{\nu-1}$. Therefore, there exists a constant $c_\nu > 0$ depending only on ν such that, for all $k \geq 1$,

$$J_0(k) \leq c_\nu k^{\nu-1}. \quad (4.19)$$

Combining Steps 2–3 gives, for $j \geq 2$ (i.e. $k = n + 1 - j \geq 1$),

$$D_{j,n} \leq \frac{h^\nu}{\Gamma(\nu)} (3\Lambda c_\nu) L_g (n + 1 - j)^{\nu-1} \Delta_j.$$

Summing over $j = 2, \dots, n$ yields

$$\sum_{j=2}^n D_{j,n} \leq \frac{h^\nu}{\Gamma(\nu)} C_{\nu,1} L_g \sum_{j=2}^n (n+1-j)^{\nu-1} \Delta_j, \quad C_{\nu,1} := 3\Lambda_{C_\nu}.$$

Step 4: initialization contribution on $[t_0, t_2]$. The term $\widehat{S}_{n+1}$ in (4.11) is fixed (chosen once and for all) and is a linear combination of finitely many nodal values G_0, G_1 with coefficients depending on n only through $J_0(n+1)$ and $J_1(n)$. Using (4.19) and the explicit formulas for J_1 (or simply the fact that $|J_1(k)| \leq C_\nu k^{\nu-1}$ for $k \geq 1$), one obtains

$$|\widehat{S}_{n+1}[u] - \widehat{S}_{n+1}[v]| \leq \frac{h^\nu}{\Gamma(\nu)} C_{\nu,2} L_g ((n+1)^{\nu-1} \Delta_0 + n^{\nu-1} \Delta_1),$$

for some $C_{\nu,2} > 0$ depending only on ν and on the chosen initialization rule. Since $\Delta_0 \leq \Delta_1 \leq \Delta_j$ for $j \geq 1$, we can bound the right-hand side by

$$|\widehat{S}_{n+1}[u] - \widehat{S}_{n+1}[v]| \leq \frac{h^\nu}{\Gamma(\nu)} C_{\nu,3} L_g \sum_{j=0}^1 (n+1-j)^{\nu-1} \Delta_j,$$

with another constant $C_{\nu,3}$ depending only on ν and the initialization.

Step 5: Conclusion. Adding the three-step part and the initialization part, we arrive at

$$|\widehat{\mathcal{I}}_{n+1}[u] - \widehat{\mathcal{I}}_{n+1}[v]| \leq \frac{h^\nu}{\Gamma(\nu)} C_\nu L_g \sum_{j=0}^n (n+1-j)^{\nu-1} \Delta_j,$$

where $C_\nu := \max\{C_{\nu,1}, C_{\nu,3}\}$ depends only on ν (and on the fixed initialization rule on $[t_0, t_2]$). This proves (4.18). $\square$

Lemma 4.7 (Discrete fractional Grönwall inequality). *Let $\nu \in (0, 1)$ and let $\{E_n\}_{n \geq 0}$ be a nonnegative sequence satisfying, for all $n \geq 1$,*

$$E_n \leq A + B h^\nu \sum_{j=0}^{n-1} (n-j)^{\nu-1} E_j, \quad (4.20)$$

for some constants $A, B \geq 0$ independent of h and n . Then, for all $n \geq 0$,

$$E_n \leq A E_\nu(B \Gamma(\nu) t_n^\nu) \leq A E_\nu(B \Gamma(\nu) T^\nu), \quad (4.21)$$

where E_ν denotes the Mittag-Leffler function.

Theorem 4.8 (Convergence of the scheme). *Assume (H1)–(H2) with $L_f < 1$, and let (H4)–(H5) and the initialization condition (4.15) hold. Let z be the unique solution of (3.1)–(3.2), and let $\{z_n\}$ be generated by (4.13). Then there exists a constant $C > 0$, independent of h , such that*

$$\max_{0 \leq n \leq N} |z(t_n) - z_n| \leq C h^3. \quad (4.22)$$

Proof of Theorem 4.8. Let z be the (unique) solution of (3.1)–(3.2). Evaluating the equivalent integral equation (4.1) at $t = t_{n+1}$ gives, for every $n \geq 0$,

$$z(t_{n+1}) = f(t_{n+1}, z(t_{n+1})) + \xi + \mathcal{I}(t_{n+1}), \quad (4.23)$$

where

$$\mathcal{I}(t_{n+1}) = \frac{1}{\Gamma(\nu)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\nu-1} g(s, z(s)) ds.$$

The numerical scheme reads, for $n \geq 2$,

$$z_{n+1} = f(t_{n+1}, z_{n+1}) + \xi + \widehat{\mathcal{I}}_{n+1}[z], \quad (4.24)$$

where $\widehat{\mathcal{I}}_{n+1}[z]$ denotes (4.11) computed from the nodal values $G_j = g(t_j, z_j)$ (together with the fixed initialization rule on $[t_0, t_2]$).

Step 1: error identity. Define the nodal errors

$$e_n := z(t_n) - z_n, \quad E_n := \max_{0 \leq j \leq n} |e_j|.$$

Subtracting (4.24) from (4.23) yields, for $n \geq 2$,

$$e_{n+1} = (f(t_{n+1}, z(t_{n+1})) - f(t_{n+1}, z_{n+1})) + (\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}[z]). \quad (4.25)$$

By the Lipschitz assumption on f in (H2) with constant $L_f < 1$, we have

$$|f(t_{n+1}, z(t_{n+1})) - f(t_{n+1}, z_{n+1})| \leq L_f |e_{n+1}|.$$

Therefore, from (4.25),

$$(1 - L_f) |e_{n+1}| \leq |\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}[z]|. \quad (4.26)$$

Step 2: split into consistency. Let $\phi(t) := g(t, z(t))$ and let $\widehat{\mathcal{I}}_{n+1}^*$ be the discrete fractional integral (4.11) built from the *exact* nodal values $\phi(t_j)$ (and the assumed initialization of order at least 3 on $[t_0, t_2]$). Add and subtract $\widehat{\mathcal{I}}_{n+1}^*$ to obtain

$$|\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}[z]| \leq |\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}^*| + |\widehat{\mathcal{I}}_{n+1}^* - \widehat{\mathcal{I}}_{n+1}[z]|.$$

Using Lemma 4.5, we have the local consistency estimate

$$|\mathcal{I}(t_{n+1}) - \widehat{\mathcal{I}}_{n+1}^*| \leq C_{\text{loc}} h^3, \quad n = 2, \dots, N-1. \quad (4.27)$$

For the second term, apply Lemma 4.6 with the two sequences $u_j = z(t_j)$ and $v_j = z_j$. Then $\Delta_j = \max_{0 \leq m \leq j} |u_m - v_m| = E_j$, and we obtain

$$|\widehat{\mathcal{I}}_{n+1}^* - \widehat{\mathcal{I}}_{n+1}[z]| \leq \frac{C_\nu L_g}{\Gamma(\nu)} h^\nu \sum_{j=0}^n (n+1-j)^{\nu-1} E_j, \quad n \geq 2. \quad (4.28)$$

Combining (4.26), (4.27), and (4.28) gives for $n \geq 2$,

$$|e_{n+1}| \leq \frac{C_{\text{loc}}}{1 - L_f} h^3 + \frac{C_\nu L_g}{(1 - L_f) \Gamma(\nu)} h^\nu \sum_{j=0}^n (n+1-j)^{\nu-1} E_j. \quad (4.29)$$

Taking the maximum over $0 \leq m \leq n+1$ and using monotonicity of $\{E_n\}$ yields

$$E_{n+1} \leq A h^3 + B h^\nu \sum_{j=0}^n (n+1-j)^{\nu-1} E_j, \quad n \geq 2, \quad (4.30)$$

with the h -independent constants

$$A := \frac{C_{\text{loc}}}{1 - L_f}, \quad B := \frac{C_\nu L_g}{(1 - L_f)\Gamma(\nu)}.$$

Step 3: incorporate errors from the starting values. By the initialization condition (4.15), there exists $C_\star > 0$ such that

$$E_0 + E_1 + E_2 \leq C_\star h^3.$$

Increasing A if necessary (still independent of h), we may assume that (4.30) holds for all $n \geq 0$ in the form

$$E_n \leq A h^3 + B h^\nu \sum_{j=0}^{n-1} (n-j)^{\nu-1} E_j, \quad n \geq 1. \quad (4.31)$$

Step 4: discrete fractional Grönwall. Applying Lemma 4.7 to (4.31) gives

$$E_n \leq A h^3 E_\nu(B\Gamma(\nu) t_n^\nu) \leq A h^3 E_\nu(B\Gamma(\nu) T^\nu), \quad n = 0, 1, \dots, N.$$

Hence, setting

$$C := A E_\nu(B\Gamma(\nu) T^\nu),$$

we obtain the desired global estimate

$$\max_{0 \leq n \leq N} |z(t_n) - z_n| = \max_{0 \leq n \leq N} E_n \leq C h^3.$$

The constant C depends on ν, T, L_f, L_g and on bounds entering C_{loc}, C_ν , but is independent of h . $\square$

4.7. Numerical simulations. In this section we present numerical experiments to illustrate the theoretical results and to validate the convergence behavior of the proposed three-step product-integration scheme. All computations are carried out on a uniform grid with step size $h = T/N$, $t_n = nh$. At each time step, the implicit hybrid relation is solved by the fixed-point iteration (4.14) with tolerance $\text{tol} = 10^{-10}$ and maximum number of iterations $\text{itmax} = 200$.

When an exact solution z_{ex} is available, we measure the maximum-norm error

$$E_\infty(N) := \max_{0 \leq n \leq N} |z_{\text{ex}}(t_n) - z_n|. \quad (4.32)$$

The observed rate under mesh refinement is computed by

$$\text{rate}(N) := \log_2 \left(\frac{E_\infty(N)}{E_\infty(2N)} \right). \quad (4.33)$$

When no exact solution is known, we use a step-doubling strategy. Let $\{z_n^{(N)}\}_{n=0}^N$ be the numerical solution computed on a mesh with N subintervals and $\{z_n^{(2N)}\}_{n=0}^{2N}$ be the solution on the refined mesh with $2N$ subintervals. We define the step-doubling difference

$$E_\infty^\Delta(N) := \max_{0 \leq n \leq N} |z_n^{(N)} - z_{2n}^{(2N)}| \quad (4.34)$$

and the corresponding observed rate

$$\text{rate}_\Delta(N) := \log_2 \left(\frac{E_\infty^\Delta(N)}{E_\infty^\Delta(2N)} \right). \quad (4.35)$$

Example 4.9. To validate the predicted third-order convergence in Theorem 4.8 under the additional smoothness assumptions, we consider a manufactured solution with a known closed form. Figure 1 further confirms this conclusion: the

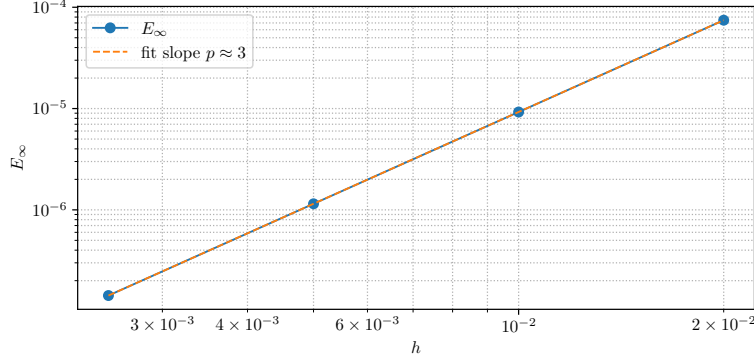


FIGURE 1. log-log plot of $E_\infty(N)$ versus h confirming third-order decay

log-log plot of $E_\infty(N)$ versus h is nearly a straight line, and the fitted slope is $p \approx 3.01$, indicating $E_\infty(N) = \mathcal{O}(h^3)$.

Fix $T = 1$, choose $0 < \nu < 1$, and set

$$f(t, z) = \alpha z, \quad 0 < \alpha < 1. \quad (4.36)$$

Let

$$z_{\text{ex}}(t) = c_0 + t^m, \quad m \geq 4, \quad (4.37)$$

and define

$$u(t) := z(t) - f(t, z(t)) = (1 - \alpha)z(t).$$

The initial datum is then chosen as

$$\xi = u(0) = (1 - \alpha)z_{\text{ex}}(0) = (1 - \alpha)c_0. \quad (4.38)$$

Using the identity ${}^C D_{0+}^\nu(t^m) = \frac{\Gamma(m+1)}{\Gamma(m+1-\nu)} t^{m-\nu}$, we set

$$g(t, z) = (1 - \alpha) \frac{\Gamma(m+1)}{\Gamma(m+1-\nu)} t^{m-\nu}, \quad (4.39)$$

which is independent of z . With these choices, z_{ex} in (4.37) is the exact solution of (3.1)–(3.2).

In the experiments below we use (unless stated otherwise):

$$\nu = 0.6, \quad \alpha = 0.3, \quad c_0 = 1, \quad m = 4, \quad T = 1, \quad \xi = (1 - \alpha)c_0 = 0.7.$$

Table 1 shows that the error decreases monotonically as the mesh is refined and the observed rates are consistently close to 3. This agrees with the expected third-order behavior of the proposed.

TABLE 1. $E_\infty(N)$ errors and observed rates.

N	h	$E_\infty(N)$	rate
50	0.02	7.450567e-05	—
100	0.01	9.245196e-06	3.0106
200	0.005	1.147594e-06	3.0101
400	0.0025	1.426352e-07	3.0082

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