Gulf Journal of Mathematics
Vol 22, Issue 2 (2026) 1-13

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v22i2.4224

NEW GENERALIZATION OF THE CLASS OF H-SUPPLEMENTED MODULES

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Abstract. Let P be a right R-module, where R is a ring with identity. In the present research, a class of modules is described similar to H-μ-supplemented and μ-lifting modules. A module P is called principally H-μ-supplemented, if there is a summand L of P such that pR is μ-equivalent to L, for every $p \in P$. Additionally, we present an extension of supplemented modules. A module P is considered to be principally μ-supplemented if, for every p in P, pR has a μ-supplement in P. A number of characteristics of these modules are shown, and it is demonstrated that both the P_μ-H-supplemented and P_μ-supplemented modules include the class of principally μ-lifting modules.

Keywords. H-supplemented modules, principally lifting modules.
2020 Mathematics Subject Classification. Primary 16D80; Secondary 16D10

1. Introduction and preliminaries

Throughout this work, R will denote an associative ring with identity and all modules are right R-modules. For convenience, the term ”direct summand” is denoted by ”d.s.” throughout the paper to simplify notation and avoid repetition. A submodule V of a module P is referred to as small in P ($V \ll P$) if whenever $P = V + L$, then $L = P$, [7]. A submodule V of P is known as supplement of L if $P = L + V$ and $L \cap V \ll V$, when all submodules of P have supplements, P is called a supplemented module, [8]. A module P is principally supplemented when all of its cyclic submodules have supplements, according to Acar and Haramanci [1]. A module P is referred to as (principally) lifting, if there exists a decomposition $P = L \oplus L'$ with $L \leqslant V$ and $L \cap L' \ll P$, for each submodule V of P, [4]. Ozcan [18], introduced the submodule $Z^*(P)$ as a dual concept of the singular submodule, $Z^*(P) = \{p \in P \text{ such that } pR \ll E(P)\}$, where $E(P)$ represents the injective hull of P. A module P is called cosingular (noncosingular) if $Z^*(P) = P$ ($Z^*(P) = 0$). Several studies have investigated some generalizations of small submodules, see for example [19]. The concept of μ-small submodules, a useful generalization of small submodule, was introduced by Kamil and Khalid [12]. A submodule V of a module P is μ-small in P ($V \ll_\mu P$) if the equality

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Date: Received: Feb 7, 2026; Revised: Feb 22, 2026; Accepted: Mar 5, 2026.

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$P = V + L$ with $\frac{P}{L}$ is cosingular implies $P = L$. Let L and V be submodules of P such that $L \leqslant V$, L is called μ-coessential submodule of V in P if $\frac{V}{L} \ll_{\mu} \frac{P}{L}$, denoted by ($L \leqslant_{\mu ce} V$ in P), V is called μ-coclosed ($V \leqslant_{\mu cc} P$) when $\frac{V}{L}$ is cosingular and $L \leqslant_{\mu ce} V$ in P, [14]. A submodule V of P is μ-supplement of L in P if $P = V + L$ and $V \cap L \ll_{\mu} \mathrm{V}$ and P is μ-supplemented, when each submodule of P has μ-supplement in P, [13].

In [10], P is referred to as μ-lifting provided that for each submodule V of P, there is a decomposition $\mathrm{P} = \mathrm{L} \oplus \mathrm{L}'$ with $\mathrm{L} \leqslant \mathrm{V}$ and $\mathrm{V} \cap \mathrm{L}' \ll_{\mu} \mathrm{L}'$, [14]. Let V and L be submodules of a module P. In [20], V and L are related by the relation β^*, denoted by $V\beta^* L$ when $\frac{T+L}{V} \ll \frac{P}{V}$ and $\frac{\mathrm{V+L}}{\mathrm{L}} \ll \frac{\mathrm{P}}{\mathrm{L}}$. In [20] and [21], a module P is H-supplemented provided that for each submodule V of $\mathrm{P}, \mathrm{V}\mu\mathrm{L}$, for some d.s L of P. A module P is known as principally goldie*-lifting module, provided that every cyclic submodule of P is μ-equivalent to a d.s in P, [6]. A module P is called principally μ-hollow (P_{μ}-hollow) if, every cyclic proper submodule in P is μ-small in P, [10].

In [10], [11], and [15], there were two new generalizations of μ-lifting modules are defined. A module P is principally μ-lifting (P_{μ}-lifting), if P satisfies the μ-lifting condition on the cyclic submodules. A module P is referred to as principally H-μ-supplemented, for short (P_{μ}-H-supplemented), provided that each submodule V of P is related with a d.s L of P by the equivalent relation μ.

Inspired by these notions, we study two new types of modules: P_{μ}-supplemented and P_{μ}-H-supplemented.

Section 2 presents several features of the relation μ and μ-small submodules. The concept of P_{μ}-H-supplemented is defined and examined in section 3. Additionally, we provide the relationship between the condition of P_{μ}-H-supplemented among various module classes and provide examples to demonstrate that, in general, the opposite implications are not satisfied.
The presentation of P_{μ}-supplemented is covered in Section 4. It is demonstrated that all P_{μ}-lifting is P_{μ}-supplemented, and we provide an example to demonstrate that the opposite is false, in general. We examine the circumstances in which the opposite implication is true. In what follows, by $\mathbb{Z}_n, \mathbb{Z}, Q$, we denote, the $\mathbb{Z}$-module of integer modulo n, integers, rational set.

2. Basic results.

This section collects the basic characteristics of μ-small submodules of a module and the relation μ on the collection of submodules, to allow for their easy referencing in later sections of the paper.

Lemma 2.1. [12] *Assume that P is a module. Consider the following conditions.*

(*i*) *Let $V \leqslant N \leqslant P$, then $N \ll_{\mu} P$ if and only if $\mathrm{V} \ll_{\mu} P$ and $\frac{N}{V} \ll_{\mu} \frac{P}{V}$.*

(*ii*) *Let $V, N \leqslant P$, then $V + N \ll_{\mu} P$ if and only if $V \ll_{\mu} P$ and $N \ll_{\mu} P$. Moreover, if $V_1, V_2, \ldots, V_n \leqslant P$ with $V_i \ll_{\mu} P, \forall i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} V_i \ll_{\mu} P$.*

(*iii*) *Let $V, N \leqslant P$ with $V \leqslant N$. If $\mathrm{V} \ll_{\mu} N$, then $\mathrm{V} \ll_{\mu} P$.*

(*iv*) *Let $\phi : P \to P'$ be a homomorphism such that $\mathrm{V} \ll_{\mu} P$, then $\phi(\mathrm{V}) \ll_{\mu} P'$.*

(*v*) *Let* $P = P_1 \oplus P_2$ *and* $V = V_1 \oplus V_2$, *where* $V_1 \leqslant P_1$ *and* $V_2 \leqslant P_2$. *Then* $V_1 \ll_\mu P_1$ *and* $V_2 \ll_\mu P_2$ *if and only if* $V \ll_\mu P$.

Lemma 2.2. *Let* P *be a module, and* $p \in P$ *and* $L \leqslant P$, *the following features are equivalent.*

(*i*) $P = pR + L$ *and* $pR \cap L \ll_\mu L$.

(*ii*) $P = pR + L$ *and for any proper submodule* N *of* L *such that* $\frac{L}{N}$ *is cosingular,* $P \neq pR + N$.

Proof. $(i) \Rightarrow (ii)$ Let $N \leqslant L$ and $P = pR + N$, where $\frac{L}{N}$ is cosingular. Then $L = (L \cap pR) + N$. As $L \cap pR \ll_\mu L$, then $L = N$.
$(ii) \Rightarrow (i)$ Let $P = pR + L$ and $N \leqslant L$ and $\frac{L}{N}$ is cosingular such that $L = (pR \cap L) + N$. Then, $P = pR + L = pR + N$. From (ii), $N = L$. Hence, $pR \cap L \ll_\mu L$. □

Kamil and Khalid [15] introduced a relation μ on the collection of submodules of a module P. Let V and L be submodules of a module P, then $\mathrm{V}\mu L$ if and only if $\frac{V+L}{V} \ll_\mu \frac{P}{V}$ and $\frac{V+L}{L} \ll_\mu \frac{P}{L}$. It is easy to see that μ is an equivalence relation. In this part, we examine several characteristics of this relationship.

Theorem 2.3. [15] *Let* V *and* L *be submodules of a module* P. *The next statements are identical.*

(*i*) $V\mu L$.

(*ii*) $V \underset{\mu ce}{\leqslant} V + L$ *in* P *and* $\mathrm{L} \underset{\mu ce}{\leqslant} V + L$ *in* P.

(*iii*) *For every* $X \leqslant P$ *satisfies* $P = V + L + X$, $\frac{P}{X}$ *is cosingular, then* $P = V + X$ *and* $P = \mathrm{L} + \mathrm{X}$.

(*iv*) *If* $P = V + A$, *for any* $A \leqslant P$ *such that* $\frac{P}{A}$ *is cosingular, then* $P = \mathrm{L} + A$ *and if* $P = \mathrm{L} + B$, *for any* $B \leqslant P$ *such that* $\frac{P}{B}$ *is cosingular, then* $P = V + B$.

Lemma 2.4. *Let* P *be a module,* $p \in P$ *and* L *a d.s of* P. *The following conditions are equivalent.*

(*i*) $pR\mu L$.

(*ii*) *If* $P = pR + L + X$ *and* $\frac{P}{X}$ *is cosingular, for any* $X \leqslant P$, *then* $P = pR + X$ *and* $P = L + X$.

Proof. $(i) \Rightarrow (ii)$ Assume that $P = pR + L + X$ with $\frac{P}{X}$ is cosingular, for some $X \leqslant P$. Then $\frac{P}{pR} = \left(\frac{pR+X}{pR}\right) + \left(\frac{pR+L}{pR}\right)\cdot$ Also, $\frac{P}{pR+X}$ is cosingular, by ([12], Corollary 2.6). Since $\frac{pR+L}{pR} \ll_\mu \frac{P}{pR}$. Thus, $P = pR + X$.
$(ii) \Rightarrow (i)$ Let $X \leqslant P$ such that $pR \leqslant X$ and $\frac{P}{pR} = \left(\frac{pR+V}{pR}\right) + \left(\frac{X}{pR}\right)$ with $\frac{P}{X}$ is cosingular. Then, $P = pR + L + X$. By (ii), $= pR + X$; and hence $P = X$. Thus, $\frac{pR+L}{pR} \ll_\mu \frac{P}{pR}$. In a similar way, we conclude $\frac{pR+L}{L} \ll_\mu \frac{P}{L}$. □

Corollary 2.5. [15] *Assume* V *and* L *are submodules of a module* P *such that* $V \leqslant L + X$ *and* $L \leqslant V + Y$. *If* X, Y *are* μ*-small in* P, *then* $V\mu L$.

Proposition 2.6. [15] *Let* P *is a module and* V, L *are submodules of* P, *then* $V\mu L$ *if and only if* $\frac{V}{X}\mu\frac{L}{X}$, *for every* $X \leqslant V \cap L$.

Lemma 2.7. [15] *Let V_1, V_2, L_1 and L_2 be submodules of a module P. If $V_1 \mu L_1$ and $V_2 \mu L_2$, then $(V_1 \oplus V_2)\, \mu\, (L_1 \oplus L_2)$.*

Lemma 2.8. [15] *Let $\varphi : P \to P'$ be R-homomorphism, then.*

- (*i*) *If $V, L \leqslant P, V\mu L$, then $\varphi(V)\mu\varphi(L)$.*
- (*ii*) *If $V, L \leqslant P', V\mu L$, then $\varphi^{-1}(V)\mu\varphi^{-1}(L)$.*

Proposition 2.9. *Let P be a module and V be a d.s of P. If $V\mu X$, where X is cyclic submodule of P with $\frac{P}{X}$ is cosingular, then V is also cyclic.*

Proof. Let $P = V \oplus V', V' \leqslant P$ and let X be a cyclic submodule of P with $\frac{P}{X}$ is cosingular. By Theorem 2.3, $P = X + V$. As $\frac{X+V'}{V'} = \frac{P}{V'} \cong V$ and X is cyclic yields that V is cyclic □

Lemma 2.10. *Let $P = P_1 \oplus P_2$, and let V and L be submodules of P_1. The next properties hold.*

- (*i*) *$V\mu L$ in P_1 if and only if $V\mu L$ in P.*
- (*ii*) *Let $P_1 \leqslant X \leqslant P$. Then $X\mu P_1$ if and only if $X \cap P_2 \ll_\mu P_2$.*
- (*iii*) *Let $X \leqslant P_1$. If P_1 is cosingular, then $X\mu P_1$ if and only if $X = P_1$.*

Proof. (*i*) Let V and L be submodules of P_1. Assume that $V\mu L$ in P_1, then $\frac{V+L}{V} \ll_\mu \frac{P_1}{V}$ and $\frac{V+L}{L} \ll_\mu \frac{P_1}{L}$. Let X be a submodule of P such that $\frac{V+L}{L} + \frac{X}{L} = \frac{P}{L}$ and $\frac{P}{X}$ is cosingular. As the cosingularity property is closed under isomorphism, then $\frac{P_1}{P_1 \cap X} \simeq \frac{P_1+X}{X} \leqslant \frac{P}{X}$; hence, $\frac{P_1}{P_1 \cap X}$ is cosingular. Then $\frac{V+L}{L} + \frac{P_1 \cap X}{L} = \frac{P_1}{L}$, so, $P_1 \cap X = P_1$, which yields that $P_1 \leqslant X$. Moreover, $\frac{X}{L} = \frac{P}{L}$. By similar proof, we conclude that $\frac{V+L}{V} \ll_\mu \frac{\mathrm{P}}{V}$. Conversely, suppose that $V\mu L$ in P. Let X be a submodule of P_1 such that $\frac{V+L}{L} + \frac{X}{L} = \frac{P_1}{L}$ and $\frac{P_1}{X}$ is cosingular. Then $\frac{P}{X} = \frac{P_1}{X} + \frac{P_2+X}{X}$ and $\frac{P}{X} \cong \frac{P_1}{P_2+X}$ is cosingular and $\frac{P}{L} = \frac{V+L}{L} + \frac{P_2+X}{L}$. By assumption, $\frac{V+L}{L} \ll_\mu \frac{\mathrm{P}}{L}$. So, $\frac{P_2+X}{L} = \frac{P}{L}$, and hence $P = P_2 + X$. Thus, $X = P_1$. Similarly, we can show that $\frac{V+L}{V} \ll_\mu \frac{P_1}{V}$.

(*ii*) Let X be a submodule of P such that $P_1 \leqslant X$ and assume that $X\mu P_1$. Then, $X = P_1 \oplus (X \cap P_2)$ and $\frac{X+P_1}{P_1} = \frac{X}{P_1} \ll_\mu \frac{\mathrm{P}}{P_1}$. As $\frac{X}{P_1} \cong X \cap P_2$ and $\frac{\mathrm{P}}{P_1} \cong P_2$, then $X \cap P_2 \ll_\mu P_2$, by Lemma 2.1. The converse is obvious.

(*iii*) Let X be a submodule of P_1 and P_1 is cosingular. Suppose that $X\mu P_1$. Then $\frac{X+P_1}{X} = \frac{P_1}{X} \ll_\mu \frac{\mathrm{P}}{X}$. Since P_1 is cosingular, $\frac{P}{P_2+X}$ is also cosingular, by ([12], Corollary 2.6). Note that $\frac{P}{X} = \frac{P_1}{X} + \frac{(P_2+X)}{X}$ yields that $P = P_2 + X$. Thus, $X = P_1$. The converse is obvious.

□

Proposition 2.11. *Assume P is a module then let V, L and T are submodules of P. If $V\mu L$ and $\mathrm{T} \ll_\mu P$, then $V\mu(L+T)$.*

Proof. Assume that $V\mu L$ and $\mathrm{T} \ll_\mu P$, then $\frac{V+L}{L} \ll_\mu \frac{P}{L}$ and $\frac{V+L}{V} \ll_\mu \frac{P}{V}$. Let W be a submodule of P such that $\frac{V+\mathrm{L}+T}{V} + \frac{W}{V} = \frac{P}{V}$ and $\frac{P}{W}$ is cosingular, then $\frac{P}{V} = \frac{V+L}{V} + \frac{T+W}{V}$ and the cosingularity of $\frac{P}{W}$ implies that of $\frac{P}{T+W}$, by ([12], Corollary 2.6). Since $\frac{V+L}{V} \ll_\mu \frac{P}{V}$ and $\frac{P}{T+W}$ is cosingular, then $P = T + W$. Moreover, $\frac{P}{W}$ is cosingular and $T \ll_\mu P$, implies $P = W$. Hence, $\frac{V+L+T}{V} \ll_\mu \frac{P}{V}$. Now, assume U is a submodule of P such that $\frac{V+L+T}{L+T} + \frac{U}{L+T} = \frac{P}{L+T}$ and $\frac{P}{U}$ is cosingular. Then $\frac{P}{L} = \frac{V+L}{L} + \frac{T+U}{L}$ and $\frac{P}{T+U}$ is cosingular because of the cosingularity of $\frac{P}{U}$. Since $\frac{V+L}{L} \ll_\mu \frac{P}{L}$, then $\frac{P}{L} = \frac{T+U}{L}$. As $T \ll_\mu P$, then $P = U$. This completes the proof. □

Proposition 2.12. *Assume* $V, L, N \leqslant P$ *with* $P = V+N = L+N, L\cap N \leqslant V\cap N$ *and* $\frac{V+L}{L} \ll_\mu \frac{P}{L}$, *then* $V\mu L$.

Proof. Note that $\frac{N}{L\cap N} \overset{\theta}{\cong} \frac{\mathrm{L+N}}{L} = \frac{P}{L}$ and $\frac{N}{V\cap N} \overset{\psi}{\cong} \frac{V+N}{V} = \frac{P}{V}$, define the map $f : \frac{N}{L\cap N} \to \frac{N}{V\cap N}$, by $f(n + L\cap N) = n + (V \cap N), n \in N, f$ is an epimorphism, and define $h : \frac{P}{L} \to \frac{P}{V}$ by $h = \psi \circ f \circ \theta$. Let $p \in P$, then $p = l + n, l \in L$ and $n \in N$. Then $\theta(p + L) = \theta(l + n + L) = n + L\cap N, f(n + L\cap N) = n + (V \cap N)$, and $\psi(n + V \cap N) = n + V = n + v + V, v \in V$. So, $h(p + L) = n + v + V$. Now, let $v + l + L \in \frac{V+L}{L}$, then there exists $l_1 + n = v + l, l_1 \in L, n \in N$. Hence, $h(v + l + L) = v + l - l_1 + V \in \frac{V+L}{L}$. Now, assume that $v_2 + l_2 + V \in \frac{V+L}{V}$, there exists $v_3 \in V$ and $n_3 \in N$ such that $l_2 = v_3 + n_3$. Hence, $v_2 + l_2 + V = n_3 + V$. Observe that $n_3 = -v_3 + l_2 \in V + L\cdot$ So, $h\left(n_3 + L\right) = (\psi \circ f \circ \theta)\left(n_3 + L\right) = \psi \circ f\left(n_3 + L \cap N\right) = \psi\left(n_3 + V \cap N\right) = n_3 + V = v_2 + l_2 + V$. Thus, $h\left(\frac{V+L}{L}\right) = \frac{V+L}{V}$. As $\frac{V+L}{L} \ll_\mu \frac{P}{L}$, then $\frac{V+L}{V} \ll_\mu \frac{P}{L}$, By Lemma 2.1, this yields that $V\mu L$. □

3. Principally H-μ-supplemented

We present P_μ-H-supplemented modules, which are inspired by the generalization of lifting and supplemented modules. Examining a few characteristics of this type of modules is the focus of this section.

Definition 3.1. A module P is referred to as principally H-μ-supplemented (briefly P_μ-H-supplemented) if, for each cyclic submodule pR of P, there is a d.s L of P such that $pR\mu L$.

Proposition 3.2. *Let* P *be a module and consider the next statements*

(*i*) *P is a principally lifting module.*
(*ii*) *P is a principally $\mathcal{G}^*$-lifting.*
(*iii*) *P is a P_μ-lifting.*
(*iv*) *P is a P_μ-H-supplemented. Then (i) ⇒(ii) ⇒ (iv) and (i) ⇒ (iii) ⇒ (iv)*

Proof. (*i*) ⇒ (*ii*) See [6].
(*ii*)⇒ (*iv*) It is obvious from the fact that μ-small submodule is a generalization of small submodule.
(*i*)⇒ (*iii*) See [10].
(*iii*) ⇒ (*iv*) Let P be a P_μ-lifting and let pR be a submodule of P. As P is P_μ-lifting, there is a d.s L of P such that $\frac{pR}{L} \ll_\mu \frac{P}{L}$: hence, $pR\mu L$.Thus, P is P_μ-H-supplemented.
(*ii*) ⇏ (*i*) The $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is principally $\mathcal{G}^*$-lifting, which is not principally lifting. Also, this is an example to show that (*iv*) ⇏ (*iii*).
(*iv*) ⇏ (*ii*) Let $S = \prod_{i=1}^{\infty} H_i$, where $H_i = \mathbb{Z}_2$, assume R is the subring of S which

is generated by $\oplus_{i=1}^{\infty} H_i$ and 1_s. As R_R is μ-hollow, it is H-μ-supplemented, and hence it is P_μ-H-supplemented, R_R is not supplemented. This is also an example shows that $(iii) \nRightarrow (i)$ $\square$

Example 3.3. Clearly, H-μ-supplemented is P_μ-H-supplemented. However, $\mathbb{Q}$ as $\mathbb{Z}$-module shows that the converse is false, in general.

Proposition 3.4. *Assume P is an indecomposable module the next properties coincide.*

- (i) *P is P_μ-hollow.*
- (ii) *P is P_μ-lifting.*
- (iii) *P is $P_\mu-$H-supplemented.*

Proof. (i)⇒(ii) See [10].
$(ii) \Rightarrow (iii)$ Proposition 3.2.
$(iii) \Rightarrow (i)$ Let pR be a proper cyclic submodule of a P_μ-H-supplemented module P, there is a d.s L of P such that $p\mathrm{R}\mu L$. As P is indecomposable, either $L = P$ or $L = 0$. If $L = P$, then $pR = P$ which is a contradiction; $L = 0$ yields that $pR \ll_\mu P$. Thus, P is P_μ-hollow. $\square$

Proposition 3.5. *Assume P is a P_μ-H-supplemented. If each submodule of P has a unique μ-coclosure, then P is P_μ-lifting.*

Proof. Let V be a cyclic submodule of a P_μ-H-supplemented module P, there is a d.s L of P such that $V\mu L$. Let X be a μ-coclosure of V,then $X \leqslant_{\mu ce} V$ in P and $X \leqslant_{\mu ce} L$ in P, hence $X \leqslant_{\mu ce} V + L$ in P, i.e, X is μ-coclosure of $V + L$. However, L is a μ-coclosure of $V + L$; therefore, $L = X \leqslant V$. Thus, P is P_μ-lifting. $\square$

Proposition 3.6. *Let End (P_R) be Abelian such that, if $V \leqslant P$, then $V = \sum_{i \in I} \psi_i(P)$, where each $\psi_i \in End\,(P_R)$. Then P is P_μ-lifting if and only if it is P_μ-H-supplemented.*

Proof. Assume pR is a cyclic submodule of P, then $pR = \sum_{i \in I} \psi_i(P)$, where $\psi_i \in End\,(P_R)$, by hypothesis. As P is P_μ-H-supplemented, there exists $g^2 = g \in End(P_R)$ such that gP is μ-supplement of V. Since End$\,(P_R)$ is Abelian, then $(1 - g)\mathrm{P} \leqslant V$; hence, $\frac{V}{(1-g)P} \ll_\mu \frac{P}{(1-g)P}$. Thus, P is P_μ-lifting. $\square$

Proposition 3.7. *In a cosingular module, the P_μ-H-supplemented and* H-μ-*supplemented properties are equivalent.*

Proof. Straightforward. $\square$

Our next results are equivalent to the conditions of P_μ-H-supplemented.

Theorem 3.8. *The following conditions are identical for a module P.*

- (i) *P is P_μ-H-supplemented.*
- (ii) *P can be expressed by $P = P_1 \oplus P_2$, where $(V + P_1) \cap P_2 \ll_\mu P_2$, for each cyclic submodule V of P.*
- (iii) *For each cyclic submodule V of P,we have $V + L = L \oplus N$, where L is a d.s of P and $N \ll_\mu P$.*

Proof. $(i) \Rightarrow (ii)$ Assume P is P_μ-H-supplemented, let V be a cyclic submodule of P, there is a decomposition $P = P_1 \oplus P_2$ such that $V \mu P_1$. Let $P_2 = (V + P_1) \cap P_2 + X$, where $X \leqslant P_2$ and $\frac{P_2}{X}$ is cosingular. Then $P = P_1 \bigoplus P_2 = P_1 + [(V + P_1) \cap P_2] + X$; hence, $\frac{P}{P_1} = \frac{P_1+X}{P_1} + \frac{[(V+P_1)\cap P_2]+P_1}{P_1}$. However, $P_1 \leqslant [(V + P_1) \cap P_2] + P_1 \leqslant V + P_1$ and $P_1 \leqslant_{\mu ce} V + L$ in P; therefore, $P_1 \leqslant_{\mu ce} [(V + P_1) \cap P_2] + P_1$ in P and $\frac{P}{X+P_1} = \frac{P_1+P_2}{X+P_1} = \frac{(P_1+X)+P_2}{X+P_1} \cong \frac{P_2}{P_2\cap(X+P_1)} = \frac{P_2}{X}$ is cosingular, this yields that $P = P_1 + X$. As $P_1 \cap X \leqslant P_1 \cap P_2 = 0$, then $P_1 \cap X = 0$. Hence $P = P_1 \oplus X, X = P_2$. Thus, $(V + P_1) \cap P_2 \ll_\mu P_2$.
$(ii) \Rightarrow (iii)$ Let V is a cyclic submodule of P. So, there is a decomposition $P = P_1 \oplus P_2$, and $(V + P_1) \cap P_2 \ll_\mu P_2$, by (ii). Observe that $V + P_1 = (V + P_1) \cap P = (V + P_1) \cap (P_1 + P_1) = P_1 \oplus [(V + P_1) \cap P_2], (V + P_1) \cap P_2 \ll_\mu P_2$.
$(iii) \Rightarrow (i)$ Let V be a cyclic submodule of $P, (V + L) = L \oplus N$, where L is a d.s of P and $N \ll\, _\mu P$, by (iii). Let $\frac{P}{L} = \frac{V+L}{L} + \frac{X}{L}, \frac{P}{X}$ is cosingular, then $P = V + L + X = L + N + X = N + X = X$, so, $\frac{V+L}{L} \ll_\mu \frac{P}{L}$. By similar way, $\frac{V+L}{V} \ll_\mu \frac{P}{V}$. Thus, P is P_μ-H-supplemented. ☐

Theorem 3.9. *A module P is P_μ-H-supplemented if and only if for each cyclic submodule V of P, there is a d.s L of P and a submodule X of P such that $V \leqslant_{\mu ce} X$ in P and $L \leqslant_{\mu ce} X$ in P*

Proof. Assume P is a P_μ-H-supplemented and V is cyclic submodule of P, there is a d.s L of P such that V $\leqslant_{\mu ce} V + L$ in P and L $\leqslant_{\mu ce} V + L$ in P. In particular, let $X = V + L$. For the converse let V be a cyclic submodule of P, there is $X \leqslant P$ and a d.s L of P such that $V \leqslant_{\mu ce} X$ in P and $L \leqslant_{\mu ce} X$ in P, then $V + L \leqslant_{\mu ce} X$ in P, by ([10], Proposition 2.6); hence, $V \leqslant_{\mu ce} V + L$ in P and $L \leqslant_{\mu ce} V + L$ in P, by ([10], Proposition 2.5). Thus, P is P_μ-H-supplemented. ☐

The next example shows if $\frac{P}{V}$ is P_μ-H-supplemented, then P may not be P_μ-H-supplemented.

Example 3.10. Assume $P = \frac{\mathbb{Z}}{p^n\mathbb{Z}}, p$ is prime, $n \in \mathbb{Z}^+$. Then P is P_μ-lifting; hence P is P_μ-H-supplemented. However, $\mathbb{Z}$ is not P_μ-H-supplemented.

Our next propositions investigate conditions when the factor of P_μ-H-supplemented is also P_μ-H-supplemented.

Proposition 3.11. *Let P be a P_μ-H-supplemented, and let $V \leqslant P$. If $\frac{V+L}{V}$ is a d.s of $\frac{P}{V}$, for any cyclic d.s L of P, then $\frac{P}{V}$ is P_μ-H-supplemented.*

Proof. Let $pR + V$ be a cyclic submodule of $\frac{P}{V}$,p$\in P$. As P is P_μ-H-supplemented, there is a d.s L of P such that $pR \mu L$. From ([15], Proposition 2.8), we conclude that $\frac{pR+V}{V} \mu \frac{L+V}{V}$. By hypothesis, $\frac{L_+V}{V}$ is a d.s of $\frac{P}{V}$.Thus, $\frac{P}{V}$ is a P_μ-H-supplemented. ☐

A module P is called distributive, if for every submodules P_1, P_2 and P_3 of P, the next equality holds $P_1 \cap (P_2 + P_3) = (P_1 \cap P_2) + (P_1 \cap P_3)$, [5] and [9]. When $\varphi(X) \leqslant X$, for each $\varphi \in$ End (P), X is said to be fully invariant, P is referred to as a duo module if all of its submodules are fully invariant, [17], [2].

Corollary 3.12. *Let P be a P_μ-H-supplemented. If P is distributive (or duo) module, then each factor module of P is P_μ- H-supplemented.*

Proof. Assume P is a distributive or duo module, and let L be a d.s of P, one can easily prove that $\frac{L+V}{V}$ is a d.s of $\frac{P}{V}$, for every $V \leqslant P$. By Proposition 3.11, $\frac{P}{V}$ is P_μ-H-supplemented. $\square$

Proposition 3.13. *Let P be a P_μ-H-supplemented, and then let V be a submodule of P. If for each $f \in \mathcal{J}(\mathrm{End}(P))$ there is $g \in \mathcal{J}\left(\mathrm{End}\left(\frac{P}{V}\right)\right)$ such that $\operatorname{Im} g \leqslant_{\mu ce} \frac{V+fP}{V}$ in $\frac{P}{V}$. Then $\frac{P}{V}$ is P_μ-H-supplemented, where $\mathcal{J}(End(P)) = \{f : P \to P|\ f$ is homomorphism with $f \circ f = f\}$.*

Proof. Assume $\frac{pR+V}{V}$ is a cyclic submodule of $\frac{P}{V}$. As P is P_μ-H-supplemented, there is $f \in \mathcal{J}(\mathrm{End}(P))$ such that $pR\mu fP$. So, there exists $g \in \mathcal{J}($ End $\left(\frac{P}{V}\right))$ such that $\operatorname{Im} g \leqslant_{\mu ce} \frac{V+fP}{V}$ in $\frac{P}{V}$; hence, $\mathrm{Img}\, \mu \frac{V+fP}{V}$ and $\frac{pR+V}{V} \mu \frac{fP+V}{V}$.However, μ is equivalence relation, [15]; therefore, $\frac{pR+V}{V} \mu\, \mathrm{Img}$ Thus, P is P_μ-H-supplemented. $\square$

Proposition 3.14. *Let P be a P_μ-H-supplemented, let $V \leqslant P$ such that, $V = (V \cap P_1) \oplus$ $(V \cap P_2)$, for every decomposition $P = P_1 \oplus P_2$. Then $\frac{P}{V}$ is P_μ-H-supplemented.*

Proof. Let $\frac{pR}{V}$ be a cyclic submodule of $\frac{P}{V}$. As P is pP_μ-H-supplemented, there exists a d.s L of P and $X \leqslant P$ such that $\frac{X}{pR} \ll_\mu \frac{P}{pR}$ and $\frac{X}{L} \ll_\mu \frac{P}{L}$. Put $P = L \oplus L', L' \leqslant P$ then V $= V \cap P = V \cap (L \oplus L') = (V \cap L) \oplus (V \cap L') = (L+V) \cap (L' + V)$, by hypothesis. So, $\frac{L+V}{V} \oplus \frac{\mathrm{L}'+\mathrm{V}}{V} = \frac{P}{V}$. Now, we have $\frac{\frac{X}{V}}{\frac{pR}{V}} \ll_\mu \frac{\frac{P}{V}}{\frac{pR}{V}}$ and $\frac{\frac{X}{V}}{\frac{L+V}{V}} \ll_\mu \frac{\frac{P}{V}}{\frac{L+V}{V}}$ and hence $\frac{P}{V}$ is P_μ-H-supplemented. $\square$

Proposition 3.15. *Let P be a module such that each cyclic submodule of P has μ-supplement that is relatively projective d.s of P, then P is P_μ-H-supplemented.*

Proof. Assume pR is a cyclic submodule of $P, p \in P$. By the assumption, there is a decomposition $P = L \oplus L'$ such that $P = pR + L$ and $pR \cap L \ll_\mu L$, and L, L are relatively projective. As L is L'-projective, $P = V \oplus L, V \leqslant pR$, by ([16], Lemma 4.47). Thus, P is P_μ-H-supplemented. $\square$

4. DECOMPOSITIONS

Direct summands of P_μ-H-supplemented may not be again P_μ-H-supplemented. Under certain conditions, P_μ-H-supplemented is inherited by d.s.

Proposition 4.1. *Let P be a P_μ-H-supplemented. If P is distributive, then each d.s of P is P_μ-H-supplemented.*

Proof. Straightforward. $\square$

Module P is considered to have the summand sum property (SSP), when the sum of each of the d.s of P is a d.s of P, P is said to have the (D_3) if P_1 and P_2 are d.s of P with $P = P_1 + P_2$, then $P_1 \cap P_2$ is d.s of P, [16] and [3].

Proposition 4.2. *Let P be a P_μ-H-supplemented. If P has the (SSP), then each d.s of P is P_μ-H-supplemented.*

Proof. Assume that $P = L \oplus L'$. It is sufficient to prove that $\frac{P}{L'}$ is P_μ-H-supplemented. Assume K is a d.s of P. As P has the (SSP), $L' + K$ is a d.s of P, let $P = (K + L') \oplus Y, Y \leqslant P$, then $\frac{P}{L'} = \frac{\mathrm{K}+L'}{L'} \oplus \frac{\mathrm{Y}+L'}{L'}$. Hence, $L \cong \frac{P}{L'}$ is P_μ-H-supplemented, by Proposition 3.11. ☐

Proposition 4.3. *Let P be a P_μ-H-supplemented with (D_3), then each d.s of P is P_μ-H-supplemented.*

Proof. Let $P = P_1 \oplus P_2$ and pR be a cyclic submodule of P_1. As P is P_μ-H-supplemented, there is a submodule X of P and $e \in \mathcal{J}(End(P))$ such that $e(P) \leqslant X$ and $\frac{X}{pR} \ll_\mu \frac{P}{pR}$ and $\frac{X}{e(P)} \ll_\mu \frac{P}{e(P)}$. Since P has the D_3-property, then $e(p) \cap P_1$ is d.s of P_1 and $\frac{X}{e(P) \cap P_1} \ll_\mu \frac{P}{e(P) \cap P_1}$. Thus, P_1 is P_μ-H-supplemented. ☐

Proposition 4.4. *Let P be a P_μ-H-supplemented, let L be a d.s of P such that, for every decomposition $P = V \oplus N$ of P, there exist submodules V' of V and N' of N such that $P = L \oplus V' \oplus N'$, then $\frac{P}{L}$ is P_μ-H-supplemented.*

Proof. Let $\frac{pR}{L} \leqslant \frac{P}{L}$, there exists a decomposition $P = V \oplus N$ such that $pR\mu V$. Hence, $\frac{pR+V}{V} \ll_\mu \frac{P}{V}$ and $\frac{pR+V}{pR} \ll_\mu \frac{P}{pR}$. Then, $P = L \oplus V' \oplus N', V' \leqslant V$ and $N' \leqslant N$, by hypothesis, it is obvious that $\frac{pR+L+V'}{pR} \ll_\mu \frac{P}{PR}$. On the other hand $\frac{P}{V} \cong N$ and $\frac{P}{L\oplus\ V'} \cong N'$ and $\frac{pR+L+V'}{L+V'} \ll_\mu \frac{P}{L+V'}$, by Lemma 2.4. Therefore, $\frac{pR}{L}\mu\frac{L\oplus V'}{L}$; hence $\frac{P}{L}$ is P_μ-H-supplemented. ☐

Theorem 4.5. *Let $P = P_1 \oplus P_2$ be a distributive (or duo) module.Then, P is P_μ-H-supplemented if and only if P_1 and P_2 are P_μ-H-supplemented.*

Proof. In two cases, P is distributive or duo, for each submodule V of $P, V = (V \cap P_1) \oplus (V \cap P_2)$. Let $p \in P$, then $p = p_1 + p_2, p_1 \in P_1$ and $p_2 \in P_2$. Then $pR = (pR \cap P_1) \oplus (pR \cap P_2)$. Hence, $p_1R = pR \cap P_1$ and $p_2R = pR \cap P_2$ are cyclic submodules of P_1 and P_2, respectively. So, there exists d.s L_1 of P_1 and L_2 of P_2 such that $p_1R\mu L_1$ and $p_2R\mu L_2$. Then, $pR\mu\,(L_1 \oplus L_2)$, according to Lemma 2.7, and $L_1 \oplus L_2$ is a d.s of P. Thus, P is P_μ-H-supplemented. The converse is obvious. ☐

A module P is said to be a weak duo, if every d.s of P is fully invariant [17].

Corollary 4.6. *Let P be a P_μ-H-supplemented. If P is a weak duo, then each d.s of P is P_μ-H-supplemented.*

Proposition 4.7. *Let $P = P_1 \oplus P_2$ be a direct sum of P_μ-H-supplemented. If* $\operatorname{ann} P_1 + \operatorname{ann} P_2 = R$*,then P is P_μ-H-supplemented.*

Proof. Let pR be a cyclic submodule of P, $p \in P$, then $p = p_1 + p_2, p_1 \in P_1$ and $p_2 \in P_2$; Hence, $pR = (pR \cap P_1) \oplus (pR \cap P_2)$ implies that $p_1R = pR \cap P_1$ and $p_2R = pR \cap P_2$ are cyclic submodules of P_1 and P_2, respectively. So, there exists d.s L_1 of P_1 and L_2 of P_2 such that $p_1R\mu L_1$ and $p_2R\mu L_2$, then $pR\mu\,(L_1 \oplus L_2)$, by Lemma 2.7. Thus, P is P_μ-H-supplemented. ☐

We will end this section with the next result.

Theorem 4.8. *Assume pR is a projection invariant cyclic submodule of a module P.*

- (i) *If P is P_μ-H-supplemented, then there is a decomposition $P = P_1 \oplus P_2, P_2 \leqslant pR$ and $\frac{pR}{P_2} \ll_\mu \frac{P}{P_2}$.*
- (ii) *If P is P_μ-H-supplemented, pR has a unique μ-coclosure and every d.s of pR has μ-coclosure in P, then $P = P_1 \oplus P_2$ such that $P_2 \leqslant pR$ and $\frac{pR}{P_2} \ll_\mu \frac{P}{P_2}$, and each of P_1 and P_2 are P_μ-H-supplemented.*

Proof. (i) Since P is P_μ-H-supplemented and $pR \leqslant P$, there is a d.s L of P such that $\frac{pR+L}{L} \ll_\mu \frac{P}{L}$ and $\frac{pR+L}{pR} \ll_\mu \frac{P}{pR}$. Let $\pi : P \to L$ be the projection map, let $\pi(P) = L = P_2$ and $(1-\pi)(P) = P_1$. As pR is projection invariant, then $\pi(pR) \leqslant pR$; hence $pR = \pi(pR) \oplus (1-\pi)(pR), \pi(pR) = pR \cap \pi(P)$ and $(1-\pi)(pR) = pR \cap (1-\pi)(P)$. Note that $P = \pi(P) \oplus (1-\pi)(P)$ implies that $P = pR + (1-\pi)(P)$. So, $\pi(P) = \pi(pR) = L \leqslant pR$ and $\frac{\pi(P)+pR}{\pi(P)} = \frac{pR}{\pi(P)} \ll_\mu \frac{P}{\pi(P)}$.

(ii) From (i), we get $P = P_1 \oplus P_2$, where $\frac{pR}{P_2} \ll_\mu \frac{P}{P_2}$ and $P_2 \leqslant pR$ and $e(P) = P_2$ and $(1-e)(P) = P_1, e \in \mathcal{J}(P)$. Let $P = X \oplus Y$, then $pR = (pR \cap X) \oplus (pR \cap Y)$. By hypothesis, $pR \cap X$ has μ-coclosure N in P and $pR \cap Y$ has μ-coclosure say N' in P; hence, $N \oplus N'$ is a μ-coclosure of pR in P. However, pR has a unique μ-coclosure; therefore, $P_2 = N \oplus N'$. Now, $X \cap (N \oplus N') = \mathrm{N} \oplus (X \cap N')$ and $Y \cap (N \oplus N') = N' \oplus (N \cap Y)$. Hence, $P_2 = N \oplus N' = (X \cap P_2) \oplus (Y \cap P_2)$. Since P_1 and P_2 are d.s of P, then P_1 and P_2 are P_μ-H-supplemented, by the construction above. □

5. PRINCIPALLY μ-SUPPLEMENTED MODULES

In this part, a new class of modules is introduced, namely, P_μ-supplemented, with some of its properties.

Definition 5.1. A module P is referred to as a principally μ-supplemented if for each cyclic submodule L of P, there is a submodule B of P such that $P = L + B$ and $L \cap B \underset{\mu}{\ll} B$, for short we will denote P_μ-supplemented.

Proposition 5.2. *Let P be a module and consider the following conditions.*

- (i) *P is a principally lifting.*
- (ii) *P is a principally supplemented.*
- (iii) *P is P_μ-lifting.*
- (iv) *P is P_μ-supplemented. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv).*

Proof. $(i) \Rightarrow (ii)$ See [1].
$(ii) \Rightarrow (iv)$ It follows from the fact that each small submodule is μ-small in P.
$(i) \Rightarrow (iii)$ see [10].
$(iii) \Rightarrow (iv)$ and $(ii) \Rightarrow (iv)$ Straightforward.
$(ii) \not\Rightarrow (i)$ Let $P = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module. Then P is principally supplemented

which is not principally lifting. Also, this is an example to show that $(iv) \nRightarrow (iii)$
$(iv) \nRightarrow (ii)$ Let $P = \prod_{i=1}^{\infty} P_i$, where $P_i = \mathbb{Z}_2$; and R is the subring of P which is generated by 1_P and $\oplus_{i=1}^{\infty} P_i$. As R is V-ring, R_R is μ-hollow; hence, it is P_μ-lifting which is not supplemented. Also, this is an example to show that $(iii) \nRightarrow (i)$ □

Example 5.3. Let $P = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module. It is easy to verify that P is principally supplemented and hence it is P_μ-supplemented. On the other hand, P is not P_μ-lifting, [10].

The following result provides a condition that makes the P_μ-supplemented and P_μ-lifting conditions equivalent.

Proposition 5.4. *Let P be a P_μ-supplemented. If P has the (SSP), then P is P_μ-lifting.*

Proof. Let $p \in P$, there is $X \leqslant P$ such that $P = pR + X$ and $pR \cap X \ll_\mu X$. There exists a d.s X_1 of P such that $X_1 \leqslant X$ and $P = pR + X_1 = X_1' \oplus X_1$. Again, the assumption, there exists a d.s X_2 of P such that $X_2 \leqslant pR$ and $P = X_2 + X_1 = X_2 \oplus X_2'$. Since P satisfies the (SSP), $X_2 \cap X_1$ is a d.s of P, let $P = (X_2 \cap X_1) \oplus K, K \leqslant P$. Then $X_1 = (X_2 \cap X_1) \oplus (K \cap X_1)$ and $P = X_2 \oplus (K \cap X_1)$. Observe that $pR \cap (K \cap X_1) \ll_\mu K \cap X_1$, as $pR \cap (K \cap X_1) \leqslant pR \cap X_1 \leqslant X_1$ and $pR \cap X_1 \ll_\mu X_1$; hence, $pR \cap X_1 \ll_\mu K \cap X_1$, because $K \cap X_1$ is a d.s of P. Thus, P is P_μ-lifting. □

Proposition 5.5. *Let P be a P_μ-H-supplemented. If P is cosingular, then P is P_μ-supplemented.*

Proof. The proof is immediate. □

Recall that a module P is called regular if for any $p \in P$, there is $\theta \in \mathrm{Hom}_R(P, R)$ with $p = p\theta(p)$, [22].

Proposition 5.6. *Every regular module is P_μ-supplemented.*

Proof. As a consequence of that each cyclic submodule of a regular module is a d.s. □

The converse implication of Proposition 5.6 is false, in general.

Example 5.7.

- (i) We know that $\mathbb{Q}$ as $\mathbb{Z}$-module is P_μ-supplemented, while it is not regular, because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.
- (ii) A submodule of P_μ-supplemented may not be again P_μ-supplemented, for example $\mathbb{Q}$ as $\mathbb{Z}$-module is P_μ-lifting; hence it is P_μ-supplemented, while the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not P_μ-supplemented since $2\mathbb{Z}$ has no P_μ-supplement in $\mathbb{Z}$.

From [13], the finite sum of μ-supplemented modules is again μ-supplemented. However, in case of P_μ-supplemented this statement is false. In several classes of modules it is true.

Theorem 5.8. *Let $P = P_1 \oplus P_2$ is a direct sum of P_μ-supplemented modules. If P is duo module, then P is P_μ-supplemented.*

Proof. Let $P = P_1 \oplus P_2$ be a duo module and let pR be a submodule of P, then $pR = (pR \cap P_1) \oplus (pR \cap P_2)$. Let $p = p_1 + p_2, p_1 \in P_1$ and $p_2 \in P_2$. Then $p_1R = pR \cap P_1$ and $p_2R = pR \cap P_2$, respectively, there exist $X_1 \leqslant P_1$ and $X_2 \leqslant P_2$ such that $P_1 = p_1R + X_1, p_1R \cap X_1 \ll_\mu X_1, P_2 = p_2R + X_2$ and $p_2R \cap X_2 \ll_\mu X_2$. Then, $P = p_1R + p_2R + X_1 + X_2 = PR + X_1 + X_2$. We prove $(pR) \cap (X_1 + X_2) \ll_\mu X_1 + X_2$.

$$\begin{aligned}(pR) \cap (X_1 + X_2) &= ((pR) \cap P_1 + (pR) \cap P_2) \cap (X_1 + X_2)\\ &\leqslant (X_1 \cap (((pR) \cap P_1) + P_2)) + (X_2 \cap (((pR) \cap P_2) + P_1))\\ &\leqslant ((pR) \cap P_1) \cap (X_1 + P_2) + ((pR) \cap P_2) \cap (X_2 + P_1)\end{aligned}$$

On the other hand $((pR) \cap P_1) \cap (X_1 + P_2) = (p_1R) \cap (X_1 + P_2) \leq X_1 \cap ((p_1R) + P_2) \leq (p_1R) \cap (X_1 + P_2)$ implies $(p_1R) \cap (X_1 + P_2) = X_1 \cap ((p_1R) + P_2) = (p_1R) \cap X_1$. Similarly $(p_2R) \cap (X_2 + P_1) = X_2 \cap ((p_2R) + P_1) = (p_2R) \cap X_2$. Since $p_1R \cap X_1 \underset{\mu}{\ll} X_1$ and $p_2R \cap X_2 \underset{\mu}{\ll} X_2$, then $p_1R \cap X_1 + p_2R \cap X_2 \underset{\mu}{\ll} X_1 + X_2$. Again by Lemma 2.1, $pR \cap (X_1 + X_2) \underset{\mu}{\ll} X_1 + X_2$. □

Theorem 5.9. *Let P be a P_μ-supplemented. If P is duo module. Then every d.s of P is P_μ-supplemented.*

Proof. Let $P = P_1 \oplus P_2$ and $p \in P_1$. There exist $X \leqslant P$ such that $P = pR + X$ and $pR \cap X \underset{\mu}{\ll} X$. Then $P_1 = pR + (P_1 \cap X)$. Since P is duo module, then $X = (X \cap P_1) \oplus (X \cap P_2)$. Now, let U be a submodule of $X \cap P_1$ with $\frac{X \cap P_1}{U}$ is cosingular, then $X = pR \cap (X \cap P_1) + U + (X \cap P_2) = (pR \cap X) + U + (X \cap P_2)$. Since $pR \cap X \underset{\mu}{\ll} X$ and $\frac{X}{U + (X \cap P_2)}$ is cosingular, then $X = U \oplus (X \cap P_2)$. It follows that $U = X \cap P_1$, that is what we have to do. □

Theorem 5.10. *Let P be a P_μ-supplemented. If P is distributive, then every d.s of P is P_μ-supplemented.*

Proof. Let $P = P_1 \oplus P_2$ and $p \in P_1$, there exists $X \leqslant P$ such that $P = pR + X$ and $pR \cap \mathrm{X} \underset{\mu}{\ll} X$. Then $P_1 = pR + (P_1 \cap X)$. It is simple to clarify that $pR \cap \mathrm{X} \underset{\mu}{\ll} P_1 \cap X$. □

ACKNOWLEDGMENT

The authors express their sincere thanks to the anonymous reviewer for conducting a thorough evaluation of this manuscript. The presentation and scientific quality of the study have been significantly bolstered by the careful comments, critical observations, and expert suggestions. We are very grateful for the reviewer's thorough evaluation and commitment.

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