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FURTHER PROPERTIES OF HERMITIAN ADJACENCY MATRIX OF MIXED GRAPHS

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Abstract. The Hermitian adjacency matrix of a mixed graph extends the classical adjacency matrix to graphs containing both edges and arcs. In this paper, we establish a unified generalised interpretation for the powers of the Hermitian adjacency matrix, where arcs may be traversed in either direction and the orientation is encoded through complex conjugate weights. We show that each entry of the kth power of the Hermitian adjacency matrix corresponds to a weighted sum of all generalised walks of length k between the corresponding vertices. Using this correspondence, we derive explicit combinatorial expansions for the determinant and for the coefficients of the characteristic polynomial in terms of spanning sub-mixed graphs. Applications to path and cycle digraphs are presented, and several spectral consequences are discussed.

Keywords. characteristic polynomial, determinant, Hermitian adjacency matrix, mixed graphs, generalised walk.
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1. Introduction and preliminaries

1.1. **Motivation.** The Hermitian adjacency matrix of a mixed graph was introduced by Liu and Li [14] as a spectral extension of the adjacency matrix to graphs containing both undirected edges and directed arcs. Its eigenvalue properties were studied by Guo and Mohar [12], and further refinements were given by Mohar [15] and Abudayah et al. [2], preserving Hermitian symmetry. In this paper, we adopt the formulation of [2], which reduces to the classical Hermitian adjacency matrix when $\alpha = i$ and to the adjacency matrix when $\alpha = 1$.

For undirected graphs and digraphs, it is well known that the entries of powers of the adjacency matrix count walks of fixed length [9, 16, 18]. Motivated by this classical correspondence, we investigate analogous interpretations for the powers of the Hermitian adjacency matrix of mixed graphs. In this setting, arcs may be traversed in either direction; when a walk follows an arc opposite to its

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orientation, its contribution is recorded through the complex conjugate weight determined by the Hermitian symmetry.

We show that each entry of the kth power of the Hermitian adjacency matrix corresponds to the weighted sum of all walks of length k between the corresponding vertices. This interpretation yields explicit combinatorial expressions for the determinant and for the coefficients of the characteristic polynomial in terms of spanning sub-mixed graphs. We further derive consequences for the spectrum of the Hermitian adjacency matrix, including bounds of the spectral radius, sufficient conditions for 0 to be one of the eigenvalue, and the Hermitian energy of a mixed graph. Consequently, several classical results for adjacency matrices of undirected and directed graphs are recovered within the Hermitian setting.

1.2. **Preliminaries.** We adopt standard terminology for undirected graphs and digraphs from [4, 6], and for directed paths and cycles from [11]. All graphs considered are finite and do not allow parallel edges or arcs.

In this paper, a digraph $\overrightarrow{\Delta}$ is an oriented digraph without parallel arcs, although directed loops are allowed. A mixed graph M is obtained by replacing each directed digon of a digraph with a single undirected edge while retaining all other arcs and directed loops; undirected loops are not allowed. An undirected graph Γ is regarded as a special case of a mixed graph that does not contain directed arcs. Throughout this paper, a digraph is understood to be oriented, so a directed digon is not allowed. On the other hand, an unoriented digraph may be viewed as a mixed graph in which each directed digon is replaced by an edge. Hence, results for mixed graphs can also be specialised to digraphs and undirected graphs. The following figure illustrates the transformation of an unoriented digraph into a mixed graph by replacing directed digons with undirected edges.

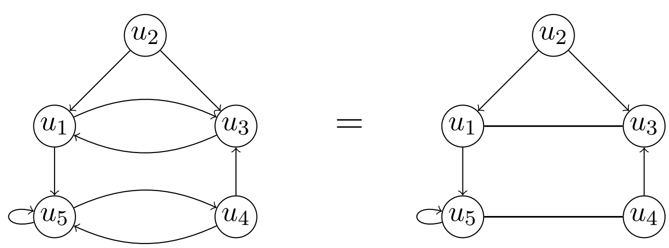

FIGURE 1. An unoriented digraph and the corresponding mixed graph.

For a mixed graph M and a vertex $\xi \in V_M$, the out- and in-neighbourhoods are defined by $N_M^+(\xi) = \{\zeta : (\xi, \zeta) \in A_M \text{ or } \{\xi, \zeta\} \in A_M\}$ and $N_M^-(\xi) = \{\zeta : (\zeta, \xi) \in A_M \text{ or } \{\zeta, \xi\} \in A_M\}$, so that each undirected edge incident to ξ contributes to both neighbourhoods. The corresponding out-degree and in-degree are $od_M(\xi) = |N_M^+(\xi)|$ and $id_M(\xi) = |N_M^-(\xi)|$. For a digraph $\overrightarrow{\Delta}$ and an undirected graph Γ, the notation $N_{\overrightarrow{\Delta}}^+(\xi)$, $N_{\overrightarrow{\Delta}}^-(\xi)$, $od_{\overrightarrow{\Delta}}(\xi)$, $id_{\overrightarrow{\Delta}}(\xi)$ and $N_\Gamma(\xi)$, $deg_\Gamma(\xi)$ is used in the usual sense; see [7, 8].

A digraph $\overrightarrow{\Lambda} = (V_{\overrightarrow{\Lambda}}, A_{\overrightarrow{\Lambda}})$ is an induced subdigraph of $\overrightarrow{\Delta}$ if $V_{\overrightarrow{\Lambda}} \subseteq V_{\overrightarrow{\Delta}}$ and $A_{\overrightarrow{\Lambda}} = \{(u,v) \in A_{\overrightarrow{\Delta}} : u, v \in V_{\overrightarrow{\Lambda}}\}$. Similarly, an For an undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$, a subgraph Ω is induced by $V_\Omega \subseteq V_\Gamma$ if $E_\Omega = u, v \in E_\Gamma : u, v \in V_\Omega$.

An induced sub-mixed graph $\Lambda = (V_\Lambda, A_\Lambda)$ of M satisfies $V_\Lambda \subseteq V_M$ and $A_\Lambda = \{\{u,v\} \in A_M,\ (w,x) \in A_M : u, v, w, x \in V_\Lambda\}$. For the definitions of induced subdigraphs and induced subgraphs, see [7].

Directed walks in digraphs and undirected walks in undirected graphs are well known, see [7]. In this paper, we introduce a unified notion of a walk in a mixed graph that extends these classical definitions and applies simultaneously to digraphs, undirected graphs, and mixed graphs. Furthermore, since in directed and undirected walks the walks cannot contain arcs that have different directions or contain both directed and undirected arcs, the walk that we unify in this paper can also contain a reverse direction of arcs or edges.

Let H be a graph (digraph, mixed graph, or undirected graph) with vertex set $V_H = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$. A generalised walk of length k from υ_i to υ_j in H is a sequence $(\upsilon_i = \vartheta_{r_0}, \vartheta_{r_1}, \ldots, \vartheta_{r_k} = \upsilon_j)$ such that for each $0 \le p \le k-1$,

$$(\vartheta_{r_p}, \vartheta_{r_{p+1}}) \in A_H, \quad (\vartheta_{r_{p+1}}, \vartheta_{r_p}) \in A_H, \quad \text{or} \quad \{\vartheta_{r_p}, \vartheta_{r_{p+1}}\} \in A_H.$$

Thus, a generalised walk may traverse forward arcs, reverse arcs, or undirected edges. A generalised walk of length k from υ_i to υ_j is denoted by $W_{ij}^{(k)}$. Moreover, for $1 \le i, j \le n$, the collection of all generalised walks of length k from υ_i to υ_j is denoted by $\mathcal{W}_{ij}^{(k)}$. A generalised path is an open generalised walk in which each vertex appears at most once, and a generalised cycle is a closed generalised path. The mixed graph M is cyclic if it contains a generalised cycle; otherwise, it is acyclic.

For a mixed graph $M = (V_M, A_M)$, the underlying graph is the undirected graph obtained by replacing each arc $(\upsilon_i, \upsilon_j) \in A_M$ with the edge $\{\upsilon_i, \upsilon_j\}$ and retaining all undirected edges of M. The terminologies and notion of the underlying graph of a digraph are well known; see [7].

Let $M = (V_M, A_M)$ be a mixed graph. A mixed graph $\Lambda = (V_\Lambda, A_\Lambda)$ is called a spanning sub-mixed graph of M if it is obtained by removing some arcs and/or undirected edges from M while keeping $V_\Lambda = V_M$. We denote by S_{M_C} a spanning sub-mixed graph whose underlying graph is isomorphic to an undirected cycle graph. For S_{M_C}, the numbers of arcs of the form $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{p+1}}, \vartheta_{r_p})$ are denoted by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{S_{M_C}}$ and $a(\vartheta_{r_{p+1}}, \vartheta_{r_p})_{S_{M_C}}$, respectively.

For a digraph $\overrightarrow{\Delta}$, define a relation $\to$ on $V_{\overrightarrow{\Delta}}$ by $x \to y$ if $x = y$ or there exist directed paths from x to y and from y to x. The equivalence classes of $\to$ induce subdigraphs of $\overrightarrow{\Delta}$. For an undirected graph Γ, define a relation $\leftrightarrow$ on V_Γ by $x \leftrightarrow y$ if $x = y$ or there exists an undirected path joining x and y; the equivalence classes induce subgraphs of Γ. For a mixed graph M, define a relation $\sim$ on V_M by $x \sim y$ if $x = y$ or there exist generalised paths from x to y and from y to x; the equivalence classes induce sub-mixed graphs of M.

In this paper, an elementary mixed graph is a mixed graph M whose equivalence classes are undirected cycles, generalised cycles, undirected paths, arcs, or directed loops. An elementary undirected graph is an undirected graph whose classes are undirected cycles, undirected paths, or undirected loops. An elementary digraph is a digraph $\overrightarrow{\Delta}$ whose classes are generalised cycles, directed loops, or arcs.

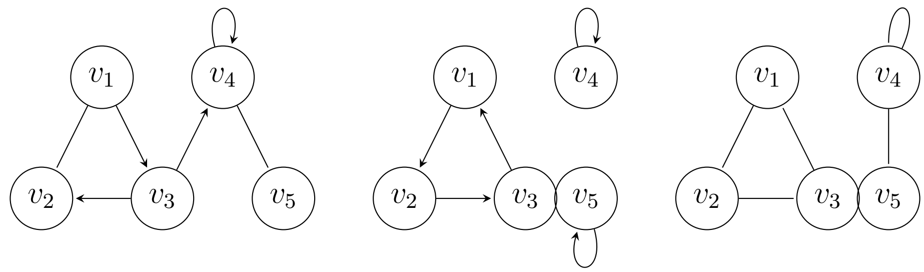

FIGURE 2. An elementary mixed graph, an elementary digraph, and an elementary undirected graph.

Let M be a mixed graph with vertex set $V_M = \{\upsilon_i : 1 \leq i \leq n\}$, and let S^{e}_M be a spanning elementary sub-mixed graph of M whose classes are $C_1, \ldots, C_\kappa$. For each $1 \leq i \leq \kappa$, denote by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_i)_{S^{\mathrm{e}}_M}}$ and $a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_i)_{S^{\mathrm{e}}_M}}$ the numbers of arcs of the forms $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{q+1}}, \vartheta_{r_q})$ in the class C_i, respectively. Define

$$\delta(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{S^{\mathrm{e}}_M} = \sum_{i=1}^{\kappa} a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_i)_{S^{\mathrm{e}}_M}} \quad \text{and} \quad \delta(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{S^{\mathrm{e}}_M} = \sum_{i=1}^{\kappa} a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_i)_{S^{\mathrm{e}}_M}}.$$

Let $\overrightarrow{\Delta}$ be a digraph with vertex set $V_{\overrightarrow{\Delta}} = \{\upsilon_i : 1 \leq i \leq n\}$, and let $S^{\mathrm{e}}_{\overrightarrow{\Delta}}$ be a spanning elementary subdigraph of $\overrightarrow{\Delta}$ whose classes are $C_1, \ldots, C_\kappa$. For each $1 \leq i \leq \kappa$, denote by $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_i)_{S^{\mathrm{e}}_{\overrightarrow{\Delta}}}}$ and $a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_i)_{S^{\mathrm{e}}_{\overrightarrow{\Delta}}}}$ the numbers of arcs of the forms $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{q+1}}, \vartheta_{r_q})$ in the class C_i, respectively. Define

$$\delta(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{S^{\mathrm{e}}_{\overrightarrow{\Delta}}} = \sum_{i=1}^{\kappa} a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_i)_{S^{\mathrm{e}}_{\overrightarrow{\Delta}}}} \quad \text{and} \quad \delta(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{S^{\mathrm{e}}_{\overrightarrow{\Delta}}} = \sum_{i=1}^{\kappa} a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_i)_{S^{\mathrm{e}}_{\overrightarrow{\Delta}}}}.$$

Let Γ be an undirected graph with vertex set $V_\Gamma = \{\upsilon_i : 1 \leq i \leq n\}$, and let S^{e}_Γ be a spanning elementary subgraph of Γ whose classes are $C_1, \ldots, C_\kappa$. Since Γ has no arcs, for each $1 \leq i \leq \kappa$ the quantities $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_i)_{S^{\mathrm{e}}_\Gamma}}$ and $a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_i)_{S^{\mathrm{e}}_\Gamma}}$ are equal to 0. Consequently,

$$\delta(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{S^{\mathrm{e}}_\Gamma} = \sum_{i=1}^{\kappa} a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_i)_{S^{\mathrm{e}}_\Gamma}} = 0, \quad \delta(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{S^{\mathrm{e}}_\Gamma} = \sum_{i=1}^{\kappa} a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_i)_{S^{\mathrm{e}}_\Gamma}} = 0.$$

2. Adjacency and Hermitian adjacency matrices

The spectral radius of a matrix A is $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A. We use the standard adjacency matrix notation for digraphs, undirected graphs, and mixed graphs; see [3].

Let M be a mixed graph of order η with arc set A_M, and let $\alpha \in \mathbb{C}$ satisfy $|\alpha| = 1$. The Hermitian adjacency matrix of M is $H(M) = (h_{ij}) \in \mathbb{C}^{\eta\times\eta}$ defined by

$$h_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in A_M, \\ \alpha, & \text{if } (v_i, v_j) \in A_M \text{ and } (v_j, v_i) \notin A_M, \\ \overline{\alpha}, & \text{if } (v_j, v_i) \in A_M \text{ and } (v_i, v_j) \notin A_M, \\ 0, & \text{otherwise,} \end{cases}$$

with $h_{kk} = 1$ whenever M contains a directed loop at v_k. When $\alpha = 1$, $H(M)$ coincides with the adjacency matrix of the underlying graph of M.

In particular, for a digraph $\overrightarrow{\Delta}$ and an undirected graph Γ, $H(\overrightarrow{\Delta})$ and $H(\Gamma)$ are obtained as special cases of the above definition, and $H(\Gamma) = A(\Gamma)$.

Motivated by [17, 18], we study the structure of the entries of $H(M)^k$ for $k \in \mathbb{N}$ and establish walk-based interpretations extending the classical adjacency-matrix power correspondences to the Hermitian setting.

2.1. **Powering Hermitian adjacency matrix.** Motivated by the classical walk interpretation of powering adjacency matrix of a digraph and an undirected graph, we establish analogous results for the Hermitian adjacency matrix of a mixed graph.

Let M be a mixed graph with Hermitian adjacency matrix $H(M) = (h_{ij})$. For a generalised walk $W_{ij}^{(k)} = (\upsilon_i = \vartheta_{r_0}, \vartheta_{r_1}, \ldots, \vartheta_{r_k} = \upsilon_j) \in \mathcal{W}_{ij}^{(k)}$, define its weight by

$$wt\Big(W_{ij}^{(k)}\Big) = h_{\vartheta_{r_0},\vartheta_{r_1}} h_{\vartheta_{r_1},\vartheta_{r_2}} \cdots h_{\vartheta_{r_{k-1}},\vartheta_{r_k}}.$$

For the walk ${W_{ij}}^{(k)} \in {\mathcal{W}_{ij}}^{(k)}$, there are arcs $\left(\vartheta_{r_m}, \vartheta_{r_{m+1}}\right) \in A_M$; arcs $\left(\vartheta_l, \vartheta_{r_{l-1}}\right) \in A_M$; or edges $\left\{\vartheta_{r_p}, \vartheta_{r_{p+1}}\right\} \in A_M$, where $0 \le m, p \le k-1$, $1 \le l \le k$, and $m \le l \le p$. Therefore, for $0 \le m \le k-1$, we have $h_{r_m, r_{m+1}} = 1$; $h_{r_m, r_{m+1}} = \alpha$; or $h_{r_m, r_{m+1}} = \overline{\alpha}$. If ${W_{ij}}^{(k)}$ is a directed path, then $wt(W_{ij})^k = \alpha^k$. If ${W_{ij}}^{(k)}$ is an almost-directed walk-$\upsilon_i - \upsilon_j$ with κ arcs of the form $\left(\vartheta_{r_l}, \vartheta_{r_{l-1}}\right)$, then $wt({W_{ij}}^{(k)}) = \alpha^{k-\kappa} \cdot \overline{\alpha}^{\kappa} = \alpha^{k-2\kappa}$. If ${W_{ij}}^{(k)}$ is a mixed walk-$\upsilon_i - \upsilon_j$ with κ edges of the form $\{\vartheta_{r_p}, \vartheta_{r_{p+1}}\}$, then $wt({W_{ij}}^{(k)}) = \alpha^{k-1}$. If ${W_{ij}}^{(k)}$ is an almost-mixed walk-$\upsilon_i - \upsilon_j$ with κ edges of the form $\{\vartheta_{r_p}, \vartheta_{r_{p+1}}\}$ and χ arcs of the form $\left(\vartheta_{r_{m+1}}, \vartheta_{r_m}\right)$, then $wt({W_{ij}}^{(k)}) = \alpha^{k-(\kappa+\chi)}\overline{\alpha}^{\kappa+\chi} = \alpha^{k-2(\kappa+\chi)}$. Hence, we obtain the following:

$$wt({W_{ij}}^{(k)}) = \alpha^{a\left(\vartheta_{r_m}, \vartheta_{r_{m+1}}\right)} \cdot \overline{\alpha}^{a\left(\vartheta_{r_l}, \vartheta_{r_{l-1}}\right)},$$

where $0 \le a\left(\vartheta_{r_m}, \upsilon_{r_{m+1}}\right), a\left(\upsilon_{r_l}, \upsilon_{r_{l-1}}\right) \le k$.

Now, for arbitrary $k \ge 2$, the entries of $H^k(M)$ are given by the following theorem..

Theorem 2.1. Let M be a mixed graph and let $H^k(M) = (h_{ij}^{(k)})$. Then for all $1 \le i, j \le n$,

$$h_{ij}^{(k)} = \sum_{W_{ij}^{(k)} \in \mathcal{W}_{ij}^{(k)}} wt\Big(W_{ij}^{(k)}\Big). \tag{2.1}$$

Proof. We proceed by mathematical induction on $k \geq 2$ to prove

$$h_{ij}^{(k)} = \sum_{W_{ij} \in \mathcal{W}_{ij}} wt\left(W_{ij}^{(k)}\right).$$

For $k = 2$, direct computation from the definition of matrix multiplication yields:

$$h_{ij}^{(2)} = \sum_{W_{ij}{}^{(2)} \in \mathcal{W}_{ij}{}^{(2)}} wt\left(W_{ij}{}^{(2)}\right).$$

Assume that for $k = \eta$, it is true that

$$h_{ij}^{(\eta)} = \sum_{W_{ij}{}^{(\eta)} \in \mathcal{W}_{ij}{}^{(\eta)}} wt\left(W_{ij}{}^{(\eta)}\right).$$

For the case $k = \eta + 1$, it will be proven that

$$h_{ij}^{(\eta+1)} = \sum_{W_{ij}{}^{(\eta+1)} \in \mathcal{W}_{ij}{}^{(\eta+1)}} wt\left(W_{ij}{}^{(\eta+1)}\right).$$

Recall that

$$h_{ij}^{(\eta+1)} = \sum_{\kappa=1}^{n} h_{i\kappa}^{(\eta)} h_{\kappa j} \quad \text{and} \quad wt\left(W_{ij}{}^{(\eta)}\right) = \alpha^{a\left(\vartheta_{r_m}, \vartheta_{r_{m+1}}\right)} \cdot \overline{\alpha}^{a\left(\vartheta_{r_l}, \vartheta_{r_{l-1}}\right)},$$

where $0 \leq a\left(\vartheta_{r_m}, v_{r_{m+1}}\right), a\left(v_{r_l}, v_{r_{l-1}}\right) \leq \eta$. To obtain $h_{i\kappa}^{(\eta)} h_{\kappa j}$, all arc configurations that involve the vertex $v_\kappa \in V_M$ and the vertex $v_j \in V_M$ must be taken into account. For $1 \leq \kappa, j \leq n$, the possible cases are as follows.

The first condition is $\{v_\kappa, v_j\} \in A_M$, which implies

$$h_{i\kappa}^{(\eta)} h_{\kappa j} = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} wt\left(W_{i\kappa}{}^{(\eta)}\right) \cdot 1 = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} \alpha^{a\left(\vartheta_{r_m}, \vartheta_{r_{m+1}}\right)_{W_{i\kappa}{}^{(\eta)}}} \cdot \overline{\alpha}^{a\left(\vartheta_{r_l}, \vartheta_{r_{l-1}}\right)_{W_{i\kappa}{}^{(\eta)}}},$$

where $0 \leq a\left(\vartheta_{r_m}, v_{r_{m+1}}\right), a\left(v_{r_l}, v_{r_{l-1}}\right) \leq \eta$. The subsequent condition is $(v_\kappa, v_j) \in A_M$; but $(v_j, v_\kappa) \notin A_M$, so that

$$h_{i\kappa}^{(\eta)} h_{\kappa j} = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} wt\left(W_{i\kappa}{}^{(\eta)}\right) \cdot \alpha = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} \alpha^{a\left(\vartheta_{r_m}, \vartheta_{r_{m+1}}\right)_{W_{i\kappa}{}^{(\eta)}}+1} \cdot \overline{\alpha}^{a\left(\vartheta_{r_l}, \vartheta_{r_{l-1}}\right)_{W_{i\kappa}{}^{(\eta)}}},$$

where $0 \leq a\left(\vartheta_{r_m}, v_{r_{m+1}}\right), a\left(v_{r_l}, v_{r_{l-1}}\right) \leq \eta$. The next condition is $(v_j, v_\kappa) \in A_M$; but $(v_\kappa, v_j) \notin A_M$, giving us

$$h_{i\kappa}^{(\eta)} h_{\kappa j} = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} wt\left(W_{i\kappa}{}^{(\eta)}\right) \cdot \overline{\alpha} = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} \alpha^{a\left(\vartheta_{r_m}, \vartheta_{r_{m+1}}\right)_{W_{i\kappa}{}^{(\eta)}}} \cdot \overline{\alpha}^{a\left(\vartheta_{r_l}, \vartheta_{r_{l-1}}\right)_{W_{i\kappa}{}^{(\eta)}}+1},$$

where $0 \leq a\left(\vartheta_{r_m}, v_{r_{m+1}}\right), a\left(v_{r_l}, v_{r_{l-1}}\right) \leq \eta$. The remaining condition implying $h_{i\kappa}^{(\eta)} h_{\kappa j} = \sum_{W_{i\kappa}{}^{(\eta)} \in \mathcal{W}_{i\kappa}{}^{(\eta)}} wt\left(W_{i\kappa}{}^{(\eta)}\right) \cdot 0 = 0$. From these cases, adding the term $h_{i\kappa}^{(\eta)} h_{\kappa j}$ for $1 \leq \kappa \leq n$ gives us:

$$h_{ij}^{(\eta+1)} = \sum_{\kappa=1}^{n} h_{i\kappa}^{(\eta)} h_{\kappa j} = \sum_{W_{ij}{}^{(\eta+1)} \in \mathcal{W}_{ij}{}^{(\eta+1)}} wt(W_{ij}{}^{(\eta+1)}).$$

Hence, we obtain Equation (2.1). □

For an undirected graph, the Hermitian adjacency matrix coincides with the usual adjacency matrix. Since undirected graphs are special cases of mixed graphs, Theorem 2.1 applies directly. In particular, $h_{ij}^{(2)}$ equals the number of undirected walks of length two from v_i to v_j, recovering the classical interpretation of matrix powers in terms of walks [9]. Moreover, Theorem 2.2 shows that the diagonal entries of $H^2(M)$ correspond to neighbourhood structures.

Theorem 2.2. Let M be a mixed graph and $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. For $1 \leq i, j \leq n$, if $H^2(M) = \left(h_{ij}^{(2)}\right)$, then

$$h_{ii}^{(2)} = |N_M^+(v_i) \bigcup N_M^-(v_i)|. \tag{2.2}$$

That is, $h_{ii}^{(2)}$ corresponds to the total number of vertices that are adjacent to or adjacent from $v_i \in V_M$.

Proof. Recall that $h_{ii}^{(2)} = \sum_{k=1}^{n} h_{ik}h_{ki}$. By the definition of $H(M)$, we have $h_{ik}h_{ki} = 1$ whenever $v_k \in N_M^+(v_i) \cup N_M^-(v_i)$, and $h_{ik}h_{ki} = 0$ otherwise. Hence $h_{ii}^{(2)}$ equals the number of vertices adjacent to or adjacent from v_i or in other words the Equation (2.2) is obtained. □

Let $H(M)$ be the Hermitian adjacency matrix of a mixed graph M of order n with characteristic polynomial

$$P(\lambda; H(M)) = \lambda^n + \sum_{k=1}^{n} (-1)^k \beta_k \lambda^{n-k}.$$

Then $\beta_1 = \operatorname{tr}(H(M))$, which equals the number of directed loops in M.

To determine β_2, we use the identity

$$\beta_2 = \frac{1}{2}\left((\operatorname{tr} H(M))^2 - \operatorname{tr}(H^2(M))\right).$$

By Theorem 2.2, each diagonal entry $h_{ii}^{(2)}$ equals the number of vertices adjacent to or adjacent from $v_i \in V_M$, so $\operatorname{tr}(H^2(M))$ is determined by the neighbourhood structure of M. This yields the following result.

Theorem 2.3. Let M be a mixed graph of order n with its Hermitian adjacency matrix $H(M)$. Suppose that $l(M)$ denotes the number of directed loops in M while $d(M)$ denotes the number of arcs and edges excluding the directed loops in M. If the characteristic polynomial of $H(M)$ is

$$P(\lambda; H(M)) = \lambda^n + \sum_{k=1}^{n} (-1)^k \beta_k \lambda^{n-k},$$

then the coefficient β_2 is obtained as follows:

$$\beta_2 = \frac{(l(M))^2 - l(M)}{2} - d(M). \tag{2.3}$$

Proof. Let h_{ii}^2 denote the main diagonal entry of the Hermitian adjacency matrix powered by 2, where $1 \leq i, j \leq n$. The characteristic polynomial of $H(M)$ can be expressed as:

$$P\left(\lambda; H\left(M\right)\right) = \lambda^n + \sum_{k=1}^{n} \left(-1\right)^k \beta_k \lambda^{n-k} = \prod_{k=1}^{n} \left(\lambda - \lambda_i\right),$$

where $\lambda_i \in \sigma\left(H(M)\right)$.

From Theorem 2.2, we acquire $h_{ii}^{(2)} = |N_M^+\left(v_i\right) \bigcup N_M^-\left(v_i\right)|$, which corresponds to the number of arcs incident to the vertex $v_i \in V_M$. Thus,

$$\sum_{i=1}^{n} h_{ii}^{(2)} = \sum_{v_i \in V_M} |N_M^+\left(v_i\right) \bigcup N_M^-\left(v_i\right)| = l(M) + 2\, d(M).$$

Using the identity $\left(\sum_{i=1}^{n} \lambda_i\right)^2 = \left(\sum_{i=1}^{n} \lambda_i^2\right) + 2\left(\sum_{1 \leq i,j \leq |V_M|} \lambda_i \lambda_j\right)$, we acquire $(tr(H(M)))^2 = tr\left(H\left(M\right)^2\right) + 2\beta_2$. Observing that $tr(H(M)) = l\left(M\right)$ and $tr\left(H\left(M\right)^2\right) = l(M) + 2\, d(M)$, solving for β_2, we obtain Equation (2.3). □

To characterise β_i for $1 \leq i \leq n$, we use the fact that β_i equals the sum of all $i \times i$ principal minors of $H(M)$. Each such minor is the determinant of the Hermitian adjacency matrix of an induced sub-mixed graph of order i. Thus, the computation of β_i reduces to evaluating determinant expansions governed by permutations of the entries of $H(M)$.

2.2. **Determinant of the Hermitian adjacency matrix.** Let $H(M) = (h_{ij})$ be the Hermitian adjacency matrix of a mixed graph M of order n. By the Leibniz formula, the determinant of $H(M)$ is given by

$$\det(H(M)) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} h_{i,\sigma(i)},$$

where $\operatorname{sgn}(\sigma) = 1$ if σ is an even permutation and $\operatorname{sgn}(\sigma) = -1$ if σ is odd. Consequently, the evaluation of $\det(H(M))$ reduces to analysing the contribution of the products $\prod_{i=1}^{n} h_{i,\sigma(i)}$ over all permutations $\sigma \in S_n$.

In this paper, we denote

$$\sigma_0 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & n \end{pmatrix}; \sigma_1 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix}.$$

Consider a generic term $\prod_{i=1}^{n} h_{i,\sigma(i)}$ in the Leibniz expansion of $\det(H(M))$. If every vertex of the mixed graph M contains a directed loop, then the diagonal contribution satisfies

$$h_{11} h_{22} \cdots h_{nn} = 1.$$

In contrast, if M contains at least one vertex without a directed loop, this product vanishes. To analyse the remaining non-diagonal contributions, it is necessary to associate each permutation $\sigma \in S_n$ with a corresponding spanning sub-mixed graph of M.

Let M be a mixed graph with vertex set V_M and arc set A_M, and let S denote a spanning sub-mixed graph induced by a permutation $\sigma \in S_n$. We define the value of S by

$$h(S) = \prod_{i=1}^{n} h_{i,\sigma(i)}.$$

In particular, spanning sub-mixed graphs whose underlying graphs are isomorphic to undirected cycles play a fundamental role in determining $\det(H(M))$. Lemma 2.4 characterises the precise conditions under which such spanning sub-mixed graphs exist and contribute nontrivially to the determinant.

Lemma 2.4. Let M be a mixed graph with a set of vertices $V_M = \{v_i : 1 \leq i \leq n\}$ and $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. If there exists a spanning sub-mixed graph S such that its underlying graph is isomorphic to an undirected cycle of length n, then

$$h_{12}h_{23}\cdots h_{(n-1)n}h_{n1} = \alpha^{\left(a\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_{M_C}}\right)} \cdot \overline{\alpha}^{\left(a\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_{M_C}}\right)} \tag{2.4}$$

and

$$h_{1n}h_{21}\cdots h_{n(n-1)} = \overline{\alpha}^{\left(a\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_{M_C}}\right)} \cdot \alpha^{\left(a\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_{M_C}}\right)}. \tag{2.5}$$

Proof. Assume that there exists a spanning sub-mixed graph S such that its underlying graph is isomorphic to an undirected cycle of length n. Then, it is a generalised cycle, or an undirected cycle of length n. Hence, $h_{12}h_{23}\cdots h_{(n-1)n}h_{n1} = wt\left(W_{ij}^{(k)}\right)$ and $h_{1n}h_{21}\cdots h_{n(n-1)} = wt\left(W_{ji}^{(k)}\right)$, which then we obtain Equations (2.4) and (2.5). ☐

Following the characterisation of spanning sub-mixed graphs whose underlying graphs are cycles, we analyse the occurrence of vertices with zero in-degree and out-degree in such subgraphs, and Lemma 2.5 specifies when these configurations force the corresponding determinant contributions to vanish.

Lemma 2.5. Let M be a mixed graph of order n and of size m, and let $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. Let S be a spanning sub-mixed graph of M. If there exists a vertex $v_k \in V_S$ such that $od_M(v_k) = id_M(v_k) = 0$, then for all $\sigma \in S_n$, we have the following:

$$\prod_{i=1}^{n} h_{i\sigma(i)} = 0. \tag{2.6}$$

Proof. Assume that there exists a vertex $v_k \in V_S$ such that $od_M(v_k) = id_M(v_k) = 0$. By definition, this implies that at least one corresponding entry in the Hermitian adjacency matrix of the mixed graph M satisfies $h_{k\sigma(k)} = 0$ where $1 \leq k, \sigma(k) \leq n$. Therefore, Equation (2.6) is acquired. ☐

These results culminate in an explicit combinatorial expansion of $\det(H(M))$ as a signed sum over spanning elementary sub-mixed graphs, thereby linking the determinant of the Hermitian adjacency matrix directly to the cyclic structure of M.

Lemma 2.6. Let M be a mixed graph with a set of vertices $V_M = \{\upsilon_i : 1 \leq i \leq n\}$ and let $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. Let S_M^e be a spanning elementary sub-mixed graph of M with κ classes. If there exists $\sigma \in S_n$ such that $\sigma \neq \sigma_0$, $\sigma \neq \sigma_1$, and $\sigma \neq \sigma_2$, then the following holds:

$$h(S_M^e) = \prod_{i=1}^{n} h_{i\sigma(i)} = \alpha^{\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_M^e}} \cdot \overline{\alpha}^{\delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_M^e}}. \tag{2.7}$$

Proof. Suppose that S_M^e is an elementary sub-mixed graph of M where $C_1, C_2, \cdots, C_\kappa$ are its classes. For every class C_ι where $1 \leq \iota \leq \kappa$, recall that the number of arcs of the form $(\vartheta_{r_p}, \vartheta_{r_{p+1}})$ and $(\vartheta_{r_{q+1}}, \vartheta_{r_q})$ are $a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_\iota)_{S_M^e}}$ and $a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_\iota)_{S_M^e}}$, respectively. Therefore, the entries $h_{r_p r_{p+1}} = \alpha$ and $h_{r_q r_{q+1}} = \overline{\alpha}$. Assume that the ι-th class has ι_η vertices, where $1 \leq \iota_\eta \leq n$. Consider the term $h_{\iota_1 \sigma(\iota_1)} h_{\iota_2 \sigma(\iota_2)} \cdots h_{\iota_\eta \sigma(\iota_\eta)}$, which corresponds to S_ι. Thus, we obtain

$$h_{\iota_1 \sigma(\iota_1)} h_{\iota_2 \sigma(\iota_2)} \cdots h_{\iota_\eta \sigma(\iota_\eta)} = \alpha^{a(\vartheta_{r_p}, \vartheta_{r_{p+1}})_{(C_\iota)_{S_M^e}}} \cdot \overline{\alpha}^{a(\vartheta_{r_{q+1}}, \vartheta_{r_q})_{(C_\iota)_{S_M^e}}}.$$

Consequently, there exists a permutation $\sigma \in S_n$ such that $\sigma \neq \sigma_0$, $\sigma \neq \sigma_1$, and $\sigma \neq \sigma_2$, which implies that $h(S_M^e) = \prod_{i=1}^{n} h_{i\sigma(i)}$ is obtained by multiplying all the terms corresponding to the classes of S_ι, where $1 \leq \iota \leq \kappa$. Therefore, we conclude that Equation (2.7) follows. $\square$

Having characterised elementary spanning sub-mixed graphs, we now determine their precise contribution to the determinant of the Hermitian adjacency matrix. Theorem 2.7 presents an explicit formula for $\det(H(M))$ expressed as a sum over spanning elementary subgraphs, with coefficients governed by the presence of vertex loops and the structure of their underlying cycles.

Theorem 2.7. Let M be a mixed graph of order n and of size m, where the set of vertices and arcs are denoted by V_M and A_M, respectively. Suppose that $H(M) = (h_{ij})$ represents the Hermitian adjacency matrix of M for $1 \leq i, j \leq n$. Let $c(S_M^e)$ be the number of classes in S_M^e, $N_{S_M^e}^l$ be the number of loops in S_M^e, and $\varpi(S_M^e)$ be the number of undirected cycles or directed cycles or mixed cycles of order ≥ 3 in S_M^e.

(a) If every vertex in M contains a directed loop, then

$$det(H(M)) = 1 + (-1)^{n-1} \cdot 2 \cdot Re\left(\alpha^{\left|a\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_{M_C}} - a\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_{M_C}}\right|}\right) +$$
$$\sum_{S_M^e} (-1)^{c(S_M^e) - N_{S_M^e}^l} \cdot 2^{\varpi(S_M^e)} \cdot Re\left(\alpha^{\left|\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_M^e} - \delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_M^e}\right|}\right). \tag{2.8}$$

(b) If M contains at least one vertex without a loop, then

$$det(H(M)) = (-1)^{n-1} \cdot 2 \cdot Re\left(\alpha^{\left|a\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_{M_C}} - a\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_{M_C}}\right|}\right) +$$

$$\sum_{S_M^e} (-1)^{c(S_M^e)-N_{S_M^e}^l} \cdot 2^{\varpi(S_M^e)} \cdot Re\left(\alpha^{|\delta\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_M^e} - \delta\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_M^e}|}\right). \tag{2.9}$$

The sum extends over all spanning elementary sub-mixed graphs S_M^e. Moreover,

$$(-1)^{n-1} \cdot 2 \cdot Re\left(\alpha^{|a\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_{M_C}} - a\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_{M_C}}|}\right) = 0,$$

if there is no spanning sub-mixed graph of M that its underlying graph is isomorphic to an undirected cycle graph. Meanwhile, if the spanning elementary sub-mixed graph S_M^e does not exist, then

$$\sum_{S_M^e} (-1)^{c(S_M^e)-N_{S_M^e}^l} \cdot 2^{\varpi(S_M^e)} \cdot Re\left(\alpha^{|\delta\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_M^e} - \delta\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_M^e}|}\right) = 0.$$

Proof. By the definition of the determinant of a matrix, we express the following.

$$\det(H(M)) = \Sigma_{\sigma\in S_n} sign(\sigma)\Pi_{i=1}^n h_{i\sigma(i)},$$

where $sign(\sigma) = 1$ if σ is an even permutation while $sign(\sigma) = -1$ if σ is an odd permutation.

Consider the term $h_{1\sigma(1)}h_{2\sigma(2)}\cdots h_{n\sigma(n)}$, where $\sigma \in S_n$. The objective is to determine spanning sub-mixed graphs of M corresponding to this term, as well as the sign of the associated permutation. For the term $h_{11}h_{22}\cdots h_{nn}$, the permutation involved is the identity, and hence $sign(\sigma) = 1$. If every vertex in M has a directed loop, then $h_{11}h_{22}\cdots h_{nn} = 1$, otherwise, if at least one vertex lacks a loop, then this product vanishes. Now, for the term $h_{12}h_{23}\cdots h_{(n-1)n}h_{n1}$ and its reverse $h_{1n}h_{21}\cdots h_{(n-1)(n-2)}h_{n(n-1)}$, the sign remains $sign(\sigma) = (-1)^n$, as they depend on n being even or odd. By Lemma 2.4, If M contains a spanning sub-mixed graph with its underlying graph is isomorphic to an undirected cycle, then

$$h_{12}h_{23}\cdots h_{(n-1)n}h_{n1} = \alpha^{\left(a\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_{M_C}}\right)} \cdot \overline{\alpha}^{\left(a\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_{M_C}}\right)}$$

and

$$h_{1n}h_{21}\cdots h_{(n-1)(n-2)}h_{n(n-1)} = \overline{\alpha}^{\left(a\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_{M_C}}\right)} \cdot \alpha^{\left(a\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_{M_C}}\right)}.$$

Summing the terms $h_{12}h_{23}\cdots h_{(n-1)n}h_{n1}$ and $h_{1n}h_{21}\cdots h_{(n-1)(n-2)}h_{n(n-1)}$ with their respective permutation signs gives

$$(-1)^{n-1} \cdot 2 \cdot Re\left(\alpha^{|a\left(\vartheta_{r_p},\vartheta_{r_{p+1}}\right)_{S_{M_C}} - a\left(\vartheta_{r_{q+1}},\vartheta_{r_q}\right)_{S_{M_C}}|}\right).$$

In contrast, whenever M does not contain a spanning sub-mixed graph with its underlying graph is isomorphic to an undirected cycle, we obtain that term is equal to zero.

By Lemma 2.5, the terms associated with any spanning sub-mixed graph containing a vertex $v_i \in V_S$ such that $id_M(v_i) = od_M(v_i) = 0$ vanish. Thus, such cases are ignored. The remaining spanning sub-mixed graphs could consist of classes such as undirected cycles, directed cycles, mixed cycles, paths, or directed loops. These classes imply that the remaining spanning graphs are all elementary. By Lemma 2.6, if there exists $\sigma \neq \sigma_0$, $\sigma \neq \sigma_1$, and $\sigma \neq \sigma_2$, we obtain the following:

$$h(S_M^e) = \prod_{i=1}^{n} h_{i\sigma(i)} = \alpha^{\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_M^e}} \cdot \overline{\alpha}^{\delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_M^e}}.$$

Recall that $sign(\sigma) = -1$ if the permutations in σ are odd and $sign(\sigma) = 1$ if the permutations in σ are even. Moreover, for the remaining term, the permutations in σ are odd or even depending on the number of classes in M with odd and even length. Observe that $c(S_M^e) = N_e + N_o$, where N_e is the number of classes that have even length while N_o is the number of classes that have odd length. Thus, we find that the permutations in σ are odd whenever $c(S_M^e) - N^l_{S_M^e}$ is an odd number. Hence, for the remaining terms, we have their respective permutation signs equal to

$$(-1)^{c(S_M^e) - N^l_{S_M^e}}.$$

If for each classes of S_M^e corresponds to exactly two distinct ways of forming a cycle in the permutation σ, summing the remaining terms with their respective permutation signs gives:

$$\sum_{S_M^e} (-1)^{c(S_M^e) - N^l_{S_M^e}} \cdot 2^{\varpi(S_M^e)} \cdot Re\left(\alpha^{\left|\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_M^e} - \delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_M^e}\right|}\right).$$

However, if the spanning elementary sub-mixed graph does not exist, then these remaining terms are equal to zero.

Depending on the nature of M, we distinguish two distinct cases, leading to the results given in Equations (2.8) and (2.9), which conclude the proof. $\square$

Theorem 2.7 applies directly to digraphs $\overrightarrow{\Delta}$. Recall that each permutation term $\prod_{i=1}^{n} h_{i\sigma(i)}$ corresponds to a spanning subdigraph of $\overrightarrow{\Delta}$. In this setting, the spanning elementary subdigraphs $S^e_{\overrightarrow{\Delta}}$ consist of directed loops, directed cycles, or single arcs. If a class $(C_k)_{S^e_{\overrightarrow{\Delta}}}$ is an arc (v_p, v_q), then $h_{pq}h_{qp} = \alpha\overline{\alpha} = 1$, which yields equal contributions from both orientations. Consequently, when all classes of $S^e_{\overrightarrow{\Delta}}$ are arcs, the associated exponent difference vanishes, and the weight satisfies

$$h\left(S^e_{\overrightarrow{\Delta}}\right) = \alpha^{\eta}\overline{\alpha}^{\eta} = 1,$$

for some $\eta \in \mathbb{N}$. In particular, path digraphs $\overrightarrow{P_n}$ and cycle digraphs $\overrightarrow{C_n}$ admit a unique spanning elementary subdigraph of this form, implying that their determinants are governed by a single nonzero contribution. This motivates a detailed investigation of $\det H(\overrightarrow{P_n})$ and $\det H(\overrightarrow{C_n})$, beginning with Theorem 2.8.

Theorem 2.8. Let $\overrightarrow{P_n}$ be a path digraph, and let $H\left(\overrightarrow{P_n}\right)$ be its Hermitian adjacency matrix. Then,

$$det\left(H\left(\overrightarrow{P_n}\right)\right) = \begin{cases} 0 & \text{if } n \text{ is an odd number,} \\ 1 & \text{if } n = 2k \text{ where } k \text{ is even,} \\ -1 & \text{if } n = 2k \text{ where } k \text{ is odd.} \end{cases}$$

Combining Theorem 2.8 with the classical results of [1], we obtain

$$\det\left(H\left(\overrightarrow{P_n}\right)\right) = \det(A(P_n))\,.$$

We next examine whether an analogous equality holds for cycle digraphs, namely whether

$$\det\left(H\left(\overrightarrow{C_n}\right)\right) = \det(A(C_n))\,,$$

which leads to Theorem 2.9.

Theorem 2.9. Let $\overrightarrow{C_n}$ be a cycle digraph, and let $H\left(\overrightarrow{C_n}\right)$ be its Hermitian adjacency matrix. Then,

$$det\left(H\left(\overrightarrow{C_n}\right)\right) = \begin{cases} 2\cdot Re\left(\alpha^n\right) & \text{if } n \,\text{is an odd number,} \\ -2\cdot Re\left(\alpha^n\right)+2 & \text{if } n=2k \,\text{where }\ k \,\text{is even,} \\ -2\cdot Re\left(\alpha^n\right)-2 & \text{if } n=2k \,\text{where }\ k \,\text{is odd.} \end{cases}$$

A path digraph $\overrightarrow{P_n}$ is defined as a digraph isomorphic to the digraph with vertex set $\{v_i : 1 \leq i \leq n\}$ and arc set $\{(v_i, v_{i+1}) : 1 \leq i \leq n-1\}$. Similarly, a cycle digraph $\overrightarrow{C_n}$ is a digraph isomorphic to the digraph with vertex set $\{v_i : 1 \leq i \leq n\}$ and arc set $\{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$.

Recall that the coefficients of the characteristic polynomial of an $n\times n$ matrix are obtained as sums of determinants of its principal submatrices. Since every principal submatrix of $H(\overrightarrow{\Delta})$ corresponds to the Hermitian adjacency matrix of an induced subdigraph of $\overrightarrow{\Delta}$, determining the characteristic polynomials of $H(\overrightarrow{P_n})$ and $H(\overrightarrow{C_n})$ requires a systematic analysis of their induced subdigraphs.

For $\overrightarrow{P_n}$, the induced subdigraphs of order ι $(1 \leq \iota \leq n)$ fall into the following classes:

(1) the null digraph N_ι, consisting of ι isolated vertices and no arcs;
(2) the path digraph $\overrightarrow{P_\iota}$ for $2 \leq \iota \leq n$; and
(3) digraphs obtained from $\overrightarrow{P_\iota}$ by deleting p arcs, where $1 \leq p \leq \iota - 2$, which may be further subdivided according to whether they contain isolated vertices.

This classification enables a complete enumeration of principal minors and forms the basis for determining the characteristic polynomial of $H(\overrightarrow{P_n})$ and, analogously, of $H(\overrightarrow{C_n})$.

We define an isolated path digraph $\overrightarrow{P_\eta^{\text{iso}}}$ as a digraph obtained by removing p arcs in a path digraph $\overrightarrow{P_\eta}$ such that there is an isolated vertex, where $1 \leq p \leq \eta$. It is trivial that $\det\left(H\left(\overrightarrow{P_\eta^i}\right)\right) = 0$ since $H\left(\overrightarrow{P_\eta^i}\right)$ has a zero row and a zero column. Clearly, $\overrightarrow{P_\eta^{\text{iso}}}$ (where $\eta \geq 3$ are some of the induced subdigraphs (of order ≥ 3) of $\overrightarrow{P_n}$ (where $n \geq 4$). This implies that we do not have to count the sum of the determinant of the induced subdigraphs of order ≥ 3 that are $\overrightarrow{P_\eta^{\text{iso}}}$.

We define a non-isolated k-path digraph $\overrightarrow{P_i^{n,k}}$ as an induced subdigraph of $\overrightarrow{P_n}$ of order i (where $4 \leq i \leq n$) that consists of $k \geq 2$ path digraphs $\overrightarrow{P_{i_\iota}}$ where $1 \leq \iota \leq k$ and $i_\iota \geq 2$. Clearly, $\sum_{\iota=1}^{k} i_\iota = i$. From Theorem 2.7, we derive the determinant of the Hermitian adjacency matrix of a non-isolated digraph, as stated in the following theorem.

Theorem 2.10. Let $\overrightarrow{P_i^{n,k}}$ be a non isolated-path digraph with $k \geq 2$ path digraphs $\overrightarrow{P_{i_\iota}}$ where $1 \leq \iota \leq k$, and let $H\left(\overrightarrow{P_i^{n,k}}\right)$ be its Hermitian adjacency matrix. Then

$$det\left(H\left(\overrightarrow{P_i^{n,k}}\right)\right) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even and there exists an odd number } i_\iota, \\ 1 & \text{if } i = 2l \text{ where } l \text{ is even and no odd number } i_\iota, \\ -1 & \text{if } i = 2l \text{ where } l \text{ is odd and no odd number } i_\iota. \end{cases}$$

From this classification, every induced subdigraph of order i $(1 \leq i \leq n)$ of $\overrightarrow{P_n}$ is isomorphic to $\overrightarrow{P_i}$, $\overrightarrow{P_i^{\text{iso}}}$, or $\overrightarrow{P_i^k}$. By Theorem 2.10, the determinant of the Hermitian adjacency matrix of a non-isolated path digraph is zero whenever its order is odd. Hence, any induced subdigraph of order i containing a connected class of odd order has a zero determinant. Consequently, when summing all $i \times i$ principal minors of $H(\overrightarrow{P_n})$, it suffices to consider induced subdigraphs isomorphic to $\overrightarrow{P_i}$ or to $\overrightarrow{P_i^k}$ whose connected classes all have even order.

We denote by $m\left(\overrightarrow{P_i}\right)$ the number of induced subdigraphs of order i of $\overrightarrow{P_n}$, and of order $1 \leq i \leq n-1$ of $\overrightarrow{C_n}$, that are isomorphic to $\overrightarrow{P_i}$; similarly, $m\left(\overrightarrow{P_i^{n,k}}\right)$ denotes the number of induced subdigraphs of order i of $\overrightarrow{P_n}$ and $\overrightarrow{C_n}$ that are isomorphic to $\overrightarrow{P_i^{n,k}}$, where i and i_ι are even.

For a cycle digraph $\overrightarrow{C_n}$, each induced subdigraph of order i with $1 \leq i \leq n-1$ is isomorphic to $\overrightarrow{P_i}$, $\overrightarrow{P_i^{\text{iso}}}$, or $\overrightarrow{P_i^k}$, while the only induced subdigraph of order n is $\overrightarrow{C_n}$. Thus, for $1 \leq i \leq n-1$, the sum of all $i \times i$ principal minors of $H(\overrightarrow{C_n})$ is determined solely by induced subdigraphs isomorphic to $\overrightarrow{P_i}$ or $\overrightarrow{P_i^k}$ with all class orders even, and the principal minor of order n is $\det\left(H(\overrightarrow{C_n})\right)$.

Recall that the setting $\alpha = 1$ in the Hermitian adjacency matrix of a digraph coincides with the adjacency matrix of its underlying undirected graph. By Theorem 2.9 and [1], the equality $\det!\left(H!\left(\overrightarrow{C_n}\right)\right) = \det!\left(A!\left(C_n\right)\right)$ holds only when $\alpha = 1$; consequently, for general α, these determinants do not coincide. We next specialise to the case where the mixed graph reduces to an undirected graph, and by applying Theorem 2.7 obtain Corollary 2.11.

Corollary 2.11. Let Γ be an undirected graph with a vertex set V_Γ consisting of n elements and an edge set E_Γ of cardinality m, and let $H(\Gamma) = (h_{ij})$ and $A(\Gamma) =$

(a_{ij}) be its Hermitian adjacency matrix and adjacency matrix, respectively, for $1 \leq i, j \leq n$. Now, let $\varpi(S_\Gamma^e)$ be the number of undirected cycles of order greater than or equal to 3 in S_Γ^e. Then, the determinant of $H(\Gamma)$ and $A(\Gamma)$ follows the conditions outlined below:

(a) If each vertex in Γ has an undirected loop, then

$$\det(H(\Gamma)) = det(A(\Gamma)) = 1 + (-1)^{n-1} \cdot 2 + \sum_{S_\Gamma^e} (-1)^{c(S_\Gamma^e) - N^l_{S_\Gamma^e}} \cdot 2^{\varpi(S_\Gamma^e)}. \quad (2.10)$$

(b) If at least one vertex in Γ has no loop, then

$$\det(H(\Gamma)) = det(A(\Gamma)) = (-1)^{n-1} \cdot 2 + \sum_{S_\Gamma^e} (-1)^{c(S_\Gamma^e) - N^l_{S_\Gamma^e}} \cdot 2^{\varpi(S_\Gamma^e)}. \quad (2.11)$$

where the sum extends over all spanning elementary subgraphs S_Γ^e. In addition, $(-1)^{n-1} \cdot 2$ vanishes if there is no spanning subgraph, that is, an undirected cycle. Meanwhile, if there is no spanning elementary subgraph, then $\sum_{S_\Gamma^e} (-1)^{c(S_\Gamma^e) - N^l_{S_\Gamma^e}} \cdot 2^{\varpi(S_\Gamma^e)} = 0$.

Proof. By definition, the Hermitian adjacency matrix of an undirected graph coincides with its adjacency matrix. Consequently, the result of Theorem 2.7 extends naturally to both matrices in the case of an undirected graph Γ. Furthermore, we observe that

$$2 \cdot Re\left(\alpha^{|a\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_{\Gamma_C}} - a\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_{\Gamma_C}}|}\right) = 2 \cdot Re\left(\alpha^0\right) = 2$$

and

$$2^{\varpi(S_\Gamma^e)} \cdot Re\left(\alpha^{|\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_\Gamma^e} - \delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_\Gamma^e}|}\right) = 2^{\varpi(S_\Gamma^e)} \cdot Re\left(\alpha^0\right) = 2^{\varpi(S_\Gamma^e)}$$

Taking into account the possible structural characteristics of Γ, we establish the determinant formulas as in Equations (2.10) and (2.11). Moreover, the determinant expressions follow directly from Theorem 2.7. $\square$

Since the coefficients of the characteristic polynomial of $H(M)$ are obtained from the determinants of the Hermitian adjacency matrices of induced sub-mixed graphs, explicit formulas for these coefficients are derived in Theorem 2.12.

Theorem 2.12. Let M be a mixed graph of order n and of size m, and let $H(M)$ be its Hermitian adjacency matrix. Now, let M' be an induced sub-mixed graph of M, of order j where $1 \leq j \leq n$. Suppose that $c\left(S_{M'}^e\right)$ denotes the number of classes in $S_{M'}^e$, $N^l_{S_M^e}$ denotes the number of directed loops in $S_{M'}^e$, and $\varpi(S_{M'}^e)$ denotes the number of undirected cycles or directed cycles or mixed cycles of order greater than or equal to 3 in $S_{M'}^e$. If the characteristic polynomial of $H(M)$ takes the form

$$P\left(\lambda; H(M)\right) = \lambda^n + \sum_{j=1}^{n} (-1)^j \beta_j \cdot \lambda^{n-j},$$

then for $1 \leq j \leq n$, the coefficients β_j adhere to the following cases.

(a) If every vertex of M' has a directed loop, then

$$\beta_j = \sum_{M'} \left(1 + (-1)^{\eta-1} \cdot 2 \cdot Re \left(\alpha^{|a\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_{M'C}} - a\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_{M'C}}|} \right) + \right.$$

$$\left. \sum_{S^e_{M'}} (-1)^{c(S^e_{M'}) - N^l_{S^e_M}} \cdot 2^{\varpi(S^e_{M'})} \cdot Re \left(\alpha^{|\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S^e_{M'}} - \delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S^e_{M'}}|} \right) \right). \quad (2.12)$$

(b) If at least one vertex of M' has no directed loop, then

$$\beta_j = \sum_{M'} \left((-1)^{\eta-1} \cdot 2 \cdot Re \left(\alpha^{|a\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_{M'C}} - a\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_{M'C}}|} \right) + \right.$$

$$\left. \sum_{S^e_{M'}} (-1)^{c(S^e_{M'}) - N^l_{S^e_M}} \cdot 2^{\varpi(S^e_{M'})} \cdot Re \left(\alpha^{|\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S^e_{M'}} - \delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S^e_{M'}}|} \right) \right). \quad (2.13)$$

where the first summation extends over all induced sub-mixed graphs M' of M, while the second iterates over all spanning elementary subgraphs $S^e_{M'}$ within M'. In addition,

$$(-1)^{\eta-1} \cdot 2 \cdot Re \left(\alpha^{|a\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S_{M'C}} - a\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S_{M'C}}|} \right)$$

vanishes if there is no spanning sub-mixed graph of M' has an undirected cycle graph as its underlying graph. On the other hand, if the spanning elementary sub-mixed graph $S^e_{M'}$ does not exist, then

$$\sum_{S^e_{M'}} (-1)^{c(S^e_{M'}) - N^l_{S^e_M}} \cdot 2^{\varpi(S^e_{M'})} \cdot Re \left(\alpha^{|\delta\left(\vartheta_{r_p}, \vartheta_{r_{p+1}}\right)_{S^e_{M'}} - \delta\left(\vartheta_{r_{q+1}}, \vartheta_{r_q}\right)_{S^e_{M'}}|} \right) = 0.$$

Proof. For any j in the range $1 \leq j \leq n$, the coefficients β_j correspond to the sum of its principal minor. Furthermore, since each principal submatrix of $H(M)$ serves as the Hermitian adjacency matrix of an induced sub-mixed graph M' (with order satisfying $1 \leq j \leq n$), it follows that these coefficients β_j can alternatively be interpreted as the sum of the Hermitian adjacency matrices of all such induced sub-mixed graphs. Invoking Theorem 2.7, which establishes $\det(H(M))$, we deduce the results shown in Equations (2.12) and (2.13). □

Applying Theorem 2.12 to undirected graphs yields the following Corollary 2.13.

Corollary 2.13. Let Γ be an undirected graph of order n and of size m, and let $H(\Gamma)$ be its Hermitian adjacency matrix. Now, let Γ' be an induced subgraph of Γ, of order j where $1 \leq j \leq n$. Suppose that $c\left(S^e_{\Gamma'}\right)$ denotes the number of classes in $S^e_{\Gamma'}$, $N^l_{S^e_\Gamma}$ denotes the number of directed loops in $S^e_{\Gamma'}$, and $\varpi(S^e_{\Gamma'})$ denotes the number of undirected cycles or directed cycles or mixed cycles of order greater than or equal to 3 in $S^e_{\Gamma'}$. If the characteristic polynomial of $H(\Gamma)$ is given by

$$P\left(\lambda; H(\Gamma)\right) = \lambda^n + \sum_{j=1}^{n} (-1)^j \varsigma_j \cdot \lambda^{n-j},$$

then for $1 \le j \le n$, the coefficients ς_j satisfy the following conditions:

(a) If every vertex in Γ' has an undirected loop, then

$$\varsigma_j = \sum_{\Gamma'} \left(1 + (-1)^{\eta-1} \cdot 2 + \sum_{S^e_{\Gamma'}} (-1)^{c(S^e_{\Gamma'}) - N^l_{S^e_{\Gamma'}}} \cdot 2^{\varpi(S^e_{\Gamma'})} \right).$$

(b) If at least one vertex in Γ' lacks an undirected loop, then

$$\varsigma_j = \sum_{\Gamma'} \left((-1)^{\eta-1} \cdot 2 + \sum_{S^e_{\Gamma'}} (-1)^{c(S^e_{\Gamma'}) - N^l_{S^e_{\Gamma'}}} \cdot 2^{\varpi(S^e_{\Gamma'})} \right).$$

The outer summation extends over all induced subgraphs Γ' of Γ, while the inner summation is taken over all spanning elementary subgraphs $S_{\Gamma'^e}$ of Γ'. Whenever Γ' contains a spanning subgraph that forms an undirected cycle, $(-1)^{\eta-1} \cdot 2$ dissolves. In contrast, $\left((-1)^{\eta-1} \cdot 2 + \sum_{S^e_{\Gamma'}} (-1)^{c(S^e_{\Gamma'}) - N^l_{S^e_{\Gamma}}} \cdot 2^{\varpi(S^e_{\Gamma'})} \right)$ dissolves whenever the spanning elementary subgraph $S^e_{\Gamma'}$) does not exist.

Proof. By Theorem 2.11, $\det(H(\Gamma))$ has been established. Applying a method analogous to the proof of Theorem 2.12, we extend these results to derive the characteristic polynomial of $H(\Gamma)$. Consequently, the following corollary holds. □

After finding the determinant of $H\left(\overrightarrow{P_n}\right)$ and $H\left(\overrightarrow{C_n}\right)$, we can derive explicit forms of their characteristic polynomials based on their determinant. Thus, we show the characteristic polynomial of $H\left(\overrightarrow{P_n}\right)$ and $H\left(\overrightarrow{C_n}\right)$ in the following Theorem 2.14 and Theorem 2.15, respectively.

Theorem 2.14. Let $\overrightarrow{P_n}$ be a path digraph and $H\left(\overrightarrow{P_n}\right)$ be its Hermitian adjacency matrix. If $O = \{2k : k \,\text{is an odd number and}\, 2k \le n\}$ and $E = \{2k : k \,\text{is an even number and}\, 2k \le n\}$, then the characteristic polynomial of $H\left(\overrightarrow{P_n}\right)$ is as follows:

$$P\left(\lambda; H\left(\overrightarrow{P_n}\right)\right) = \lambda^n + \sum_{i \in E} \left(m\left(\overrightarrow{P_i}\right) + m\left(\overrightarrow{P_i^k}\right) \right) \lambda^{n-i} - \sum_{j \in O} \left(m\left(\overrightarrow{P_j}\right) + m\left(\overrightarrow{P_j^k}\right) \right) \lambda^{n-j}.$$

Theorem 2.15. Let $\overrightarrow{C_n}$ be a cycle digraph and $H\left(\overrightarrow{C_n}\right)$ be its Hermitian adjacency matrix. Suppose that $O = \{2k : k \,\text{is an odd number and}\, 2k \le n-1\}$ and $E = \{2k : k \,\text{is an even number and}\, 2k \le n-1\}$=. If n is an odd number, then the characteristic polynomial of $\overrightarrow{C_n}$ is as follows:

$$P\left(\lambda; H\left(\overrightarrow{C_n}\right)\right) = \lambda^n + \sum_{i \in E} \left(m\left(\overrightarrow{P_i}\right) + m\left(\overrightarrow{P_i^k}\right) \right) \lambda^{n-i} - \sum_{j \in O} \left(m\left(\overrightarrow{P_j}\right) + m\left(\overrightarrow{P_j^k}\right) \right) \lambda^{n-j} + 2 \cdot Re\left(\alpha^n\right).$$

If $n = 2l$ where l is even, then the characteristic polynomial of $\overrightarrow{C_n}$ is as follows:

$$P\left(\lambda; H\left(\overrightarrow{C_n}\right)\right) = \lambda^n + \sum_{i \in E} \left(m\left(\overrightarrow{P_i}\right) + m\left(\overrightarrow{P_i^k}\right) \right) \lambda^{n-i} - \sum_{j \in O} \left(m\left(\overrightarrow{P_j}\right) + m\left(\overrightarrow{P_j^k}\right) \right) \lambda^{n-j} - 2 \cdot Re\left(\alpha^n\right) + 2.$$

If $n = 2l$ where l is odd, then the characteristic polynomial of $\overrightarrow{C_n}$ is as follows:

$$P\left(\lambda; H\left(\overrightarrow{C_n}\right)\right) = \lambda^n + \sum_{i \in E}\left(m\left(\overrightarrow{P_i}\right) + m\left(\overrightarrow{P_i^k}\right)\right)\lambda^{n-i} - \sum_{j \in O}\left(m\left(\overrightarrow{P_j}\right) + m\left(\overrightarrow{P_j^k}\right)\right)\lambda^{n-j} - 2 \cdot Re\left(\alpha^n\right) - 2.$$

3. Eigenvalues of Hermitian adjacency matrix

In this section, we investigate the eigenvalues properties of the Hermitian adjacency matrix. Since $H(M)$ is Hermitian, all its eigenvalues are real and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, with $\lambda_1 = \rho(H(M))$; see [13].

We will also use the standard Frobenius norm bounds for the spectral radius,

$$\sqrt{\frac{\sum_{i,j}|a_{ij}|^2}{n}} \;\leq\; \rho(A) \;\leq\; \sqrt{\sum_{i,j}|a_{ij}|^2},$$

which hold for any $A \in \mathbb{C}^{n \times n}$; see [19, 10].

Applying the standard Frobenius norm bounds to $H(M)$ yields the following estimate of its spectral radius.

Theorem 3.1. Let M be a mixed graph of order n and size m and let $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. For $1 \leq i \leq n$, suppose that $\lambda_i \in \sigma(H(M))$ where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$, then we obtain the following:

$$\sqrt{\frac{m}{n}} \leq \rho(H(M)) \leq \sqrt{m}. \tag{3.1}$$

Proof. For $1 \leq i \leq n$, suppose that $\lambda_i \in \sigma(H(M))$ where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$. Applying the Frobenius norm bounds to $H(M)$, we obtain $\sqrt{\frac{\sum_{1 \leq i,j \leq n}|h_{ij}|^2}{n}} \leq \rho(H(M)) \leq \sqrt{\sum_{1 \leq i,j \leq n}|h_{ij}|^2}$. Based on the definition of the Hermitian adjacency matrix, we obtain $|h_{ij}|^2 = 1$ whenever there is an arc or edge connecting the vertex $v_i \in V_M$ and the vertex $v_j \in V_M$; and ${h_{ij}}^2 = 0$ otherwise. This implies $\sum_{1 \leq i,j \leq n}|h_{ij}|^2 = m$. Consequently, we obtain Equation (3.1). □

Theorem 3.1 establishes bounds for the spectral radius of the Hermitian adjacency matrix based on the order and size of the graph. Although it estimates the largest absolute eigenvalue, it does not identify specific eigenvalues. The focus now shifts to conditions that result in zero being one of the eigenvalues of the Hermitian adjacency matrix. We further derive a sufficient condition that $0 \in \sigma(H(M))$ is based on the structure of the mixed graph M, as presented in Theorem 3.2 below.

Theorem 3.2. Let M be a mixed graph of order n and size m; and let $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. If each of the following conditions is satisfied:

(a) there exists a loopless vertex in the mixed graph M,
(b) there is no spanning sub-mixed graph of M that its underlying graph is isomorphic to an undirected cycle, and
(c) spanning elementary sub-mixed graphs S_M^e does not exist,

then $0 \in \sigma(H(M))$.

Proof. Based on Theorem 2.7, if there exists a loopless vertex in the mixed graph M, then

$$det(H(M)) = (-1)^{n-1} \cdot 2 \cdot Re\left(\alpha^{|a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{S_{M_C}} - a(\vartheta_{r_{q+1}},\vartheta_{r_q})_{S_{M_C}}|}\right) +$$

$$\sum_{S_M^e} (-1)^{c(S_M^e) - N^l_{S_M^e}} \cdot 2^{\varpi(S_M^e)} \cdot Re\left(\alpha^{|\delta(\vartheta_{r_p},\vartheta_{r_{p+1}})_{S_M^e} - \delta(\vartheta_{r_{q+1}},\vartheta_{r_q})_{S_M^e}|}\right).$$

Moreover, if there is no spanning sub-mixed graph of M that its underlying graph is isomorphic to an undirected cycle graph, then

$$(-1)^{n-1} \cdot 2 \cdot Re\left(\alpha^{|a(\vartheta_{r_p},\vartheta_{r_{p+1}})_{S_{M_C}} - a(\vartheta_{r_{q+1}},\vartheta_{r_q})_{S_{M_C}}|}\right) = 0,$$

while, if the spanning elementary sub-mixed graph S_M^e does not exist, then

$$\sum_{S_M^e} (-1)^{c(S_M^e) - N^l_{S_M^e}} \cdot 2^{\varpi(S_M^e)} \cdot Re\left(\alpha^{|\delta(\vartheta_{r_p},\vartheta_{r_{p+1}})_{S_M^e} - \delta(\vartheta_{r_{q+1}},\vartheta_{r_q})_{S_M^e}|}\right) = 0.$$

Therefore, if conditions (a), (b), and (c) are satisfied, then $\det(H(M)) = 0$. Recall that if $\det H(M) = 0$, then $0 \in \sigma(H(M))$. Consequently, if each of the conditions is satisfied, then $0 \in \sigma(H(M))$. □

Extending the skew-energy concept of Cavers et al. [5], we define the Hermitian energy of a mixed graph by $\varepsilon_{H(M)} = \sum_{i=1}^{n} |\lambda_i|$. An upper bound for $\varepsilon_{H(M)}$ follows directly from Theorem 3.1 and is presented in Corollary 3.3.

Corollary 3.3. Let M be a mixed graph of order n and size m and let $H(M) = (h_{ij})$ be its Hermitian adjacency matrix for $1 \leq i, j \leq n$. For $1 \leq i \leq n$, suppose that $\lambda_i \in \sigma(H(M))$ where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$, then $\varepsilon_{H(M)} \leq n\sqrt{m}$.

Proof. Based on Theorem 3.1, we obtain $|\lambda_i| \leq \sqrt{m}$. Thus, the upper bound of the sum of $|\lambda_i|$ is obtained, that is, $\varepsilon_{H(M)} \leq n\sqrt{m}$. □

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