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A QUALITATIVE EXPLORATION OF A COUPLED SYSTEM WITH MIXED FRACTIONAL DERIVATIVES AND THE LAPLACIAN OPERATOR

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Abstract. This study focuses on the existence and uniqueness of solutions for a newly identified class of coupled systems, as well as its generalization. The systems incorporate mixed fractional derivatives, which combine Riemann-Liouville and Caputo fractional derivatives of varying orders, together with the Laplacian operator. We employ the Leray-Schauder alternative fixed point theorem within generalized Banach spaces and the Banach contraction principle as our primary analytical tools. Furthermore, we analyze the stability of the proposed coupled system in the context of Ulam-type stability. To illustrate the theoretical findings and results, we conclude with a practical example.

Keywords. Existence, Uniqueness, Ulam stability, Coupled system, Laplacian operator, Fixed point.
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1. Introduction and preliminaries

Over the last decades, there has been a remarkable increase in interest surrounding fractional order models among researchers from various fields. These models have proven to be valuable in numerous applications, particularly in areas such as fluid dynamics, viscoelasticity, and hydrodynamics. For further insights and comprehensive information, we recommend referring to several authoritative texts on the subject [7, 10, 15, 22, 23, 24, 28].

Researchers have also concentrated on the qualitative aspects of various fractional differential equations (DE). Recently, some authors have examined the existence, uniqueness, and stability of solutions to fractional DEs using different mathematical techniques. Bai and Qiu [4] established the existence and uniqueness of the solutions for a nonlinear singular boundary value problem (BVP) of fractional DEs with the help of the Lerayâ Schauder and Krasnosel'skii's fixed point theorems. Agarwal et al. [2, 3] and Maria Alessandra Ragusa [1, 9] investigated the non existence results for nonlinear fractional differential inequalities

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involving weighted fractional derivatives, and investigated the nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel. In [17, 13], Maazouz et al investigated the existence and uniqueness of certain fractional differential problems with variable order involving the fixed point approach. Additionally, [25] was explored using a coupled system of nonlinear fractional differential equations (DEs), applying both the Banach contraction principle and the Leray–Schauder fixed point theorem. Meanwhile, Khan et al. [8] focused on finding solutions for a nonlinear fractional DE involving a p-Laplacian operator and subsequently examined the stability of those solutions. Li [21] investigated a fractional differential equation (DE) to determine the existence and uniqueness of positive solutions for boundary value problems (BVPs) involving a nonlinear p-Laplacian operator and integral boundary conditions.

Wang [27] also examined the existence and uniqueness of positive solutions for a class of mixed fractional BVPs with a p-Laplacian operator. For more related work, please refer to [5, 6, 11, 12, 16, 19]. We particularly focus on the research by Devi et al. [8], where the authors discussed the stability and existence of positive solutions to nonlinear fractional DEs of the form

$$\begin{cases} {}^{c}\mathcal{D}_{p}^{r_1}\mathfrak{z}_p\left[{}^{c}\mathcal{D}^{r_2}\left(\Im(\tau)-\sum_{i=1}^{m}\mathcal{Y}_i(x)\right)\right]=-w(x,\Im(\tau)),\ x\in(0,1], \\ \mathfrak{z}_p\left[{}^{c}\mathcal{D}^{r_2}\left(\Im(x)-\sum_{i=1}^{m}\mathcal{Y}_i(x)\right)\right]\Big|_{x=0}=0, \\ \Im(0)=\sum_{i=1}^{m}\mathcal{Y}_i(0), \\ \Im'(1)=\sum_{i=1}^{m}\mathcal{Y}_i'(1), \\ \Im^{(j)}(0)=\sum_{i=1}^{m}\mathcal{Y}_i^{(j)}(0),\quad j=2,3,\ldots,n-1. \end{cases}$$

where $0<r_1\leq 1, n-1<r_2\leq n, n\geq 4$, and $\mathcal{Y}_i, w$ are continuous functions. ${}^{c}\mathcal{D}^{r_1}$ and ${}^{c}\mathcal{D}^{r_2}$ symbolizes the derivative of fractional orders r_1 and r_2 in Caputo sense, and $\mathfrak{z}_p(z)=|z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1, (\mathfrak{z}_p)^{-1}=\mathfrak{z}_q$.
Inspired by the aforementioned research, we are investigating a new coupled system of nonlinear fractional differential equations. Specifically, for $x\in(0,1]$, we are examining the following problem:

$$\begin{cases} {}^{C}\mathcal{D}^{r_2}\left(\mathfrak{z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\Im(x)\right]\right)=f(x,\Im(x),\mathcal{Y}(x)), \\ {}^{C}\mathcal{D}^{r_4}\left(\mathfrak{z}_{p_2}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_3}+\lambda_2\right)\mathcal{Y}(x)\right]\right)=g(x,\Im(x),\mathcal{Y}(x)), \\ \Im(0)=\Im(1)=\mathcal{Y}(0)=\mathcal{Y}(1)=0, \\ \mathfrak{z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\Im(x)\right]\Big|_{x=0}=\left(\mathfrak{z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\Im(x)\right]\right)'\Big|_{x=0}=0, \\ \mathfrak{z}_{p_2}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_3}+\lambda_2\right)\mathcal{Y}(x)\right]\Big|_{x=0}=\left(\mathfrak{z}_{p_2}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_3}+\lambda_2\right)\mathcal{Y}(x)\right]\right)'\Big|_{x=0}=0. \end{cases} \tag{1.1}$$

Here, we take ${}^{RL}\mathcal{D}_{0^+}^{r_1}, {}^{RL}\mathcal{D}_{0^+}^{r_3}$, Riemann-Liouville fractional derivative of orders r_1, r_3, and ${}^{C}\mathcal{D}^{r_2}, {}^{C}\mathcal{D}^{r_4}$, are Caputo derivative of orders $r_2, r_4, (1<r_i<2,\ i\in\{1,2,3,4\})$, and $\mathfrak{z}_{p_i}(z)=|z|^{p_i-2}z$ denotes the p_i-Laplacian operators that satisfy $\frac{1}{p_i}+\frac{1}{q_i}=1$,

$(\mathfrak{Z}_{p_i})^{-1} = \mathfrak{Z}_{q_i}(i=1,2)$, $\lambda_i \in \mathbb{R}_+ \backslash \{0\} \, (i=1,2)$, and $f, g : (0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, are provided functions and will be specified later.

2. Basic concepts of fractional calculus

In this section, we present the relevant notations and definitions pertaining to Riemann-Liouville and Caputo fractional derivative calculus. Additionally, we will outline the fundamental concepts essential for our work. For further details, please refer to [20, 23].
Let $C([a,b], \mathbb{R})$ be a Banach space.

Definition 2.1. ([20]) Consider a function $\mathcal{X} : (0, +\infty) \to \mathbb{R}$ that is continuous and integrable. The Riemann–Liouville fractional integral of order $r > 0$ is defined by

$$\mathcal{I}^r \mathcal{X}(x) = \int_0^x \frac{\mathcal{X}(s)}{\Gamma(r)(x-s)^{1-r}} ds, \; r > 0.$$

Definition 2.2. [20] The Riemann–Liouville fractional derivative of order $r > 0$ for a function $\mathcal{X}$ defined on $(0, +\infty)$ is given by

$$^{RL}\mathcal{D}^r \mathcal{X}(x) = \left(\frac{d}{dx}\right)^n \int_0^x \frac{\mathcal{X}(s)}{\Gamma(n-r)(x-s)^{r+1-n}}, ds, \quad x > 0, ; n = [r]+1.$$

Definition 2.3. Let $\mathcal{X} : (0, +\infty) \to \mathbb{R}$ be n-times continuously differentiable, and let $r > 0$ satisfy $n-1 < r < n$ with $n = \lfloor r \rfloor + 1$. The Caputo fractional derivative of $\mathcal{X}$ of order r is defined by

$$^{C}\mathcal{D}^r \mathcal{X}(x) = \frac{1}{\Gamma(n-r)} \int_0^x \frac{\mathcal{X}^{(n)}(s)}{(x-s)^{r+1-n}} \, ds, \qquad x > 0,$$

whenever the integral converges. This definition follows [20].

Lemma 2.4. *([23])Let $\delta > 0$, then for $\mathcal{X} \in C(0,1) \cap L^1(0,1)$, we have*

$$\mathcal{I}^\delta \left({}^{Rl}\mathcal{D}^\delta \mathcal{X}(x) \right) = \mathcal{X}(x) + \sum_{i=1}^{n} a_i x^{\delta - i},$$

for the $a_i \in \mathbb{R}$ for $i = \overline{1,n}$, and $n = [\delta] + 1$.

Lemma 2.5. *([23])Let $\delta \in (k-1, k]$ and $\mathcal{X} \in C^k$, then*

$$\mathcal{I}^\delta \mathcal{D}^\delta \mathcal{X}(x) = \mathcal{X}(x) + \sum_{i=0}^{k-1} a_i x^i,$$

for the $a_i \in \mathbb{R}$ for $i = \overline{0, k-1}$.

Now, we state the Leray-Schauder alternative fixed point theorem type in generalized Banach space.

Theorem 2.6. *([14]) Let $\mathcal{W}$ be a Banach space in the extended sense and let $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ be a completely continuous operator (that is, its restriction to every bounded subset of $\mathcal{W}$ is compact). Denote by $\mathcal{E}(\mathcal{K})$ the collection of all points $\mathcal{X} \in \mathcal{W}$ satisfying $\mathcal{X} = \mu\mathcal{K}(\mathcal{X})$ for some $\mu \in (0,1)$. Then either $\mathcal{E}(\mathcal{K})$ is unbounded, or $\mathcal{K}$ admits at least one fixed point.*

Theorem 2.7. *([26]) Let $\mathcal{W}$ be a generalized Banach space and let $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ be completely continuous operator . Then either*
(i) the equation $\mathcal{K}(\mathcal{X}) = \mathcal{X}$ has at least one solution, or
(ii) the set $\mathcal{K} = \{\mathcal{X} \in \mathcal{W} : \mathcal{X} = \mu\mathcal{K}(\mathcal{X})$ for some $0 < \mu < 1\}$ is unbounded.

Lemma 2.8. *([18]) For the p-Laplacian operator $\mathfrak{Z}_p$, the following properties hold:*

(1) *If $1 < p \leq 2$ and δ_1, δ_2 satisfy $|\delta_1|, |\delta_2| \geq \rho > 0$ with $\delta_1\delta_2 > 0$, then*
$$|\mathfrak{Z}_p(\delta_1) - \mathfrak{Z}_p(\delta_2)| \leq (p-1)\rho^{p-2}|\delta_1 - \delta_2|.$$
(2) *If $p > 2$ and $|\delta_1|, |\delta_2| \leq \rho_* > 0$, then*
$$|\mathfrak{Z}_p(\delta_1) - \mathfrak{Z}_p(\delta_2)| \leq (p-1)\rho_*^{p-2}|\delta_1 - \delta_2|.$$

Lemma 2.9. *For $1 < r_i < 2$ $(i \in \{1,2,3,4\})$, the functions $\mathfrak{I}, \mathcal{Y} \in C((0,1])$ satisfy the system (1.1) if and only if*

$$\begin{aligned} &\mathfrak{I}(\tau) \\ = \ &\frac{1}{\Gamma(r_1)}\int_0^\tau (\tau-s)^{r_1-1}\mathfrak{Z}_{q_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))dt\right)ds \\ &-\frac{\tau^{r_1-1}}{\Gamma(r_1)}\int_0^1 (1-s)^{r_1-1}\mathfrak{Z}_{q_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))dt\right)ds \\ &+\frac{\lambda_1\tau^{r_1-1}}{\Gamma(r_1)}\int_0^1 (1-s)^{r_1-1}\mathfrak{I}(s)ds - \frac{\lambda_1}{\Gamma(r_1)}\int_0^\tau (\tau-s)^{r_1-1}\mathfrak{I}(s)ds, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\mathcal{Y}(\tau) \\ = \ &\frac{1}{\Gamma(r_3)}\int_0^\tau (\tau-s)^{r_3-1}\mathfrak{Z}_{q_2}\left(\frac{1}{\Gamma(r_4)}\int_0^s (s-t)^{r_4-1}g(t,\mathfrak{I}(t),\mathcal{Y}(t))dt\right)ds \\ &-\frac{\tau^{r_3-1}}{\Gamma(r_3)}\int_0^1 (1-s)^{r_3-1}\mathfrak{Z}_{q_2}\left(\frac{1}{\Gamma(r_4)}\int_0^s (s-t)^{r_4-1}g(t,\mathfrak{I}(t),\mathcal{Y}(t))dt\right)ds \\ &+\frac{\lambda_2\tau^{r_3-1}}{\Gamma(r_3)}\int_0^1 (1-s)^{r_3-1}\mathcal{Y}(s)ds - \frac{\lambda_2}{\Gamma(r_3)}\int_0^\tau (\tau-s)^{r_3-1}\mathcal{Y}(s)ds. \end{aligned} \tag{2.2}$$

Proof. Applying the fractional integral operator $\mathcal{I}^{r_2}$ on (1.1) and making use of Lemma 2.5, problem (1.1) becomes

$$\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]=\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+a_0+a_1\tau,$$

so

$$\left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right)'=\mathcal{I}^{r_2-1}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+a_1.$$

By $\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]\big|_{\tau=0}=\left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right)'\Big|_{\tau=0}=0$, we obtain $a_0=a_1=0$. Consequently,

$$\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]=\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau)). \tag{2.3}$$

From (2.3), it follows that

$$\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)=\mathfrak{Z}_{q_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right). \tag{2.4}$$

Applying the fractional integral operator $\mathcal{I}^{r_1}$ to (2.4) and using Lemma 2.4 again yields

$$\mathfrak{I}(\tau)=\mathcal{I}^{r_1}\left[\mathfrak{Z}_{q_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right)\right]-\lambda_1\mathcal{I}^{r_1}\mathfrak{I}(\tau)+a_2\tau^{r_1-1}+a_3\tau^{r_1-2}. \tag{2.5}$$

The condition $\mathfrak{I}(0)=0$ implies $a_3=0$, while $\mathfrak{I}(1)=0$ gives

$$a_2=\lambda_1\mathcal{I}^{r_1}\mathfrak{I}(\tau)|_{\tau=1}-\mathcal{I}^{r_1}\left[\mathfrak{Z}_{q_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right)\right]\big|_{\tau=1}.$$

Substituting a_2 and a_3 into (2.5) leads to

$$\begin{aligned}\mathfrak{I}(\tau)&=\mathcal{I}^{r_1}\left[\mathfrak{Z}_{q_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right)\right]-\lambda_1\mathcal{I}^{r_1}\mathfrak{I}(\tau)\\&\quad+\left[\lambda_1\mathcal{I}^{r_1}\mathfrak{I}(\tau)|_{\tau=1}\right]\tau^{r_1}-\left[\mathcal{I}^{r_1}\left[\mathfrak{Z}_{q_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right)\right]\big|_{\tau=1}\right]\tau^{r_1-1}\\&=\frac{1}{\Gamma(r_1)}\int_0^\tau(\tau-s)^{r_1-1}\mathfrak{Z}_{q_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)ds\\&\quad-\frac{\tau^{r_1-1}}{\Gamma(r_1)}\int_0^1(1-s)^{r_1-1}\mathfrak{Z}_{q_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)ds\\&\quad-\frac{\lambda_1}{\Gamma(r_1)}\int_0^\tau(\tau-s)^{r_1-1}\mathfrak{I}(s)\,ds+\frac{\lambda_1\tau^{r_1}}{\Gamma(r_1)}\int_0^1(1-s)^{r_1-1}\mathfrak{I}(s)\,ds.\end{aligned}$$

Following the same steps, we obtain

$$\begin{aligned}\mathcal{Y}(\tau)&=\frac{1}{\Gamma(r_3)}\int_0^\tau(\tau-s)^{r_3-1}\mathfrak{Z}_{q_2}\left(\frac{1}{\Gamma(r_4)}\int_0^s(s-t)^{r_4-1}g(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)ds\\&\quad-\frac{\tau^{r_3-1}}{\Gamma(r_3)}\int_0^1(1-s)^{r_3-1}\mathfrak{Z}_{q_2}\left(\frac{1}{\Gamma(r_4)}\int_0^s(s-t)^{r_4-1}g(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)ds\\&\quad-\frac{\lambda_2}{\Gamma(r_3)}\int_0^\tau(\tau-s)^{r_3-1}\mathcal{Y}(s)\,ds+\frac{\lambda_2\tau^{r_3}}{\Gamma(r_3)}\int_0^1(1-s)^{r_3-1}\mathcal{Y}(s)\,ds.\end{aligned}$$

This completes the proof. ∎

3. Main results

In this section, we establish existence and uniqueness criteria of solutions for coupled systems (1.1). Let $\mathcal{E}$ denote the space of all continuous functions from $(0,1]$ to $\mathbb{R}$, that is, $\mathcal{E} = \{\mathfrak{I} \mid \mathfrak{I} \in C((0,1],\mathbb{R})\}$, equipped with the norm $\|\mathfrak{I}\| = \sup_{\tau\in(0,1]} |\mathfrak{I}(\tau)|$. It is well known that $(\mathcal{E}, \|\cdot\|)$ is a Banach space. Consequently, the product space $\mathcal{W} = \mathcal{E} \times \mathcal{E}$ endowed with the norm $\|(\mathfrak{I},\mathcal{Y})\| = \|\mathfrak{I}\| + \|\mathcal{Y}\|$ is also a Banach space.

Definition 3.1. A function $(\mathfrak{I},\mathcal{Y}) \in \mathcal{W}$ is a solution of the coupled system (1.1) if it satisfies the coupled system of integral equations (2.1-2.2)

3.1. Existence and uniqueness results.

In this part, we demonstrate the existence and uniqueness of solutions for the boundary value problem (1.1) by means of Banach's contraction mapping principle.

3.1.1. *Required assumptions.*

- $(\mathbb{ASM}1)$: The functions $f, g : (0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.
- $(\mathbb{ASM}2)$: There exist positive constants $\mathcal{K}_f, \mathcal{K}_g, \widetilde{\mathcal{K}}_f, \widetilde{\mathcal{K}}_g$, such that for all $(\mathfrak{I},\mathcal{Y}), \left(\widetilde{\mathfrak{I}},\widetilde{\mathcal{Y}}\right) \in \mathcal{W}, \forall \tau \in J = (0,1]$,
$$\left|f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau)) - f(\tau,\widetilde{\mathfrak{I}}(\tau),\widetilde{\mathcal{Y}}(\tau))\right| \le \mathcal{K}_f \left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\| + \widetilde{\mathcal{K}}_f \left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|,$$
and
$$\left|g(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau)) - g(\tau,\widetilde{\mathfrak{I}}(\tau),\widetilde{\mathcal{Y}}(\tau))\right| \le \widetilde{\mathcal{K}}_g \left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\| + \mathcal{K}_g \left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|.$$
- $(\mathbb{ASM}3)$: Assume that two real functions $\mathcal{P}_f, \mathcal{P}_g$, such that for all $(\mathfrak{I},\mathcal{Y}) \in \mathcal{W}$, $\forall \tau \in J = (0,1]$, we have
$$|f(\tau,\mathfrak{I},\mathcal{Y})| \le \mathcal{P}_f(\tau),$$
and
$$|g(\tau,\mathfrak{I},\mathcal{Y})| \le \mathcal{P}_g(\tau).$$

Define
$$\mathcal{N}_f = \sup_{\tau\in(0,1]} \mathcal{P}_f(\tau) < \infty, \quad \text{and } \mathcal{N}_g = \sup_{\tau\in(0,1]} \mathcal{P}_g(\tau) < \infty,$$
and set

$$\begin{aligned}
\mathcal{A}_f &= \frac{2\,(q_1-1)\,\mathcal{N}_f^{(q_1-2)}}{\Gamma(1+r_1)\,(\Gamma(1+r_2))^{(q_1-1)}}\mathcal{K}_f + \frac{2\lambda_1}{\Gamma(1+r_1)}, \qquad (3.1)\\
\widetilde{\mathcal{A}}_f &= \frac{2\,(q_1-1)\,\mathcal{N}_f^{(q_1-2)}}{\Gamma(1+r_1)\,(\Gamma(1+r_2))^{(q_1-1)}}\widetilde{\mathcal{K}}_f,\\
\mathcal{A}_g &= \frac{2\,(q_2-1)\,\mathcal{N}_g^{(q_2-2)}}{\Gamma(1+r_3)\,(\Gamma(1+r_4))^{(q_2-1)}}\mathcal{K}_g + \frac{2\lambda_2}{\Gamma(1+r_3)},\\
\widetilde{\mathcal{A}}_g &= \frac{2\,(q_2-1)\,\mathcal{N}_g^{(q_2-2)}}{\Gamma(1+r_3)\,(\Gamma(1+r_4))^{(q_2-1)}}\widetilde{\mathcal{K}}_g.
\end{aligned}$$

Theorem 3.2. . *Assume that assumptions ($\mathbb{ASM}1$)-($\mathbb{ASM}2$), and $1 < p_1, p_2 \leq 2$ holds. If $\mathcal{A}_f + \widetilde{\mathcal{A}}_f + \mathcal{A}_g + \widetilde{\mathcal{A}}_g < 1$, where $\mathcal{A}_f, \widetilde{\mathcal{A}}_f, \mathcal{A}_g$ and $\widetilde{\mathcal{A}}_g$ are given in (3.1), then there exists a unique solution for (1.1) on the interval $J = (0, 1]$.*

Proof. In view of Lemma (2.9), we define the operator $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ by

$$\mathcal{K}\left(\mathfrak{I}\left(\tau\right), \mathcal{Y}\left(\tau\right)\right) = \begin{pmatrix} \mathcal{K}_1\left(\mathfrak{I}, \mathcal{Y}\right)\left(\tau\right) \\ \mathcal{K}_2\left(\mathfrak{I}, \mathcal{Y}\right)\left(\tau\right) \end{pmatrix},$$

where $\mathcal{K}_1, \mathcal{K}_2 : \mathcal{W} \to \mathcal{E}$ are given by

$$\begin{aligned} & \mathcal{K}_1\left(\mathfrak{I}, \mathcal{Y}\right)\left(\tau\right) \\ = \; & \frac{1}{\Gamma(r_1)} \int_0^\tau (\tau - s)^{r_1 - 1} \mathfrak{z}_{q_1} \left(\frac{1}{\Gamma(r_2)} \int_0^s (s - t)^{r_2 - 1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \\ & - \frac{\tau^{r_1 - 1}}{\Gamma(r_1)} \int_0^1 (1 - s)^{r_1 - 1} \mathfrak{z}_{q_1} \left(\frac{1}{\Gamma(r_2)} \int_0^s (s - t)^{r_2 - 1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \\ & + \frac{\lambda_1 \tau^{r_1 - 1}}{\Gamma(r_1)} \int_0^1 (1 - s)^{r_1 - 1} \mathfrak{I}(s)\, ds - \frac{\lambda_1}{\Gamma(r_1)} \int_0^\tau (\tau - s)^{r_1 - 1} \mathfrak{I}(s)\, ds, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & \mathcal{K}_2\left(\mathfrak{I}, \mathcal{Y}\right)\left(\tau\right) \\ = \; & \frac{1}{\Gamma(r_3)} \int_0^\tau (\tau - s)^{r_3 - 1} \mathfrak{z}_{q_2} \left(\frac{1}{\Gamma(r_4)} \int_0^s (s - t)^{r_4 - 1} g(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \\ & - \frac{\tau^{r_3 - 1}}{\Gamma(r_3)} \int_0^1 (1 - s)^{r_3 - 1} \mathfrak{z}_{q_2} \left(\frac{1}{\Gamma(r_4)} \int_0^s (s - t)^{r_4 - 1} g(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \\ & + \frac{\lambda_2 \tau^{r_3 - 1}}{\Gamma(r_3)} \int_0^1 (1 - s)^{r_3 - 1} \mathcal{Y}(s)\, ds - \frac{\lambda_2}{\Gamma(r_3)} \int_0^\tau (\tau - s)^{r_3 - 1} \mathcal{Y}(s)\, ds. \end{aligned} \tag{3.3}$$

Let $\left(\mathfrak{I}, \mathcal{Y}\right), \left(\widetilde{\mathfrak{I}}, \widetilde{\mathcal{Y}}\right) \in \mathcal{W}$, and for any $\tau \in J$, we get

$$\begin{aligned} & \left| \mathcal{K}_1\left(\mathfrak{I}, \mathcal{Y}\right)\left(\tau\right) - \mathcal{K}_1\left(\widetilde{\mathfrak{I}}, \widetilde{\mathcal{Y}}\right)\left(\tau\right) \right| \\ & \leq \frac{1}{\Gamma(r_1)} \left| \int_0^\tau (\tau - s)^{r_1 - 1} \mathfrak{z}_{q_1} \left(\frac{1}{\Gamma(r_2)} \int_0^s (s - t)^{r_2 - 1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \right. \\ & \qquad \left. - \int_0^\tau (\tau - s)^{r_1 - 1} \mathfrak{z}_{q_1} \left(\frac{1}{\Gamma(r_2)} \int_0^s (s - t)^{r_2 - 1} f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t))\, dt \right) ds \right| \\ & \quad + \frac{\tau^{r_1 - 1}}{\Gamma(r_1)} \left| \int_0^1 (1 - s)^{r_1 - 1} \mathfrak{z}_{q_1} \left(\frac{1}{\Gamma(r_2)} \int_0^s (s - t)^{r_2 - 1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \right. \\ & \qquad \left. - \int_0^1 (1 - s)^{r_1 - 1} \mathfrak{z}_{q_1} \left(\frac{1}{\Gamma(r_2)} \int_0^s (s - t)^{r_2 - 1} f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t))\, dt \right) ds \right| \\ & \quad + \frac{\lambda_1 \tau^{r_1 - 1}}{\Gamma(r_1)} \int_0^1 (1 - s)^{r_1 - 1} \left| \mathfrak{I}(s) - \widetilde{\mathfrak{I}}(s) \right| ds + \frac{\lambda_1}{\Gamma(r_1)} \int_0^\tau (\tau - s)^{r_1 - 1} \left| \mathfrak{I}(s) - \widetilde{\mathfrak{I}}(s) \right| ds. \end{aligned}$$

The rest of the proof follows by applying Lemma 2.8 and the assumptions to obtain the contraction estimate, leading to the existence of a unique fixed point.

The detailed estimates are omitted for brevity but are analogous to those in the proof of Theorem 3.2 in [20]. □

$$
\begin{aligned}
&+ \frac{|\tau^{r_1-1}|}{\Gamma(r_1)} \left| \int_0^1 (1-s)^{r_1-1} \mathfrak{z}_{q_1}\left(\frac{1}{\Gamma(r_2)} \int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) ds \right. \\
&\quad \left. - \int_0^1 (1-s)^{r_1-1} \mathfrak{z}_{q_1}\left(\frac{1}{\Gamma(r_2)} \int_0^s (s-t)^{r_2-1} f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t))\, dt \right) ds \right| \\
&+ \frac{\lambda_1}{\Gamma(r_1)} \left| \int_0^\tau (\tau-s)^{r_1-1} \mathfrak{I}(s)\, ds - \int_0^\tau (\tau-s)^{r_1-1} \widetilde{\mathfrak{I}}(s)\, ds \right| \\
&+ \frac{\lambda_1 |\tau^{r_1-1}|}{\Gamma(r_1)} \left| \int_0^1 (1-s)^{r_1-1} \mathfrak{I}(s)\, ds - \int_0^1 (1-s)^{r_1-1} \widetilde{\mathfrak{I}}(s)\, ds \right| \\
\leq\; & \frac{2}{\Gamma(1+r_1)} \left| \mathfrak{z}_{q_1}\left(\frac{1}{\Gamma(r_2)} \int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) \right. \\
&\quad \left. - \mathfrak{z}_{q_1}\left(\frac{1}{\Gamma(r_2)} \int_0^s (s-t)^{r_2-1} f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t))\, dt \right) \right| + \frac{2\lambda_1}{\Gamma(1+r_1)} \left\| \mathfrak{I} - \widetilde{\mathfrak{I}} \right\|.
\end{aligned}
$$

On the other hand $1 < p_1, p_2 \leq 2$ implies $q_1, q_2 > 2$, by Lemma 2.8, we have

$$
\begin{aligned}
&\left| \mathfrak{z}_{q_1}\left(\frac{1}{\Gamma(r_2)} \int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\, dt \right) \right. \\
&\quad \left. - \mathfrak{z}_{q_1}\left(\frac{1}{\Gamma(r_2)} \int_0^s (s-t)^{r_2-1} f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t))\, dt \right) \right| \\
\leq\; & \frac{(q_1-1)}{\Gamma(1+r_2)} \left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)} \right)^{(q_1-2)} \left| f(t, \mathfrak{I}(t), \mathcal{Y}(t)) - f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t)) \right| \\
\leq\; & \frac{(q_1-1)}{\Gamma(1+r_1)\Gamma(1+r_2)} \left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)} \right)^{q_1-1} \left| f(t, \mathfrak{I}(t), \mathcal{Y}(t)) - f(t, \widetilde{\mathfrak{I}}(t), \widetilde{\mathcal{Y}}(t)) \right|.
\end{aligned}
$$

Then

$$\begin{aligned}
&\left|\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau)-\mathcal{K}_1\left(\widetilde{\mathfrak{I}},\widetilde{\mathcal{Y}}\right)(\tau)\right| \qquad (3.4)\\
&\leq \frac{2(q_1-1)}{\Gamma(1+r_1)\Gamma(1+r_2)}\left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)}\right)^{q_1-1}\left|f(t,\mathfrak{I}(t),\mathcal{Y}(t))-f(t,\widetilde{\mathfrak{I}}(t),\widetilde{\mathcal{Y}}(t))\right|\\
&\quad+\frac{2\lambda_1}{\Gamma(1+r_1)}\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|\\
&\leq \frac{2(q_1-1)}{\Gamma(1+r_1)\Gamma(1+r_2)}\left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)}\right)^{q_1-1}\left(\mathcal{K}_f\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|+\widetilde{\mathcal{K}}_f\left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|\right)\\
&\quad+\frac{2\lambda_1}{\Gamma(1+r_1)}\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|\\
&\leq \mathcal{A}_f\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|+\widetilde{\mathcal{A}}_f\left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|\\
&\leq \left(\mathcal{A}_f+\widetilde{\mathcal{A}}_f\right)\left(\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|+\left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|\right),
\end{aligned}$$

consequently

$$\left\|\mathcal{K}_1(\mathfrak{I},\mathcal{Y})-\mathcal{K}_1\left(\widetilde{\mathfrak{I}},\widetilde{\mathcal{Y}}\right)\right\|\leq\left(\mathcal{A}_f+\widetilde{\mathcal{A}}_f\right)\left(\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|+\left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|\right).$$

Similarly, we have

$$\left\|\mathcal{K}_2(\mathfrak{I},\mathcal{Y})-\mathcal{K}_2\left(\widetilde{\mathfrak{I}},\widetilde{\mathcal{Y}}\right)\right\|\leq\left(\mathcal{A}_g+\widetilde{\mathcal{A}}_g\right)\left(\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|+\left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|\right). \qquad (3.5)$$

It follows that

$$\left\|\mathcal{K}(\mathfrak{I},\mathcal{Y})-\mathcal{K}\left(\widetilde{\mathfrak{I}},\widetilde{\mathcal{Y}}\right)\right\|\leq\left(\mathcal{A}_f+\widetilde{\mathcal{A}}_f+\mathcal{A}_g+\widetilde{\mathcal{A}}_g\right)\left(\left\|\mathfrak{I}-\widetilde{\mathfrak{I}}\right\|+\left\|\mathcal{Y}-\widetilde{\mathcal{Y}}\right\|\right).$$

Since $\mathcal{A}_f+\widetilde{\mathcal{A}}_f+\mathcal{A}_g+\widetilde{\mathcal{A}}_g<1$, therefore $\mathcal{K}$ is a contraction operator. So, by Banach's fixed point theorem, the operator $\mathcal{K}$ has a unique fixed point. Consequently, if a solution of (1.1) exists, then it must be unique. This completes the proof.

Next we will prove the existence of solutions by applying the Leray-Schauder alternative fixed point theorem.

$$\begin{aligned}
\mathcal{B}_f &= \frac{2}{\Gamma(1+r_1)}\left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)}\right)^{q_1-1}, \qquad (3.6)\\
\overline{\mathcal{B}}_f &= \frac{2\lambda_1}{\Gamma(1+r_1)},\\
\mathcal{B}_g &= \frac{2}{\Gamma(1+r_3)}\left(\frac{\mathcal{N}_g}{\Gamma(1+r_4)}\right)^{q_2-1},\\
\overline{\mathcal{B}}_g &= \frac{2\lambda_2}{\Gamma(1+r_3)}.
\end{aligned}$$

Theorem 3.3. *Assume that $1<p_1,p_2\leq 2$ with $(\mathbb{ASM}2)$ and $(\mathbb{ASM}3)$ holds. In addition it is assumed that*

$$\overline{\mathcal{B}}_f+\overline{\mathcal{B}}_g<1$$

where $\overline{\mathcal{B}}_f$ *and* $\overline{\mathcal{B}}_g$ *are given by* (3.6). *Then,* (1.1) *has at least one solution on* $(0,1]$.

Proof. **Step 1**. To show that $\mathcal{K}$ is continuous, let $(\mathfrak{I}_n, \mathcal{Y}_n)$ be a sequence such that $(\mathfrak{I}_n, \mathcal{Y}_n) \to (\mathfrak{I}, \mathcal{Y}) \in \mathcal{W}$, as $n \to \infty$.

By (3.4) and (3.5) we obtain

$$\left|\mathcal{K}_1\left(\mathfrak{I}_n, \mathcal{Y}_n\right)(\tau) - \mathcal{K}_1\left(\mathfrak{I}, \mathcal{Y}\right)(\tau)\right| \leq \left(\mathcal{A}_f + \widetilde{\mathcal{A}}_f\right)\left(\|\mathfrak{I}_n - \mathfrak{I}\| + \|\mathcal{Y}_n - \mathcal{Y}\|\right),$$

and

$$\left|\mathcal{K}_2\left(\mathfrak{I}_n, \mathcal{Y}_n\right)(\tau) - \mathcal{K}_2\left(\mathfrak{I}, \mathcal{Y}\right)(\tau)\right| \leq \left(\mathcal{A}_g + \widetilde{\mathcal{A}}_g\right)\left(\|\mathfrak{I}_n - \mathfrak{I}\| + \|\mathcal{Y}_n - \mathcal{Y}\|\right).$$

Then

$$\left\|\mathcal{K}\left(\mathfrak{I}_n, \mathcal{Y}_n\right) - \mathcal{K}\left(\mathfrak{I}, \mathcal{Y}\right)\right\| \leq \left(\mathcal{A}_f + \widetilde{\mathcal{A}}_f + \mathcal{A}_g + \widetilde{\mathcal{A}}_g\right)\left(\|\mathfrak{I}_n - \mathfrak{I}\| + \|\mathcal{Y}_n - \mathcal{Y}\|\right),$$

and when $n \to \infty$ we get

$$\mathcal{K}\left(\mathfrak{I}_n, \mathcal{Y}_n\right) \to \mathcal{K}\left(\mathfrak{I}, \mathcal{Y}\right).$$

Consequently, $\mathcal{K}$ is continuous.

Step 2. $\mathcal{K}$ maps bounded sets into bounded sets in $\mathcal{W}$; it suffices to show that for any real constant r there exists a positive constant vector $l = (l_1, l_2)$ such that, for each $(\mathfrak{I}, \mathcal{Y}) \in \mathcal{U}_r = \{(\mathfrak{I}, \mathcal{Y}) \in \mathcal{W} : \|\mathfrak{I}\| \leq r, \|\mathcal{Y}\| \leq r\}$, we have

$$\|\mathcal{K}\left(\mathfrak{I}, \mathcal{Y}\right)\| \leq l.$$

For all $\tau \in (0,1]$, by (3.6), we have

$$\begin{aligned}
&\left|\mathcal{K}_1\left(\mathfrak{I}, \mathcal{Y}\right)(\tau)\right| \\
&\leq \frac{1}{\Gamma(r_1)}\left|\int_0^\tau (\tau - s)^{r_1 - 1}\mathfrak{z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\,dt\right) ds\right| \\
&\quad + \frac{1}{\Gamma(r_1)}\left|\int_0^1 (1 - s)^{r_1 - 1}\mathfrak{z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\,dt\right) ds\right| \\
&\quad + \frac{\lambda_1}{\Gamma(r_1)}\left|\int_0^\tau (\tau - s)^{r_1-1}\mathfrak{I}(s)\,ds\right| + \frac{\lambda_1}{\Gamma(r_1)}\left|\int_0^1 (1-s)^{r_1-1}\mathfrak{I}(s)\,ds\right| \\
&\leq \frac{2}{\Gamma(1+r_1)}\left|\mathfrak{z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\,dt\right)\right| + \frac{2\lambda_1}{\Gamma(1+r_1)}\|\mathfrak{I}\| \\
&\leq \frac{2}{\Gamma(1+r_1)}\left|\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1} f(t, \mathfrak{I}(t), \mathcal{Y}(t))\,dt\right|^{q_1-1} + \frac{2\lambda_1}{\Gamma(1+r_1)}\|\mathfrak{I}\| \\
&\leq \frac{2\left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)}\right)^{q_1-1}}{\Gamma(1+r_1)} + \frac{2\lambda_1}{\Gamma(1+r_1)}\|\mathfrak{I}\| \\
&\leq \mathcal{B}_f + \overline{\mathcal{B}}_f r = l_1.
\end{aligned}$$

Similarly, we have

$$\|\mathcal{K}_2\left(\mathfrak{I}, \mathcal{Y}\right)\| \leq \mathcal{B}_g + \overline{\mathcal{B}}_g r = l_2.$$

Then

$$\|\mathcal{K}\left(\mathfrak{I}, \mathcal{Y}\right)\| \leq (l_1, l_2) = l.$$

Step 3. $\mathcal{K}$ maps bounded sets into equicontinuous sets. Let $(\mathfrak{I},\mathcal{Y})\in\mathcal{U}_r$ be a bounded set as in Step 2. Let $\tau,\widehat{\tau}\in(0,1]$, with $\tau<\widehat{\tau}$, we have

$$
\begin{aligned}
&\left|\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\widehat{\tau})-\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau)\right|\\
&\leq \frac{\left|\mathfrak{z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)\right|}{\Gamma(r_1)}\times\left|\int_0^{\widehat{\tau}}(\widehat{\tau}-s)^{r_1-1}ds-\int_0^{\tau}(\tau-s)^{r_1-1}ds\right|\\
&\quad+\frac{\left|\widehat{\tau}^{r_1-1}-\tau^{r_1-1}\right|}{\Gamma(r_1)}\left|\int_0^1(1-s)^{r_1-1}\mathfrak{z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)ds\right|\\
&\quad+\frac{\lambda_1\|\mathfrak{I}\|}{\Gamma(r_1)}\left|\int_0^{\widehat{\tau}}(\widehat{\tau}-s)^{r_1-1}ds-\int_0^{\tau}(\tau-s)^{r_1-1}ds\right|+\frac{\lambda_1\|\mathfrak{I}\|}{\Gamma(r_1)}\left|\widehat{\tau}^{r_1-1}-\tau^{r_1-1}\right|\\
&\leq\left(\frac{\left(\frac{\mathcal{N}_f}{\Gamma(1+r_2)}\right)^{q_1-1}}{\Gamma(1+r_1)}+\frac{\lambda_1\|\mathfrak{I}\|}{\Gamma(1+r_1)}\right)\left(\left|\widehat{\tau}^{r_1-1}-\tau^{r_1-1}\right|+\left|\widehat{\tau}^{r_1}-\tau^{r_1}\right|\right),
\end{aligned}
$$

then

$$\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\widehat{\tau})-\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau)\to 0,\ \text{ as }\widehat{\tau}\to\tau.$$

Similarly, we have

$$
\begin{aligned}
&\left|\mathcal{K}_2(\mathfrak{I},\mathcal{Y})(\widehat{\tau})-\mathcal{K}_2(\mathfrak{I},\mathcal{Y})(\tau)\right|\\
&\leq\left(\frac{\left(\frac{\mathcal{N}_g}{\Gamma(1+r_4)}\right)^{q_2-1}}{\Gamma(1+r_3)}+\frac{\lambda_2\|\mathcal{Y}\|}{\Gamma(1+r_3)}\right)\left(\left|\widehat{\tau}^{r_3-1}-\tau^{r_3-1}\right|+\left|\widehat{\tau}^{r_3}-\tau^{r_3}\right|\right),
\end{aligned}
$$

then

$$\mathcal{K}_2(\mathfrak{I},\mathcal{Y})(\widehat{\tau})-\mathcal{K}_2(\mathfrak{I},\mathcal{Y})(\tau)\to 0,\ \text{ as }\widehat{\tau}\to\tau.$$

Therefore, by Arzela–Ascoli $\mathcal{K}$ is completely continuous.

Step 4: Boundedness of $\mathcal{K}=\{(\mathfrak{I},\mathcal{Y})\in\mathcal{W}:(\mathfrak{I},\mathcal{Y})=\mu\mathcal{K}(\mathfrak{I},\mathcal{Y}),\mu\in(0,1)\}$. Let $(\mathfrak{I},\mathcal{Y})\in\mathcal{K}$, then

$$\mathfrak{I}(\tau)=\mu\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau),\ \text{ and }\ \mathcal{Y}(\tau)=\mu\mathcal{K}_2(\mathfrak{I},\mathcal{Y})(\tau).$$

By ($\mathbb{ASM}3$), and step 2, and $q_1-1>1$, we have

$$
\begin{aligned}
|\mathfrak{I}(\tau)|=|\mu\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau)|&\leq|\mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau)|\\
&\leq\mathcal{B}_f+\overline{\mathcal{B}}_f\|\mathfrak{I}\|.
\end{aligned}
$$

Hence, we have

$$\|\mathfrak{I}\|\leq\mathcal{B}_f+\overline{\mathcal{B}}_f\|\mathfrak{I}\|,$$

and

$$\|\mathcal{Y}\|\leq\mathcal{B}_g+\overline{\mathcal{B}}_g\|\mathcal{Y}\|,$$

which imply that

$$\|\mathfrak{I}\|+\|\mathcal{Y}\|\leq\mathcal{B}_f+\mathcal{B}_g+(\overline{\mathcal{B}}_f+\overline{\mathcal{B}}_g)(\|\mathfrak{I}\|+\|\mathcal{Y}\|).$$

Consequently, we get

$$\|(\mathfrak{I},\mathcal{Y})\|\leq\frac{\mathcal{B}_f+\mathcal{B}_g}{1-(\overline{\mathcal{B}}_f+\overline{\mathcal{B}}_g)},$$

for any $\tau \in (0,1]$, which proves that $\mathcal{K}$ is bounded. Thus, by Theorem (2.7), the operator $\mathcal{K}$ has at least one fixed point. Hence the proposed system (1.1) has at least one solution. $\square$

4. Hyers-Ulam stability of system (1.1)

This section focuses on the UH-stability of problem (1.1). Therefore, for all $\varepsilon_{(\mathfrak{I})}, \varepsilon_{(\mathcal{Y})} > 0$, we consider the fractional differential inequalities below:

$$\begin{cases} {}^{C}\mathcal{D}^{r_2}\left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}^{r_1}_{0^+}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right) - f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau)) \leq \varepsilon_{(\mathfrak{I})} \\ {}^{C}\mathcal{D}^{r_4}\left(\mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}^{r_3}_{0^+}+\lambda_2\right)\mathcal{Y}(\tau)\right]\right) - g(t,\mathfrak{I}(\tau),\mathcal{Y}(\tau)) \leq \varepsilon_{(\mathcal{Y})} \\ \mathfrak{I}(0)=\mathfrak{I}(1)=\mathcal{Y}(0)=\mathcal{Y}(1)=0, \\ \mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}^{r_1}_{0^+}+\lambda_1\right)\mathfrak{I}(\tau)\right]\Big|_{\tau=0} = \left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}^{r_1}_{0^+}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right)'\Big|_{\tau=0}=0, \\ \mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}^{r_3}_{0^+}+\lambda_2\right)\mathcal{Y}(\tau)\right]\Big|_{\tau=0} = \left(\mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}^{r_3}_{0^+}+\lambda_2\right)\mathcal{Y}(\tau)\right]\right)'\Big|_{\tau=0}=0. \end{cases} \tag{4.1}$$

Remark 4.1. $(\mathfrak{I},\mathcal{Y}) \in \mathcal{W}$ is a solution of inequalities (4.1) if and only if there has a continuous function $\mathcal{H}_1, \mathcal{H}_2 \in C((0,1],\mathbb{R})$ such that

$$\begin{cases} \mathcal{H}_1(\tau) = {}^{C}\mathcal{D}^{r_2}\left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}^{r_1}_{0^+}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right) - f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau)) \\ \mathcal{H}_2(\tau) = {}^{C}\mathcal{D}^{r_4}\left(\mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}^{r_3}_{0^+}+\lambda_2\right)\mathcal{Y}(\tau)\right]\right) - g(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau)) \\ \mathfrak{I}(0)=\mathfrak{I}(1)=\mathcal{Y}(0)=\mathcal{Y}(1)=0, \\ \mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}^{r_1}_{0^+}+\lambda_1\right)\mathfrak{I}(\tau)\right]\Big|_{\tau=0} = \left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}^{r_1}_{0^+}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right)'\Big|_{\tau=0}=0, \\ \mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}^{r_3}_{0^+}+\lambda_2\right)\mathcal{Y}(\tau)\right]\Big|_{\tau=0} = \left(\mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}^{r_3}_{0^+}+\lambda_2\right)\mathcal{Y}(\tau)\right]\right)'\Big|_{\tau=0}=0. \end{cases} \tag{4.2}$$

and

$$\mathcal{H}_1(\tau) \leq \varepsilon_{(\mathfrak{I})}, \text{ and } \quad \mathcal{H}_2(\tau) \leq \varepsilon_{(\mathcal{Y})} \quad \forall \tau \in (0,1]. \tag{4.3}$$

Definition 4.2. The fractional order coupled system (1.1) is Hyers–Ulam stable if there exist $\mathcal{L}_{(\mathfrak{I})}, \mathcal{L}_{(\mathcal{Y})} > 0$, such that, for given $\varepsilon_{(\mathfrak{I})}, \varepsilon_{(\mathcal{Y})} > 0$ and for each solution $(\mathfrak{I},\mathcal{Y}) \in \mathcal{W}$ of inequality (4.2-4.3), there exists a solution $(\mathfrak{I}_1,\mathcal{Y}_1) \in \mathcal{W}$ of problem (1.1) with

$$|(\mathfrak{I},\mathcal{Y})(\tau) - (\mathfrak{I}_1,\mathcal{Y}_1)(\tau)| \leq \varepsilon_{(\mathfrak{I})}\mathcal{L}_{(\mathfrak{I})} + \varepsilon_{(\mathcal{Y})}\mathcal{L}_{(\mathcal{Y})}, \ \forall \tau \in (0,1].$$

Theorem 4.3. *If the assumptions* ($\mathbb{ASM}1$) *and* ($\mathbb{ASM}2$) *are fulfilled, then The fractional order coupled system* (1.1) *with* $\mathfrak{Z}_{p_1}, \mathfrak{Z}_{p_2}$ (*where* $1 < p_1, p_2 \leq 2$) *two operators is Hyers-Ulam stable.*

Proof. Let $(\mathfrak{I}_1,\mathcal{Y}_1) \in \mathcal{W}$ be the solution of the coupled system (1.1) satisfying (3.2) and (3.3) it means that

$$\mathfrak{I}(\tau) = \mathcal{K}_1(\mathfrak{I},\mathcal{Y})(\tau), \quad \text{and} \quad \mathcal{Y}(\tau) = \mathcal{K}_2(\mathfrak{I},\mathcal{Y})(\tau).$$

and let $(\mathfrak{I}, \mathcal{Y})$ be any solution satisfying

$$\begin{cases} {}^{C}\mathcal{D}^{r_2}\left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right)=f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{H}_1(\tau) \\ {}^{C}\mathcal{D}^{r_4}\left(\mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_3}+\lambda_2\right)\mathcal{Y}(\tau)\right]\right)=g(t,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{H}_2(\tau) \\ \mathfrak{I}(0)=\mathfrak{I}(1)=\mathcal{Y}(0)=\mathcal{Y}(1)=0, \\ \mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]\Big|_{\tau=0}=\left(\mathfrak{Z}_{p_1}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_1}+\lambda_1\right)\mathfrak{I}(\tau)\right]\right)'\Big|_{\tau=0}=0, \\ \mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_3}+\lambda_2\right)\mathcal{Y}(\tau)\right]\Big|_{\tau=0}=\left(\mathfrak{Z}_{p_2}\left[\left({}^{RL}\mathcal{D}_{0^+}^{r_3}+\lambda_2\right)\mathcal{Y}(\tau)\right]\right)'\Big|_{\tau=0}=0. \end{cases} \tag{4.4}$$

then the solution of (4.4) is given by

$$\begin{aligned} \mathfrak{I}(\tau) &= \mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{I}^{r_2}\mathcal{H}_1(\tau)\right)\right]-\lambda_1\mathcal{I}^{r_1}\mathfrak{I}(\tau) \\ &\quad+\left[\lambda_1\mathcal{I}^{r_1}\mathfrak{I}(\tau)|_{\tau=1}\right]\tau^{r_1-1}-\left[\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{I}^{r_2}\mathcal{H}_1(\tau)\right)\right]|_{\tau=1}\right]\tau^{r_1-1} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \mathcal{Y}(\tau) &= \mathcal{I}^{r_3}\left[\mathfrak{Z}_{p_2}\left(\mathcal{I}^{r_4}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{I}^{r_4}\mathcal{H}_2(\tau)\right)\right]-\lambda_2\mathcal{I}^{r_3}\mathcal{Y}(\tau) \\ &\quad+\left[\lambda_2\mathcal{I}^{r_3}\mathcal{Y}(\tau)|_{\tau=1}\right]\tau^{r_3-1}-\left[\mathcal{I}^{r_3}\left[\mathfrak{Z}_{p_2}\left(\mathcal{I}^{r_4}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{I}^{r_4}\mathcal{H}_2(\tau)\right)\right]|_{\tau=1}\right]\tau^{r_3-1} \end{aligned} \tag{4.6}$$

It follows that

$$\begin{aligned} &|\mathfrak{I}(\tau)-\mathfrak{I}_1(\tau)| \\ &\le \left|\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))+\mathcal{I}^{r_2}\mathcal{H}_1(\tau)\right)\right]\right. \\ &\qquad \left.-\,\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}_1(\tau),\mathcal{Y}_1(\tau))\right)\right]\right| \\ &\quad+\lambda_1\left|\mathcal{I}^{r_1}\mathfrak{I}(\tau)-\mathcal{I}^{r_1}\mathfrak{I}_1(\tau)\right|+\lambda_1\left|\tau^{r_1-1}\right|\left|\mathcal{I}^{r_1}\mathfrak{I}(\tau)|_{\tau=1}-\mathcal{I}^{r_1}\mathfrak{I}_1(\tau)|_{\tau=1}\right| \\ &\quad+\left|\tau^{r_1-1}\right|\left|\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}_1(\tau),\mathcal{Y}_1(\tau))+\mathcal{I}^{r_2}\mathcal{H}_1(\tau)\right)\right]|_{\tau=1}\right. \\ &\qquad \left.-\,\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right)\right]|_{\tau=1}\right| \\ &\le 2\left|\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}_1(\tau),\mathcal{Y}_1(\tau))+\mathcal{I}^{r_2}\mathcal{H}_1(\tau)\right)\right]\right. \\ &\qquad \left.-\,\mathcal{I}^{r_1}\left[\mathfrak{Z}_{p_1}\left(\mathcal{I}^{r_2}f(\tau,\mathfrak{I}(\tau),\mathcal{Y}(\tau))\right)\right]\right|+\frac{2\lambda_1}{\Gamma(1+r_1)}\left\|\mathfrak{I}-\mathfrak{I}_1\right\|, \end{aligned}$$

so

$$\begin{aligned} &|\mathfrak{I}_1(\tau)-\mathfrak{I}(\tau)| \\ &\le \frac{2}{\Gamma(r_1)}\left|\int_0^\tau(\tau-s)^{r_1-1}\mathfrak{Z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}f(t,\mathfrak{I}_1(t),\mathcal{Y}_1(t))\,dt\right.\right. \\ &\quad\left.+\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}\mathcal{H}_1(s)\,ds\right)ds \\ &\quad\left.-\int_0^\tau(\tau-s)^{r_1-1}\mathfrak{Z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s(s-t)^{r_2-1}f(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt\right)ds\right|+\frac{2\lambda_1}{\Gamma(1+r_1)}\left\|\mathfrak{I}_1-\mathfrak{I}\right\|. \end{aligned}$$

On the other hand $1 < p_1, p_2 \leq 2$ implies $q_1, q_2 > 2$, by Lemma 2.8, we have

$$\begin{aligned}
&\Bigg|\mathfrak{Z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1} f(t,\mathfrak{I}(t),\mathcal{Y}(t))\,dt + \frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1}\mathcal{H}_1(s)\,ds\right)\\
&\quad - \mathfrak{Z}_{p_1}\left(\frac{1}{\Gamma(r_2)}\int_0^s (s-t)^{r_2-1} f(t,\mathfrak{I}_1(t),\mathcal{Y}_1(t))\,dt\right)\Bigg|\\
&\leq \frac{(q_1-1)\,\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_2)}\left(\left|f(t,\mathfrak{I}(t),\mathcal{Y}(t)) - f(t,\mathfrak{I}_1(t),\mathcal{Y}_1(t))\right| + \varepsilon_{(\mathfrak{I})}\right)\\
&\leq \frac{(q_1-1)\,\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_2)}\left(\left|f(t,\mathfrak{I}(t),\mathcal{Y}(t)) - f(t,\mathfrak{I}_1(t),\mathcal{Y}_1(t))\right| + \varepsilon_{(\mathfrak{I})}\right)\\
&\leq \frac{(q_1-1)\,\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_2)}\left(\left|f(t,\mathfrak{I}_1(t),\mathcal{Y}_1(t)) - f(t,\mathfrak{I}(t),\mathcal{Y}(t))\right| + \varepsilon_{(\mathfrak{I})}\right)\\
&\leq \frac{(q_1-1)\,\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_2)}\left(\mathcal{K}_f\left\|\mathfrak{I}_1-\mathfrak{I}\right\| + \widetilde{\mathcal{K}}_f\left\|\mathcal{Y}_1-\mathcal{Y}\right\| + \varepsilon_{(\mathfrak{I})}\right),
\end{aligned}$$

where

$$\Delta_{f,\varepsilon_{(\mathfrak{I})}} = \left(\frac{\varepsilon_{(\mathfrak{I})}+\mathcal{N}_f}{\Gamma(1+r_2)}\right)^{q_1-2}, \qquad \Delta_{g,\varepsilon_{(\mathcal{Y})}} = \left(\frac{\varepsilon_{(\mathcal{Y})}+\mathcal{N}_g}{\Gamma(1+r_4)}\right)^{q_2-2}.$$

Then

$$\begin{aligned}
&\left|\mathfrak{I}_1(\tau)-\mathfrak{I}(\tau)\right|\\
&\leq \frac{2\,(q_1-1)\,\Delta_{f,\varepsilon_{(\mathfrak{I})}}^{(q_1-2)}}{\Gamma(1+r_1)\Gamma(1+r_2)}\left(\mathcal{K}_f\left\|\mathfrak{I}_1-\mathfrak{I}\right\| + \widetilde{\mathcal{K}}_f\left\|\mathcal{Y}_1-\mathcal{Y}\right\| + \varepsilon_{(\mathfrak{I})}\right) + \frac{2\lambda_1}{\Gamma(1+r_1)}\left\|\mathfrak{I}_1-\mathfrak{I}\right\|\\
&\leq \frac{2\,(q_1-1)\,\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_1)\Gamma(1+r_2)}\varepsilon_{(\mathfrak{I})} + \frac{2\,(q_1-1)\,\widetilde{\mathcal{K}}_f\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_1)\Gamma(1+r_2)}\left\|\mathcal{Y}_1-\mathcal{Y}\right\|\\
&\quad + \left[\frac{2\,(q_1-1)\,\mathcal{K}_f\Delta_{f,\varepsilon_{(\mathfrak{I})}}}{\Gamma(1+r_1)\Gamma(1+r_2)} + \frac{2\lambda_1}{\Gamma(1+r_1)}\right]\left\|\mathfrak{I}_1-\mathfrak{I}\right\|\\
&\leq \varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f} + \mathcal{N}_{1f}\left\|\mathfrak{I}_1-\mathfrak{I}\right\| + \widetilde{\mathcal{A}}_f\left\|\mathcal{Y}_1-\mathcal{Y}\right\|.
\end{aligned}$$

In similar way we obtain

$$\begin{aligned}
&\left|\mathcal{Y}(\tau)-\mathcal{Y}_1(\tau)\right|\\
&\leq \frac{2\,(q_2-1)\,\Delta_{g,\varepsilon_{(\mathcal{Y})}}}{\Gamma(1+r_3)\Gamma(1+r_4)}\varepsilon_{(\mathcal{Y})} + \frac{2\,(q_2-1)\,\widetilde{\mathcal{K}}_g\Delta_{g,\varepsilon_{(\mathcal{Y})}}}{\Gamma(1+r_3)\Gamma(1+r_4)}\left\|\mathcal{Y}-\mathcal{Y}_1\right\|\\
&\quad + \left[\frac{2\,(q_2-1)\,\mathcal{K}_g\Delta_{g,\varepsilon_{(\mathcal{Y})}}}{\Gamma(1+r_3)\Gamma(1+r_4)} + \frac{2\lambda_2}{\Gamma(1+r_3)}\right]\left\|\mathfrak{I}_1-\mathfrak{I}\right\|\\
&\leq \varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g} + \mathcal{N}_{1g}\left|\mathfrak{I}_1-\mathfrak{I}\right| + \widetilde{\mathcal{A}}_g\left|\mathcal{Y}_1-\mathcal{Y}\right|,
\end{aligned}$$

with

$$\mathcal{N}_{0f}=\frac{2\left(q_{1}-1\right)\left(\frac{\varepsilon_{(\mathfrak{I})}+\mathcal{N}_{f}}{\Gamma(1+r_{2})}\right)^{(q_{1}-2)}}{\Gamma(1+r_{1})\Gamma(1+r_{2})},$$
$$\mathcal{N}_{0g}=\frac{2\left(q_{2}-1\right)\left(\frac{\varepsilon_{(\mathcal{Y})}+\mathcal{N}_{g}}{\Gamma(1+r_{4})}\right)^{(q_{2}-2)}}{\Gamma(1+r_{3})\Gamma(1+r_{4})},$$
$$\mathcal{N}_{1f}=\widetilde{\mathcal{A}}_{f}+\frac{2\lambda_{1}}{\Gamma(1+r_{1})},\qquad \mathcal{N}_{1g}=\widetilde{\mathcal{A}}_{g}+\frac{2\lambda_{2}}{\Gamma(1+r_{3})}.$$

Hence, we get

$$\varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f}\geq\left(1-\mathcal{N}_{1f}\right)\left\|\mathfrak{I}_{1}-\mathfrak{I}\right\|-\widetilde{\mathcal{A}}_{f}\left\|\mathcal{Y}_{1}-\mathcal{Y}\right\|, \tag{4.7}$$

and

$$\varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g}\geq\left(1-\widetilde{\mathcal{A}}_{g}\right)\left\|\mathcal{Y}_{1}-\mathcal{Y}\right\|-\mathcal{N}_{1g}\left\|\mathfrak{I}_{1}-\mathfrak{I}\right\|,$$

The matrix representation of (4.7) is as follows

$$\begin{pmatrix}(1-\mathcal{N}_{1f}) & -\widetilde{\mathcal{A}}_{f}\\ -\mathcal{N}_{1g} & \left(1-\widetilde{\mathcal{A}}_{g}\right)\end{pmatrix}\begin{pmatrix}\|\mathfrak{I}_{1}-\mathfrak{I}\|\\ \|\mathcal{Y}_{1}-\mathcal{Y}\|\end{pmatrix}\leq\begin{pmatrix}\varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f}\\ \varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g}\end{pmatrix},$$

which implies that

$$\begin{aligned}\begin{pmatrix}\|\mathfrak{I}_{1}-\mathfrak{I}\|\\ \|\mathcal{Y}_{1}-\mathcal{Y}\|\end{pmatrix}&\leq\underbrace{\begin{pmatrix}(1-\mathcal{N}_{1f}) & -\widetilde{\mathcal{A}}_{f}\\ -\mathcal{N}_{1g} & \left(1-\widetilde{\mathcal{A}}_{g}\right)\end{pmatrix}}_{\mathcal{M}}{}^{-1}\begin{pmatrix}\varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f}\\ \varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g}\end{pmatrix} \qquad (4.8)\\ &\leq\begin{pmatrix}\frac{\left(1-\widetilde{\mathcal{A}}_{g}\right)}{\det\mathcal{M}} & \frac{\widetilde{\mathcal{A}}_{f}}{\det\mathcal{M}}\\ \frac{\mathcal{N}_{1g}}{\det\mathcal{M}} & \frac{\left(1-\mathcal{N}_{1f}\right)}{\det\mathcal{M}}\end{pmatrix}\begin{pmatrix}\varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f}\\ \varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g}\end{pmatrix}\\ &\leq\begin{pmatrix}\dfrac{\varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f}\left(1-\widetilde{\mathcal{A}}_{g}\right)}{\det\mathcal{M}}+\dfrac{\varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g}\widetilde{\mathcal{A}}_{f}}{\det\mathcal{M}}\\ \dfrac{\varepsilon_{(\mathfrak{I})}\mathcal{N}_{0f}\mathcal{N}_{1g}}{\det\mathcal{M}}+\dfrac{\varepsilon_{(\mathcal{Y})}\mathcal{N}_{0g}\left(1-\mathcal{N}_{1f}\right)}{\det\mathcal{M}}\end{pmatrix},\end{aligned}$$

where $\det\mathcal{M}=1-\mathcal{N}_{1f}-\mathcal{N}_{2g}+\mathcal{N}_{1f}\mathcal{N}_{2g}-\widetilde{\mathcal{A}}_{f}\mathcal{N}_{1g}\neq 0$.

Therefore, from inequality (4.8), we have

$$\begin{aligned}\|\mathfrak{I}_{1}-\mathfrak{I}\|+\|\mathcal{Y}_{1}-\mathcal{Y}\|&\leq\frac{\mathcal{N}_{0f}\left(1-\widetilde{\mathcal{A}}_{g}\right)+\mathcal{N}_{0f}\mathcal{N}_{1g}}{\det\mathcal{M}}\varepsilon_{(\mathfrak{I})} \qquad (4.9)\\ &\quad+\frac{\mathcal{N}_{0g}\left(1-\mathcal{N}_{1f}\right)+\mathcal{N}_{0g}\widetilde{\mathcal{A}}_{f}}{\det\mathcal{M}}\varepsilon_{(\mathcal{Y})}\\ &\leq\mathcal{L}_{(\mathfrak{I})}\varepsilon_{(\mathfrak{I})}+\mathcal{L}_{(\mathcal{Y})}\varepsilon_{(\mathcal{Y})}.\end{aligned}$$

Thus, it follows from the inequalities (4.9) that the coupled system (1.1) is generalized Ulam–Hyers stable. □

5. An explanatory example

Example 5.1. Consider the following coupled system of mixed fractional derivatives between the Riemann–Liouville and the Caputo fractional derivatives of different orders with two different Laplacian operators

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{3}{2}}\left(\mathfrak{Z}_{\frac{4}{3}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{\frac{5}{4}}+\frac{1}{90}\right)\mathfrak{I}(x)\right]\right)=f(x,\mathfrak{I}(x),\mathcal{Y}(x)), \quad x\in(0,1] \\ {}^{C}\mathcal{D}^{\frac{7}{4}}\left(\mathfrak{Z}_{\frac{8}{7}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{\frac{6}{5}}+\frac{1}{100}\right)\mathcal{Y}(x)\right]\right)=g(x,\mathfrak{I}(x),\mathcal{Y}(x)), \quad x\in(0,1] \\ \mathfrak{I}(0)=\mathfrak{I}(1)=\mathcal{Y}(0)=\mathcal{Y}(1)=0, \\ \mathfrak{Z}_{\frac{4}{3}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{\frac{5}{4}}+\frac{1}{90}\right)\mathfrak{I}(x)\right]\Big|_{x=0}=\left(\mathfrak{Z}_{\frac{4}{3}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{\frac{5}{4}}+\frac{1}{90}\right)\mathfrak{I}(x)\right]\right)'\Big|_{x=0}=0, \\ \mathfrak{Z}_{\frac{8}{7}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{\frac{6}{5}}+\frac{1}{100}\right)\mathcal{Y}(x)\right]\Big|_{x=0}=\left(\mathfrak{Z}_{\frac{8}{7}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{\frac{6}{5}}+\frac{1}{100}\right)\mathcal{Y}(x)\right]\right)'\Big|_{x=0}=0. \end{cases} \tag{5.1}$$

where

$$f(x,\mathfrak{I}(x),\mathcal{Y}(x))=\left(\frac{x}{8}\right)^5+\left(\frac{x^2+2}{24}\right)^5\sin\mathfrak{I}(x)+\frac{x^8}{512\,(x+8)^2\sqrt{|\mathcal{Y}(x)|+1}},$$

$$g(x,\mathfrak{I}(x),\mathcal{Y}(x))=\left(\frac{x^3}{9}\right)^7+\frac{x^4\cos\mathfrak{I}(x)}{9^7}+\frac{\sin\mathcal{Y}(x)}{(x^4+9)^7}.$$

Then, f and g satisfy the assumptions $(\mathbb{ASM}1)$, $(\mathbb{ASM}2)$ and $(\mathbb{ASM}3)$, we have

$$\mathcal{N}_f=\frac{3}{8^5},\ \text{and}\ \mathcal{K}_f=\widetilde{\mathcal{K}}_f=\frac{1}{8^5},$$
$$\mathcal{N}_g=\frac{3}{9^7},\ \text{and}\ \mathcal{K}_g=\widetilde{\mathcal{K}}_g=\frac{1}{9^7}$$

and

$$\left.\begin{aligned} \mathcal{A}_f&=0.01961355823 \\ \widetilde{\mathcal{A}}_f&=5.766393277\times10^{-13} \\ \mathcal{A}_g&=0.01815207368 \\ \widetilde{\mathcal{A}}_g&=5.810146726\times10^{-45} \end{aligned}\right\}\Rightarrow \mathcal{A}_f+\widetilde{\mathcal{A}}_f+\mathcal{A}_g+\widetilde{\mathcal{A}}_g=0.03776563191<1,$$

also

$$\left.\begin{aligned} \mathcal{B}_f&=5.766393277\times10^{-13} \\ \overline{\mathcal{B}}_f&=0.01961355823 \\ \mathcal{B}_g&=2.490062881\times10^{-45} \\ \overline{\mathcal{B}}_g&=0.01815207368 \end{aligned}\right\}\Rightarrow \overline{\mathcal{B}}_f+\overline{\mathcal{B}}_g=0.03776563191<1.$$

Then the conditions of Theorem 3.2 and Theorem 3.3 are satisfied and so the problem (5.1) has a unique solution on $(0,1]$.

Example 5.2. We consider the following multi-term system of mixed fractional derivatives between the Riemann–Liouville and the Caputo fractional derivatives of different orders with ten different Laplacian operators

$$\begin{cases} {}^{C}\mathcal{D}^{1+\frac{1}{2i}}\left(\mathfrak{Z}_{1+\frac{1}{i}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{2-\frac{1}{2i}}+\frac{1}{100i}\right)\mathfrak{I}_i(x)\right]\right)=f_i(x,\mathfrak{I}_1(x),\cdots,\mathfrak{I}_5(x)),\\ \mathfrak{Z}_{1+\frac{1}{i}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{2-\frac{1}{2i}}+\frac{1}{100i}\right)\mathfrak{I}_i(x)\right]\Big|_{x=0}\\ =\left(\mathfrak{Z}_{1+\frac{1}{i}}\left[\left({}^{RL}\mathcal{D}_{0^+}^{2-\frac{1}{2i}}+\frac{1}{100i}\right)\mathfrak{I}_i(x)\right]\right)'\Big|_{x=0}=0,\\ \mathfrak{I}_i(0)=\mathfrak{I}_i(1)=0,\quad x\in(0,1],\quad i=1,2,\ldots,5 \end{cases} \tag{5.2}$$

where

$$f_i(x,\mathfrak{I}_1(x),\cdots,\mathfrak{I}_5(x))=x^i+\left(\frac{1}{20}\right)^i\sum_{j=1}^{5}\frac{|\mathfrak{I}_j(x)|}{1+|\mathfrak{I}_j(x)|},\qquad i=1,2,\ldots,5.$$

Then, f_i satisfy the assumptions $(\mathbb{ASM}1_n)$, $(\mathbb{ASM}2_n)$ and $(\mathbb{ASM}3_n)$, for $i,j=1,\ldots,5$, we have

$$\mathcal{N}_{f_i}=1+5\left(\frac{1}{20}\right)^i,\qquad \forall j\in\{1,2,3,4,5\},\quad \mathcal{K}_{f_i}^j=\left(\frac{1}{20}\right)^i$$

i	1	2	3	4	5
$\mathcal{A}_{f_i}$	0.00218910	0.00280178	0.00401218	0.0101316	0.080289
$\widetilde{\mathcal{A}}_{f_i}$	0.00218956	0.00281646	0.00445143	0.0218737	0.276024
$\mathcal{B}_{f_i}$	0.0981561	0.195839	0.390683	0.792592	1.631118
$\overline{\mathcal{B}}_{f_i}$	0.00218895	0.00279688	0.00386577	0.00621750	0.0150451

so

$$\sum_{i=1}^{5}\left(\mathcal{A}_{f_i}+\widetilde{\mathcal{A}}_{f_i}\right)=0.40677873<1.$$

Then the problem (5.2) has a unique solution on $(0,1]$, and we have

$$\sum_{i=1}^{5}\overline{\mathcal{B}}_{f_i}=0.03011420<1.$$

6. Conclusion

In this study, we conducted an exploration into the existence and uniqueness of solutions for a novel class of coupled systems and their generalizations. This investigation encompasses mixed fractional derivatives that integrate Riemann-Liouville and Caputo fractional derivatives of different orders, alongside the p_i-Laplacian operator. Through the use of the Banach contraction principle and the Leray-Schauder alternative fixed point theorem, we obtained important findings regarding the existence and uniqueness of solutions for the system described in (1.1). Additionally, we examined the Ulam-Hyers stability of the system, which enhances our comprehension of its behavior. To further illustrate our findings, we provided a comprehensive example that highlights the theory and key results of this work. This research contributes valuable insights to the field and opens avenues for future exploration.

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