

TWO STABLE MIXED FINITE ELEMENT APPROXIMATIONS FOR THE OBSTACLE PROBLEM OF THE EULER-BERNOULLI BEAM

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ABSTRACT. We present two finite element approximations of a mixed formulation for the obstacle problem of the Euler-Bernoulli beam problem. The two numerical approximations are based on conforming methods with dual Lagrange multiplier. The first uses a suitable stable pair of finite element spaces. The second proposed scheme employs biorthogonal basis functions. We derive a priori and a posteriori error estimates. Numerical tests using a primal-dual active set algorithm and the classical Uzawa iteration are given that validate and illustrate our approach.

Keywords. Fourth order variational inequalities, Mixed Finite element method, a priori estimates, a posteriori estimate

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1. INTRODUCTION

The Euler-Bernoulli model is one of the most widely used mathematical models for thin, slender bodies. The Euler-Bernoulli model (without obstacle) can be derived as a special case by dimension reduction from three-dimensional hyperelasticity using the gamma-convergence technique (see [15], [3]). Starting from three-dimensional elasticity (Signorini problem), the unilateral constraint case was rigorously justified by means of asymptotic methods in [17]. It can also be seen as the limit of the Timoshenko beam when the thickness approaches zero (see [20]).

The mathematical analysis of the continuous problem for fourth-order variational inequalities can be done in several standard ways. However, their numerical counterparts face numerous difficulties. First, it is not easy to develop an effective discretization scheme. Within the finite element framework, several standard approaches exist. We could use conforming finite element methods. But even for the equality case, conforming methods for fourth-order problems require C^1 elements, which are complicated, and this gets even more complicated when we consider higher-dimensional problems. We could also consider non-conforming methods, but they do not naturally form hierarchies. Hence, they are not efficient for capturing smoother solutions. Moreover, it requires considerable ingenuity to develop non-conforming methods that work for general problems, and very little is known about non-conforming elements in higher dimensions (see [19]). We could also use mixed finite element methods by splitting the fourth-order equation into two second-order equations. For some model problems, the mixed formulation can miss the leading singularity, and solutions obtained from such mixed methods do not converge to the exact solution

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(see [2]). Also, it is not easy to find finite element pairs that satisfy the Babuška-Brezzi inf-sup condition for stability. Even if we have the correct mixed formulation and identify a stable finite element pair, we still need to solve a saddle point problem. In contrast, the original problem is symmetric positive definite. Another major difficulty is that the discrete problems we obtain are more ill-conditioned than those for second-order problems. For the fourth-order problems, the condition number grows at the order of h^{-4} , instead of h^{-2} for second-order problems. We need discretizations that overcome the drawbacks of standard methods while allowing fast solvers to handle the ill-conditioning of our discrete problems.

Furthermore, for the fourth-order variational inequalities, the situation is even worse. The exact solution is sought in a closed, convex set, and it is challenging to construct finite element spaces incorporating the necessary constraints, namely, the inequality constraints, for a general convex set. Despite being a classical problem, research on fourth-order variational inequalities remains an area of active interest, as evidenced by recent contributions (see [5]). In general, for fourth-order variational inequalities, the complementarity system of the variational inequality exists only in the weak sense. The easiest way to overcome this difficulty is to use penalisation, but the condition number of the matrix grows as $\sim \varepsilon^{-1} \times h^{-4}$, revealing a multiplicative effect of the penalization parameter and the mesh size (see [14]) and for the penalty method it is not easy to find the correct penalty parameter ([18]). An efficient approach to overcoming the shortcomings of standard methods is the so-called C^0 -interior penalty method [5]. It is a discontinuous Galerkin method for fourth-order problems that uses standard finite element spaces for second-order problems. This method has some advantages over classical approaches for fourth-order problems. At the end of this introduction, let us mention the regularity issue for fourth-order variational inequalities. Here, ψ represents a rigid obstacle and f is the external load. The complementarity system plays a crucial role in the analysis of such a variational inequality. Unfortunately, for fourth-order variational inequalities, the full H^4 -regularity of the solution is not always fulfilled (see [5],[10]). But, if $u \in H^4(\Omega)$, then u satisfies the complementarity system of the variational inequality in a strong sense:

$$\begin{cases} u'''' - f \geq 0 & \text{a.e. on } \Omega, \\ u - \psi \geq 0 & \text{a.e. on } \Omega, \\ (u'''' - f)(u - \psi) = 0 & \text{a.e. on } \Omega, \end{cases} \quad (1.1)$$

and in that case, the numerical analysis of the obstacle problem can proceed as in the second-order case. However, in general $u \notin H_{\text{loc}}^4(\Omega)$ even if Ω, f and ψ are smooth (see [8]). The exact solution given in Section 2.2 below, indicates that the smoothness threshold is $C^{2,1/2}(\Omega)$ or $H^{7/2-\varepsilon}(\Omega)$, $\varepsilon > 0$. The absence of H_{loc}^4 regularity implies that the variational inequality's complementarity system exists only in a weak sense; moreover, $u'''' - f$ is only a Borel measure, which complicates the numerical analysis of the fourth-order obstacle problem.

In this paper, we will present two efficient approaches: a mixed conforming method based on a stable pair of spaces [10] and a variationally consistent method that relies on a biorthogonal basis for the multiplier [18], and derive both a priori and a posteriori error estimates.

The paper is organized as follows: Section 2 introduces the model and establishes the well-posedness and regularity of the solution. In Section 3, we present the typical conforming finite element scheme and a priori error estimate in the case of full regularity. Section 4 presents the first mixed finite element scheme and establishes its stability, plus a priori error analysis and a posteriori error analysis of residual type for this scheme. Next,

in Section 5, we introduce the second scheme, the new variationally consistent scheme, and show its stability. Two numerical approximation algorithms are described in Section 6. Finally, comprehensive numerical simulations are presented in Section 7.

2. THE CONTINUOUS PROBLEM

Let $\Omega = (0, \ell)$, $\ell > 0$, $f \in L^2(\Omega)$ and let ψ represent a rigid obstacle. We introduce the obstacle problem associated with the Euler–Bernoulli beam. ¹:

$$\begin{cases} \text{Find } u \in \mathcal{K} \text{ satisfies:} \\ \int_0^\ell u''(v'' - u'') dx \geq \int_0^\ell f(v - u) dx, \text{ for all } v \in \mathcal{K} \end{cases} \quad (2.1)$$

where,

$$\mathcal{K} := \{v \in V := H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \quad (2.2)$$

2.1. Existence and uniqueness.

We prove the existence and uniqueness of the solution to (2.1), by applying Stampacchia's theorem (see [6] for instance). Since the problem (2.1) is equivalent to the problem of minimizing the functional

$$J(v) := \frac{1}{2} \int_\Omega |v''|^2 dx - \int_\Omega f v dx$$

over the closed convex set $\mathcal{K}$, then we can apply the direct method of calculus of variations to establish the existence of a solution. The main ingredients for the proof are the strict convexity and continuity of J that imply the weak lower semicontinuity and weak closedness of $\mathcal{K}$.

Theorem 2.1. *Problem (2.1) admits a unique solution.*

2.2. Regularity of the solution. It is possible to show that solutions of problem (2.1) obey higher regularity properties. If the data are sufficiently smooth, namely, $f \in L^2(\Omega)$ and $\psi \in H^3(\Omega)$, then continuity of the second derivative of the solution, i.e., $u \in H^3(\Omega)$, can be proved.

Proposition 2.2. *Let $f \in L^2(0, \ell)$ and $\psi \in H^3(0, \ell)$. Then the solution u of (2.1), belongs to $H^3(0, \ell)$.*

Proof. Let us first denote $\mathcal{C}$ to the contact set, i.e $\mathcal{C} := \{x \in \Omega; u(x) = \psi(x)\}$ and $\mathcal{C}^c$ the noncontact set, i.e $\mathcal{C}^c := \{x \in \Omega; u(x) > \psi(x)\}$. Then we have

$$\begin{cases} u = \psi & \text{in } \mathcal{C} \\ u'''' = f & \text{in } \mathcal{C}^c \end{cases} \quad (2.3)$$

It thus suffices to consider a point $\xi \in \mathcal{F} = \partial\mathcal{N} \cap \mathcal{C}$ and show that u'' is continuous at ξ . By definition there exists $\varepsilon > 0$ such that $u(\xi - x) = \psi(\xi - x)$ and $u(\xi + x) > \psi(\xi + x)$ either for all $-\varepsilon < x \leq 0$ or for all $0 \leq x < \varepsilon$. Without loss of generality, we consider the first case. For every nonnegative function $\phi \in C^2(\Omega)$ with $\text{supp } \phi \subset B_\varepsilon(\xi)$, we have $u + \phi \geq \psi$ and thus

$$\begin{aligned} 0 &\leq \int_{\xi-\varepsilon}^{\xi+\varepsilon} u'' \phi'' dx - \int_{\xi-\varepsilon}^{\xi+\varepsilon} f \phi dx \\ &= \int_{\xi}^{\xi+\varepsilon} \psi'' \phi'' dx - \int_{\xi}^{\xi+\varepsilon} f \phi dx + (\psi''(\xi) - u''(\xi)) \phi'(\xi). \end{aligned}$$

¹Since the constants do not influence the mathematical analysis, they are omitted in what follows.

Since $\phi \geq 0$ is arbitrary, by taking the limit $\varepsilon \rightarrow 0$, we find that

$$\psi''(\xi) \leq u''(\xi) \quad (2.4)$$

Note that the inequality (2.4) holds for higher dimension see [8]. The function $w = (u - \psi)'$ is nonpositive², continuous and satisfies :

$$w(\xi) = 0 \quad \text{and} \quad w'(\xi) \geq 0, \quad (\text{by (2.4)})$$

Then ξ is a local maximum for the function w . Thus, we must have $w'(\xi) = 0$, i.e.,

$$u''(\xi) = \psi''(\xi)$$

which implies that u'' is continuous at ξ . □

The previous proposition claims that under reasonable smoothness assumptions, the solution of (2.1) does belong to the Sobolev space $H^3(\Omega)$. However, the analytic expression of the solution given in the next example seems to indicate that, in general, the solution fails to belong to the Sobolev space $H^4(\Omega)$, even for very smooth data.

Example 2.3. Let $\Omega = (0, 1)$, $f = 0$ and $\psi(x) = 4x^3 - 9x^2 + 6x - 1$, then the closed convex set $\mathcal{K}$ reads,

$$\mathcal{K} := \{v \in H_0^2(0, 1), \quad v(x) \geq \psi(x)\}.$$

The exact solution is given by,

$$u(x) = \begin{cases} -4x^3 + 3x^2 & \text{if } 0 \leq x \leq 0.5 \\ 4x^3 - 9x^2 + 6x - 1 & \text{if } 0.5 \leq x \leq 1 \end{cases} \quad (2.5)$$

$$u''(x) = \begin{cases} -24x + 6 & \text{if } 0 \leq x \leq 0.5 \\ +24x - 18 & \text{if } 0.5 \leq x \leq 1 \end{cases} \quad (2.6)$$

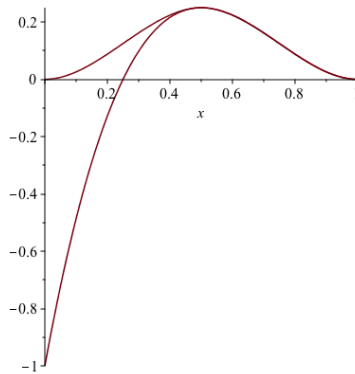


FIGURE 1. The solution $u(x)$ and the obstacle $\psi(x)$.

A straightforward calculation shows that the smoothness threshold is $C^{2,1/2}(\bar{\Omega})$ or $H^{7/2-\varepsilon}(\Omega)$, $\varepsilon > 0$, but not $C^3(\bar{\Omega})$.

²Since w is positive and decreasing to zero.

3. FINITE ELEMENT APPROXIMATION

Let's consider a discretisation of the interval $[0, \ell]$,

$$0 = x_0 < x_1 < \dots < x_n = \ell, \quad x_i - x_{i-1} = h_i.$$

and the finite space

$$V_h := \{v_h \in V : v_h|_{[x_{i-1}, x_i]} \in \mathbb{P}_3([x_{i-1}, x_i]), \forall i = 1, \dots, n\} = \mathbb{P}_3^{\text{Hermit}} \cap H_0^2(\Omega) \quad (3.1)$$

Recall that for the reference element $[-1, 1]$, the Hermite basis ϕ_k and $\psi_k, k = 0, 1$ can be obtained via Lagrange's interpolants. Indeed, let

$$L_0(x) = \frac{1}{2}(1-x) \quad \text{and} \quad L_1(x) = \frac{1}{2}(x+1)$$

be the Lagrange basis. We consider the polynomials ϕ_k and $\psi_k, k = 0, 1$, defined by

$$\phi_k(x) = [L_k(x)]^2 (1 - 2L'_k(x_k)(x - x_k)), \quad (3.2)$$

$$\psi_k(x) = [L_k(x)]^2 (x - x_k). \quad (3.3)$$

Then

$$\phi_0(x) = \frac{1}{4}(1-x)^2[1 + (x+1)], \quad (3.4)$$

$$\phi_1(x) = \frac{1}{4}(x+1)^2[1 - (x-1)], \quad (3.5)$$

$$\psi_0(x) = \frac{1}{4}(1-x)^2(x+1), \quad (3.6)$$

$$\psi_1(x) = \frac{1}{4}(x+1)^2(x-1). \quad (3.7)$$

3.1. A priori error estimate in case of full regularity. Let us assume $\psi \in H^4(\Omega)$. The convex $\mathcal{K}$ can be represented approximately by:

$$\mathcal{K}_h = \{v_h \in V_h : v_h \geq \psi_h\}$$

where, $\psi_h = \mathcal{I}_h^H(\psi)$ represents the Hermite interpolant of ψ in V_h . It's straightforward to check that $\mathcal{K}_h$ is a non-empty, closed, convex set in V_h .

We define the discrete problem as follows

$$\begin{cases} \text{Find } u_h \in \mathcal{K}_h \text{ such that} \\ a(u_h, v_h - u_h) \geq \langle f, v_h - u_h \rangle \quad \forall v_h \in \mathcal{K}_h \end{cases} \quad (3.8)$$

Theorem 3.1. *The problem (3.8) is well-posed.*

Proof. Apply Stampacchia's theorem. □

Theorem 3.2. *Let u and u_h denote the solutions of the continuous problem (2.1) and the discrete problem (3.8), respectively. Suppose that $u \in H^4(\Omega)$, $f \in L^2(\Omega)$, and $\psi \in H^4(\Omega)$. Then, there exists a positive constant C such that:*

$$\|u - u_h\|_{H^2(\Omega)} \leq Ch^2 (|u|_{H^4(\Omega)} + \|f\|_{L^2(\Omega)} + |\psi|_{H^4(\Omega)}).$$

Proof. It is well known that $\exists C > 0$ such that the Hermite interpolant satisfies:

$$\|(w - \mathcal{I}_h^H(w))''\|_{L^2(\Omega)} = |w - \mathcal{I}_h^H(w)|_{H^2(\Omega)} \leq Ch^2 |w|_{H^4(\Omega)}, \quad \forall w \in H^4(\Omega) \quad (3.9)$$

$$|w - \mathcal{I}_h^H(w)|_{H^1(\Omega)} \leq Ch^3 |w|_{H^4(\Omega)}, \quad \forall w \in H^4(\Omega) \quad (3.10)$$

$$|w - \mathcal{I}_h^H(w)|_{L^2(\Omega)} \leq Ch^4 |w|_{H^4(\Omega)}, \quad \forall w \in H^4(\Omega) \quad (3.11)$$

To obtain an a priori error estimate, we proceed as in the obstacle problem case.

$$\begin{aligned} |u - u_h|_{H^2(\Omega)}^2 &= a(u - u_h, u - u_h) \\ &= \underbrace{a(u - u_h, u - \mathcal{I}_h^H u)}_I + \underbrace{a(u - u_h, \mathcal{I}_h^H u - u_h)}_{II} \end{aligned}$$

For term I, using the Cauchy-Schwarz inequality and the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we obtain:

$$\begin{aligned} a(u - u_h, u - \mathcal{I}_h^H u) &\leq \frac{|u - u_h|_{H^2(\Omega)}^2}{2} + \frac{|u - \mathcal{I}_h^H u|_{H^2(\Omega)}^2}{2} \\ &\leq \frac{|u - u_h|_{H^2(\Omega)}^2}{2} + C h^4 |u|_{H^4(\Omega)}^2 \end{aligned}$$

For term II, we have,

$$a(u - u_h, \mathcal{I}_h^H u - u_h) = a(u, \mathcal{I}_h^H u - u_h) - a(u_h, \mathcal{I}_h^H u - u_h).$$

Using the definition of the discrete problem (3.8) we have:

$$-a(u_h, \mathcal{I}_h^H u - u_h) \leq -\langle f, \mathcal{I}_h^H u - u_h \rangle.$$

Then, we get,

$$\begin{aligned} a(u - u_h, \mathcal{I}_h^H u - u_h) &\leq a(u, \mathcal{I}_h^H u - u_h) - \langle f, \mathcal{I}_h^H u - u_h \rangle \\ &= (u'', (\mathcal{I}_h^H u - u_h)'') - \langle f, \mathcal{I}_h^H u - u_h \rangle \\ &= (u''''', \mathcal{I}_h^H u - u_h) - \langle f, \mathcal{I}_h^H u - u_h \rangle \\ &= (\lambda, \mathcal{I}_h^H u - u_h) \end{aligned}$$

where $\lambda = u'''' - f \geq 0$,

$$\begin{aligned} (\lambda, \mathcal{I}_h^H u - u_h) &= (\lambda, \mathcal{I}_h^H(u - \psi) - (u - \psi)) \\ &\quad + (\lambda, u - \psi) \\ &\quad + (\lambda, \mathcal{I}_h^H \psi - u_h) \end{aligned}$$

we have $\mathcal{I}_h^H \psi - u_h \leq 0$ and $\lambda \geq 0$ then,

$$(\lambda, \mathcal{I}_h^H \psi - u_h) \leq 0$$

Since $(\lambda, u - \psi) = 0$ so it is sufficient to bound the first term:

$$(\lambda, (u - \psi) - \mathcal{I}_h^H(u - \psi))$$

Since $u \in H^4(\Omega)$, it follows that $\lambda \in L^2(\Omega)$, thus

$$\begin{aligned} (\lambda, (u - \psi) - \mathcal{I}_h^H(u - \psi)) &\leq \|\lambda\|_{L^2(\Omega)} \|(u - \psi) - \mathcal{I}_h^H(u - \psi)\|_{L^2(\Omega)} \\ &\leq C \|\lambda\|_{L^2(\Omega)} h^4 |u - \psi|_{H^4(\Omega)}, \quad (\text{from (3.11)}) \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{|u - u_h|_{H^2(\Omega)}^2}{2} &\leq C \left(h^4 |u|_{H^4(\Omega)}^2 + h^0 \|\lambda\|_{L^2(\Omega)} h^4 |u - \psi|_{H^4(\Omega)} \right) \\
&= C \left(h^4 |u|_{H^2(\Omega)}^2 + h^2 \|\lambda\|_{L^2(\Omega)} h^2 |u - \psi|_{H^4(\Omega)} \right) \\
&\leq C \left(h^4 |u|_{H^4(\Omega)}^2 + \frac{h^4 \|\lambda\|_{L^2(\Omega)}^2}{2} + \frac{h^4 |u - \psi|_{H^4(\Omega)}^2}{2} \right) \\
&\leq C \left(h^4 |u|_{H^4(\Omega)}^2 + \frac{h^4 \left(\|f\|_{L^2(\Omega)}^2 + |u|_{H^4(\Omega)}^2 \right)}{2} + \frac{h^4 \left(|u|_{H^4(\Omega)}^2 + |\psi|_{H^4(\Omega)}^2 \right)}{2} \right).
\end{aligned}$$

Hence,

$$\|u - u_h\|_{H^2(\Omega)} \leq C h^2 \left(|u|_{H^4(\Omega)} + |\psi|_{H^4(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

□

The a priori result obtained in Theorem 3.2 is based on the H^4 regularity assumption; without this assumption, the result is false. However, even for very smooth data, namely, constant loads and constant obstacles, the H^4 -regularity is not fulfilled. Indeed, for constant $f < 0$ and $\psi < 0$, the analytic expression of the solution derived in [4] shows that the third derivative of the solution exhibits a jump of Heaviside type at the free boundary. However, specific manufactured solutions can be constructed such that the H^4 regularity is satisfied. As can be shown by the following example.

Example 3.3. Consider the Euler–Bernoulli obstacle problem on the interval $\Omega = (0, 1)$.

$$u'''' \geq f \quad \text{in } (0, 1), \quad (3.12)$$

$$u \geq \psi \quad \text{in } (0, 1), \quad (3.13)$$

$$(u'''' - f)(u - \psi) = 0 \quad \text{in } (0, 1), \quad (3.14)$$

with the boundary conditions

$$u(0) = u(1) = 0.008 \quad (3.15)$$

$$u'(0) = -0.108, u'(1) = 0.108 \quad (3.16)$$

With $\psi = 0$, $\xi = 0.3$ and the loading f given by

$$f(x) = \begin{cases} 24, & x < \xi, \\ 0, & \xi \leq x \leq 1 - \xi, \\ 24, & x > 1 - \xi. \end{cases}$$

Then, the exact solution reads:

$$u(x) = \begin{cases} (\xi - x)^4, & 0 \leq x < \xi, \\ 0, & \xi \leq x \leq 1 - \xi, \\ (x - (1 - \xi))^4, & 1 - \xi < x \leq 1. \end{cases}$$

This solution is of class $C^3[0, 1]$, hence it belongs to $H^4(0, 1)$. Note that, for this example, the Lagrange multiplier vanishes throughout Ω , indicating that the obstacle is inactive and the variational inequality has the same solution as the corresponding variational equality; this situation is very particular and rarely encountered in practice.

4. MIXED FORMULATION

Since the full regularity of the solution is not always guaranteed, the last a priori error estimate does not hold in general. Therefore, we introduce an alternative formulation of the problem, following [9, 10], which exploits the Lagrange multiplier from the complementarity system, which should be understood in the weak sense, since $u^{(4)} - f \in H^{-2}(\Omega)$ is a positive measure and, in general, does not belong to $L^2(\Omega)$. The corresponding equivalent (weak) complementary form can then be expressed as follows:

$$\begin{cases} u^{(4)} - f \geq 0 \text{ a.e in } \Omega \\ u - \psi \geq 0 \text{ a.e in } \Omega \\ \langle u^{(4)} - f, u - \psi \rangle = 0 \text{ a.e in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \quad (4.1)$$

Let us introduce the following spaces

$$\begin{aligned} V &:= H_0^2(\Omega), \\ Q &:= V' = H^{-2}(\Omega) \end{aligned}$$

and the closed convex set

$$\Lambda := \{\mu \in Q : \langle \mu, v \rangle \geq 0, \quad \forall v \in V, \quad v \geq 0 \text{ a.e in } \Omega\}$$

Let λ denote the Lagrange multiplier, defined by

$$\langle \lambda, v \rangle = a(u, v) - \mathcal{L}(v), \quad \forall v \in V.$$

Therefore we obtain, for $v \in V, v \geq 0$ a.e in Ω ,

$$\begin{cases} u^{(4)} - f = \lambda \text{ a.e in } \Omega \\ \langle \lambda, v \rangle \geq 0 \text{ a.e in } \Omega \\ u - \psi \geq 0 \text{ a.e in } \Omega \\ \langle \lambda, u - \psi \rangle = 0 \text{ a.e in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \quad (4.2)$$

Through standard manipulations, one can readily verify that (4.2) is equivalent to the following mixed variational inequality:

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda \text{ such that:} \\ a(u, v) - \langle \lambda, v \rangle = (f, v) \quad \forall v \in V \\ \langle \mu - \lambda, u \rangle \geq \langle \mu - \lambda, \psi \rangle \quad \forall \mu \in \Lambda \end{cases} \quad (4.3)$$

Problem (2.1) and problem (4.3) are equivalent in following sense [7]

Theorem 4.1. *If (u, λ) is a solution of (4.3), then u also solves (2.1).*

Now we prove the well-posedness of the mixed variational inequality.

Theorem 4.2. *The problem (4.3) admits a unique solution (u, λ) .*

Proof. We need to establish the existence of a constant $\beta > 0$ such that

$$\inf_{\mu \in Q} \sup_{v \in V} \frac{\langle \mu, v \rangle}{\|\mu\|_Q \|v\|_V} \geq \beta, \quad \mu, v \neq 0. \quad (4.4)$$

Indeed, let $\mu \in Q = (H_0^2(\Omega))'$ with $\mu \neq 0$. By Riesz representation theorem, there exists $v_\mu \neq 0 \in H_0^2(\Omega) = V$ such that:

$$\forall v \in V : \langle \mu, v \rangle = (v, v_\mu)_V \quad \text{and} \quad \|v_\mu\|_V = \|\mu\|_Q.$$

Then

$$\sup_{v \in V} \langle \mu, v \rangle \geq \langle \mu, v_\mu \rangle = (v_\mu, v_\mu)_V = \|v_\mu\|_V^2 = \|v_\mu\|_V \|\mu\|_Q.$$

Therefore,

$$\inf_{\mu \in Q} \sup_{v \in V} \frac{\langle \mu, v \rangle}{\|v\|_V \|\mu\|_Q} \geq 1, \quad \mu, v \neq 0.$$

□

4.1. Compact reformulation. Let $\mathcal{H} := V \times Q$. We define the bilinear form $\mathcal{A} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and the linear form $\mathcal{L} : \mathcal{H} \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} \mathcal{A}((v, \mu); (w, \xi)) &= a(v, w) - \langle \mu, w \rangle - \langle \xi, v \rangle = (v'', w'') - \langle \mu, w \rangle - \langle \xi, v \rangle \\ \mathcal{L}(w, \xi) &= (f, w) - \langle \xi, \psi \rangle \end{aligned}$$

Problem (4.3) can be rewritten in the following compact form:

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda & \text{such that} \\ \mathcal{A}((u, \lambda); (v, \mu - \lambda)) \leq \mathcal{L}(v, \mu - \lambda), & \forall (v, \mu) \in V \times \Lambda \end{cases} \quad (4.5)$$

Theorem 4.3. *For all $(v, \mu) \in V \times Q$ there exists $w \in V$ such that :*

$$\mathcal{A}((v, \mu); (w, -\mu)) \gtrsim (\|v\|_2 + \|\mu\|_{-2})^2 \quad (4.6)$$

$$\|w\|_2 \lesssim \|v\|_2 + \|\mu\|_{-2} \quad (4.7)$$

Proof. Let the pair $(v, \mu) \in \mathcal{H}$. By Riesz representation theorem, there exists a unique element $q \in V$ such that

$$\langle \mu, z \rangle = (q, z)_V = (q'', z'') + (q', z') + (q, z) \quad \forall z \in V. \quad (4.8)$$

$$\|q\|_2 = \|\mu\|_{-2}. \quad (4.9)$$

We take $w = v - q$. Using (4.8), (4.9), Poincaré's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} \mathcal{A}((v, \mu); (w, -\mu)) &= (v'', w'') - \langle \mu, w \rangle + \langle \mu, v \rangle \\ &= (v'', v'') - (v'', q'') + \langle \mu, q \rangle \\ &\geq \|v''\|_0^2 - \|v''\|_0 \|q''\|_0 + (q, q)_V \\ &\geq |v|_2^2 - |v|_2 \|q\|_2 + \|q\|_2^2 \\ &\geq \frac{1}{2} (|v|_2 + \|\mu\|_{-2})^2. \end{aligned} \quad (4.10)$$

The inequality (4.7) follows from the triangle inequality

$$\|w\|_2 = \|v - q\|_2 \leq \|v\|_2 + \|q\|_2 = \|v\|_2 + \|\mu\|_{-2}.$$

□

4.2. The discrete problem for mixed-compact formulation. Let us define the finite-dimensional space Q_h and the convex set Λ_h as follows:

$$\begin{cases} Q_h := \{\mu_h \in Q : \mu_h|_{[x_{i-1}, x_i]} \in \mathbb{P}_0([x_{i-1}, x_i]), \forall i = 1, \dots, n\} \\ \Lambda_h := \{\mu_h \in Q_h : \mu_h \geq 0 \quad \text{in } \Omega\}. \end{cases} \quad (4.11)$$

We consider the discrete problem of (4.5)

$$\begin{cases} \text{Find } (u_h, \lambda_h) \in V_h \times \Lambda_h \text{ s.t.:} \\ \mathcal{A}((u_h, \lambda_h); (v_h, \mu_h - \lambda_h)) \leq \mathcal{L}(v_h, \mu_h - \lambda_h) \quad \forall (v_h, \mu_h) \in V_h \times \Lambda_h \end{cases} \quad (4.12)$$

To ensure stability of the mixed finite element method, the spaces must satisfy the Babuška–Brezzi (inf-sup) condition:

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \gtrsim \|\xi_h\|_{-2} \quad (4.13)$$

To prove that the inf-sup condition holds for the chosen spaces, we introduce the following discrete H^{-2} norm:

$$\|\xi_h\|_{-2,h}^2 = \sum_{i=1}^n h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 = (h^2 \|\xi_h\|_0)^2, \quad \forall \xi_h \in Q_h. \quad (4.14)$$

Remark 4.4. The discrete norm (4.14) is related to the continuous norm $\|\cdot\|_{-2}$ via an inverse estimate in the the following sense, $\|\xi_h\|_{-2,h} \lesssim \|\xi_h\|_{-2}$ for all $\xi_h \in Q_h$. The factor h^4 (for the square of the norm) is a direct consequence of the standard inverse estimate in finite element spaces. It will be used in an intermediate step to prove the inf-sup condition. We begin by showing stability in the discrete norm (4.14), and then apply Lemma 4.7 (below) to get the stability in the continuous norm. The advantage of using the mesh-dependent norm is that the discrete negative norm can be computed elementwise, whereas the continuous norm is global.

Lemma 4.5. (Inverse inequality). There exists $C > 0$, such that

$$\|v_h\|_2 \leq Ch^{-2} \|v_h\|_0, \quad \forall v \in V_h. \quad (4.15)$$

Proof. It is well known that there exists a constant $c > 0$ such that

$$\|v_h'\|_0 \leq ch^{-1} \|v_h\|_0.$$

Therefore

$$\|v_h\|_2 \leq c_p \|v_h\|_2 = c_p \|v_h''\|_0 \leq cc_p h^{-1} \|v_h'\|_0 \leq \underbrace{c^2 c_p}_{=C} h^{-2} \|v_h\|_0.$$

□

Lemma 4.6. There exists two positive constants C_1, C_2 such that:

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h} \quad (4.16)$$

Proof. We recall the definition of the negative norm:

$$\|\xi\|_{-2} := \sup_{v \in V} \frac{\langle \xi, v \rangle}{\|v\|_V}, \quad \forall \xi \in Q. \quad (4.17)$$

Since $Q_h \subset Q$, it follows that for all $\xi_h \in Q_h$ (4.17) implies the existence of $v \in H_0^2(\Omega)$, $v \neq 0$, and a constant $C > 0$ such that :

$$\langle \xi_h, v \rangle \geq C \|v\|_2 \|\xi_h\|_{-2}.$$

Let v_h denote the Hermite interpolant of v . Then we obtain :

$$\begin{aligned} \langle \xi_h, v_h \rangle &= \langle \xi_h, v_h - v \rangle + \langle \xi_h, v_h \rangle \\ &\geq \langle \xi_h, v_h - v \rangle + C \|v\|_2 \|\xi_h\|_{-2} \\ &\geq - (h^{-2} \|v_h - v\|_0) (h^2 \|\xi_h\|_0) + C \|v\|_2 \|\xi_h\|_{-2} \\ &= \|v\|_2 \left(C \|\xi_h\|_{-2} - \frac{h^{-2} \|v_h - v\|_0}{\|v\|_2} \|\xi_h\|_{-2,h} \right). \end{aligned}$$

We recall the following properties of the Hermite interpolant: there exist constants $K_1, K_2 > 0$ such that

$$\|v_h - v\|_0 \leq K_1 h^2 \|v\|_2, \quad \|v_h\|_2 \leq K_2 \|v\|_2.$$

It then follows that

$$\frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h}.$$

Therefore

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h}.$$

□

Lemma 4.7. *There exists a constant $C_3 > 0$ such that*

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C_3 \|\xi_h\|_{-2,h}. \quad (4.18)$$

Proof. Let $\{\chi_i\}_{i=1}^n$, where $\chi_i = 1_{]x_{i-1}, x_i[}$, be the basis of Q_h , and let $\xi_h \in Q_h$ be written as

$$\xi_h = \sum_{k=1}^n \xi_h^k \chi_k.$$

We distinguish two cases:

i) Case $n = 2k$. We choose $v_h \in V_h$ as

$$v_h = h^4 \sum_{i=1}^k (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1}.$$

Hence we obtain

$$\begin{aligned} \langle \xi_h, v_h \rangle &= h^4 \sum_{i=1}^k \langle \xi_h^{2i-1} \chi_{2i-1} + \xi_h^{2i} \chi_{2i}, (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1} \rangle \\ &= h^4 \sum_{i=1}^k \xi_h^{2i-1} \int_{x_{2i-2}}^{x_{2i-1}} (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1} dx \\ &\quad + \xi_h^{2i} \int_{x_{2i-1}}^{x_{2i}} (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1} dx \\ &= h^4 \sum_{i=1}^k \xi_h^{2i-1} \int_{x_{2i-2}}^{x_{2i-1}} \frac{1}{2} (\xi_h^{2i-1} + \xi_h^{2i}) - \frac{1}{12} (\xi_h^{2i} - \xi_h^{2i-1}) dx \\ &\quad + \xi_h^{2i} \int_{x_{2i-1}}^{x_{2i}} \frac{1}{2} (\xi_h^{2i-1} + \xi_h^{2i}) + \frac{1}{12} (\xi_h^{2i} - \xi_h^{2i-1}) dx \\ &\gtrsim h^4 \sum_{i=1}^k h (\xi_h^{2i-1} + \xi_h^{2i})^2 + h (\xi_h^{2i} - \xi_h^{2i-1})^2 \\ &= h^4 \sum_{i=1}^k h ((\xi_h^{2i-1})^2 + (\xi_h^{2i})^2) \\ &= h^4 \sum_{i=1}^k \int_{x_{2i-2}}^{x_{2i}} (\xi_h)^2 dx \\ &= \|\xi_h\|_{-2,h}^2 \end{aligned}$$

At the same time

$$\begin{aligned}
\|v_h\|_0^2 &= h^8 \sum_{i=1}^k \int_{x_{2i-2}}^{x_{2i}} \left((\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1} \right)^2 dx \\
&= h^8 \sum_{i=1}^k \int_{x_{2i-2}}^{x_{2i}} (\xi_h^{2i-1} + \xi_h^{2i})^2 (\Phi_{2i-1})^2 + \left(\frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \right)^2 (\Psi_{2i-1})^2 dx \\
&= h^8 \sum_{i=1}^k \int_{x_{2i-2}}^{x_{2i}} \frac{26}{35} (\xi_h^{2i-1} + \xi_h^{2i})^2 + \frac{2}{105} (\xi_h^{2i} - \xi_h^{2i-1})^2 dx \\
&\lesssim h^8 \sum_{i=1}^k \int_{x_{2i-2}}^{x_{2i}} (\xi_h^{2i-1} + \xi_h^{2i})^2 + (\xi_h^{2i} - \xi_h^{2i-1})^2 dx \\
&\lesssim h^4 \|\xi_h\|_{-2,h}^2.
\end{aligned}$$

By using the estimate from (4.15), we get

$$\|v_h\|_2 \leq Ch^{-2} \|v_h\|_0 \lesssim \|\xi_h\|_{-2,h}^2$$

Hence we deduce

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_{2,\Omega}} \geq C \|\xi_h\|_{-2,h}^2$$

ii) Case $n = 2k + 1$. For any $\xi_h \in Q_h$ we have either

$$\sum_{i=1}^{2k} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 \geq \frac{1}{2} \|\xi_h\|_{-2,h}^2 \quad \text{or} \quad \sum_{i=2}^{2k+1} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 \geq \frac{1}{2} \|\xi_h\|_{-2,h}^2$$

In the first case, we choose the same $v_h \in V_h$ as in the previous construction, and in the second, we take

$$v_h = h^4 \sum_{i=1}^k (\xi_h^{2i} + \xi_h^{2i+1}) \Phi_{2i} + \frac{\xi_h^{2i+1} - \xi_h^{2i}}{h} \Psi_{2i}.$$

In either case, we obtain

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_{2,\Omega}} \geq \frac{1}{2} C \|\xi_h\|_{-2,h}^2.$$

Finally taking $C_3 = \frac{1}{2}C$, then for all $n \geq 2$

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_{2,\Omega}} \geq C_3 \|\xi_h\|_{-2,h}^2.$$

□

We now have all the prerequisites to prove the inf-sup condition for the norm $\|\cdot\|_{-2}$

Theorem 4.8. *There exists a constant $C > 0$, such that for any $\xi_h \in Q_h$ we have*

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C \|\xi_h\|_{-2}.$$

Proof. Let $\alpha > 0$, combining Lemma (4.6) and Lemma (4.7), we obtain

$$\begin{aligned} \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} &= \alpha \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} + (1 - \alpha) \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \\ &\geq \alpha \left(C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h} \right) + (1 - \alpha) C_3 \|\xi_h\|_{-2,h} \\ &= \alpha C_1 \|\xi_h\|_{-2} + (C_3 - \alpha (C_2 + C_1)) \|\xi_h\|_{-2,h}. \end{aligned} \quad (4.19)$$

If we take $\alpha = \frac{C_3}{C_1 + C_2}$, the second term vanishes, and we then obtain

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq \frac{C_1 C_3}{C_1 + C_2} \|\xi_h\|_{-2}.$$

□

Theorem 4.9. *The problem (4.12) admits a unique solution.*

Proof. Since we have:

$$\inf_{\xi_h \in Q_h} \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2 \|\xi_h\|_{-2}} \geq C > 0. \quad (4.20)$$

It then follows that the mixed finite element problem admits a unique solution (u_h, λ_h) (see [9, 16]). □

4.3. A priori error estimate. First, we need the following theorem about the discrete stability estimate, implied by the inf-sup condition

Theorem 4.10. *For all $(v_h, \xi_h) \in \mathcal{H}$, there exists $w_h \in V_h$ such that :*

$$\mathcal{A}((v_h, \xi_h); (w_h, -\xi_h)) \gtrsim (\|v_h\|_2 + \|\xi_h\|_{-2})^2 \quad (4.21)$$

$$\|w_h\|_2 \lesssim \|v_h\|_2 + \|\xi_h\|_{-2} \quad (4.22)$$

Proof. Let $\xi_h \in Q_h$. Let us consider the following auxiliary problem:

$$\begin{cases} \text{Find } y_h \in V_h \text{ s.t} \\ a(y_h, z_h) + (y_h, z_h)_{H^1} = \langle \xi_h, z_h \rangle, \quad \forall z_h \in V_h \end{cases}$$

This is a standard elliptic variational equality, which admits a unique solution satisfying

$$\|y_h\|_2^2 = \langle \xi_h, y_h \rangle.$$

Since the inf-sup condition (4.13) is satisfied, it then follows that

$$\|\xi_h\|_{-2} \lesssim \sup_{z_h \in V_h} \frac{\langle \xi_h, z_h \rangle}{\|z_h\|_V} = \sup_{z_h \in V_h} \frac{a(y_h, z_h) + (y_h, z_h)_{H^1}}{\|z_h\|_V} \leq \|y_h\|_2.$$

However

$$\|y_h\|_2 = \frac{\|y_h\|_2^2}{\|y_h\|_2} = \frac{\langle \xi_h, y_h \rangle}{\|y_h\|_2} \leq \frac{\|\xi_h\|_{-2} \|y_h\|_2}{\|y_h\|_2} = \|\xi_h\|_{-2}.$$

Now, if we take $w_h = v_h - y_h$,

$$\begin{aligned}
\mathcal{A}((v_h, \xi_h); (w_h, -\xi_h)) &= \mathcal{A}((v_h, \xi_h); (v_h - y_h, -\xi_h)) \\
&= a(v_h, v_h) - a(v_h, y_h) - \langle \xi_h, v_h - y_h \rangle + \langle \xi_h, v_h \rangle \\
&= a(v_h, v_h) - a(v_h, y_h) + \langle \xi_h, y_h \rangle \\
&\geq \|v_h''\|_0^2 - \int_{\Omega} v_h'' y_h'' + \|y_h\|_2^2 \\
&\geq \frac{1}{2} (\|v_h\|_0^2 + \|y_h\|_0^2) \\
&\gtrsim \frac{1}{2} (\|v_h\|_2^2 + \|y_h\|_2^2) \\
&\gtrsim (\|v_h\|_2^2 + \|\xi_h\|_{-2}^2)
\end{aligned}$$

and we have

$$\|w_h\|_2 \leq \|v_h\|_2 + \|y_h\|_2 \lesssim \|v_h\|_2 + \|\xi_h\|_{-2}.$$

□

Theorem 4.11. *Let (u, λ) be the solution of the continuous problem (4.5), and let (u_h, λ_h) be the solution of the discrete problem (4.12). Then, the following error estimate holds:*

$$\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_2 + \inf_{\xi_h \in \Lambda_h} \left(\|\lambda - \xi_h\|_{-2} + \sqrt{\langle \xi_h, u - \psi \rangle} \right) \quad (4.23)$$

Proof. Let $(v_h, \xi_h) \in V_h \times \Lambda_h$. By the previous theorem, there exists $w_h \in V_h$ such that

$$\|w_h\|_1 \lesssim \|u_h - v_h\|_1 + \|\lambda_h - \xi_h\|_{-1}$$

and

$$(\|u_h - v_h\|_1 + \|\lambda_h - \xi_h\|_{-1})^2 \lesssim \mathcal{A}((u_h - v_h, \lambda_h - \xi_h); (w_h, \xi_h - \lambda_h)).$$

From the discrete problem and the bilinearity of $\mathcal{A}$, we obtain

$$\begin{aligned}
\mathcal{A}((u_h - v_h, \lambda_h - \xi_h); (w_h, \xi_h - \lambda_h)) &= \mathcal{A}((u_h, \lambda_h); (w_h, \xi_h - \lambda_h)) - \mathcal{A}((v_h, \xi_h); (w_h, \xi_h - \lambda_h)) \\
&\leq \mathcal{A}((u - v_h, \lambda - \xi_h); (w_h, \xi_h - \lambda_h)) + \mathcal{L}(w_h, \xi_h - \lambda_h) \\
&\quad - \mathcal{A}((u, \lambda); (w_h, \xi_h - \lambda_h)) \\
&= \mathcal{A}((u - v_h, \lambda - \xi_h); (w_h, \xi_h - \lambda_h)) + \langle \xi_h - \lambda_h, u - \psi \rangle \\
&= \mathcal{A}((u - v_h, \lambda - \xi_h); (w_h, \xi_h - \lambda_h)) + \langle \xi_h, u - \psi \rangle \\
&\quad + \underbrace{\langle -\lambda_h, u - \psi \rangle}_{\leq 0} \\
&\leq \mathcal{A}((u - v_h, \lambda - \xi_h); (w_h, \xi_h - \lambda_h)) + \langle \xi_h, u - \psi \rangle
\end{aligned}$$

The bilinear form $\mathcal{A}$ is continuous, therefore

$$\begin{aligned}
\mathcal{A}((u - v_h, \lambda - \xi_h); (w_h, \xi_h - \lambda_h)) &\leq (\|u - v_h\|_2 + \|\lambda - \xi_h\|_{-2}) (\|w_h\|_2 + \|\lambda_h - \xi_h\|_{-2}) \\
&\lesssim (\|u - v_h\|_2 + \|\lambda - \xi_h\|_{-2}) (\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2})
\end{aligned}$$

By combining the previous estimates, we obtain

$$\begin{aligned}
& (\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2})^2 \\
& \lesssim \mathcal{A}((u - v_h, \lambda - \xi_h); (w_h, \xi_h - \lambda_h)) + \langle \xi_h, u - \psi \rangle \\
& \lesssim (\|u - v_h\|_2 + \|\lambda - \xi_h\|_{-2}) (\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2}) + \langle \xi_h, u - \psi \rangle \\
& \leq \frac{(\|u - v_h\|_2 + \|\lambda - \xi_h\|_{-2})^2}{2} + \frac{(\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2})^2}{2} + \langle \xi_h, u - \psi \rangle \\
& \lesssim \frac{(\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2})^2}{2} + \left(\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2} + \sqrt{\langle \xi_h, u - \psi \rangle} \right)^2.
\end{aligned}$$

This implies

$$\|u_h - v_h\|_2 + \|\lambda_h - \xi_h\|_{-2} \lesssim \|u - v_h\|_2 + \|\lambda - \xi_h\|_{-2} + \sqrt{\langle \xi_h, u - \psi \rangle}.$$

The triangle inequality gives:

$$\begin{aligned}
\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} & \leq \|u - v_h\|_2 + \|u_h - v_h\|_2 + \|\lambda - \xi_h\|_{-1} + \|\lambda_h - \xi_h\|_{-2} \\
& \lesssim \|u - v_h\|_2 + \|\lambda - \xi_h\|_{-1} + \sqrt{\langle u - \psi, \xi_h \rangle}.
\end{aligned}$$

Finally, since the choice of (v_h, ξ_h) was arbitrary we conclude

$$\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_2 + \inf_{\xi_h \in \Lambda_h} \left(\|\lambda - \xi_h\|_{-2} + \sqrt{\langle \xi_h, u - \psi \rangle} \right).$$

□

4.4. A posteriori Error Estimate. Next, we derive an a posteriori error estimate. To this end, we define the following error indicators:

$$\begin{aligned}
\eta_i^2 &= h^4 \|\lambda_h + f\|_{0, [x_{i-1}, x_i]}^2, \quad \eta^2 = \sum_{i=1}^{n-1} \eta_i^2 \\
S^2 &= h^{-4} \|(\psi - u_h)_+\|_0^2 + \langle \lambda_h, (\psi - u_h)_+ \rangle, \quad \text{where } (\psi - u_h)_+ = \max\{\psi - u_h, 0\}
\end{aligned}$$

We also define the (squared) local oscillation of f on $I_i =]x_{i-1}, x_i[$

$$\text{osc}_i(f)^2 := h_i^2 \inf_{q \in \mathbb{P}_0} \|f - q\|_{L^2(I_i)}^2.$$

and the global oscillation of f

$$\text{osc}(f) = \left(\sum_i \text{osc}_i(f)^2 \right)^{1/2}.$$

We now introduce and prove some useful lemmas.

Lemma 4.12. *There exists $w \in V$ and $w_h \in V_h$ such that*

$$\|u - u_h\|_2^2 + \|\lambda - \lambda_h\|_{-2}^2 \lesssim \mathcal{L}(w - w_h, \lambda_h - \lambda) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) \quad (4.24)$$

Proof. We know from theorem (4.10), there exists $w \in V$ such that:

$$\mathcal{A}((u - u_h, \lambda - \lambda_h); (w, \lambda_h - \lambda)) \gtrsim (\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2})^2.$$

Let w_h denote the Hermite interpolant of w . It then follows that

$$\begin{aligned}
& (\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2})^2 \\
& \lesssim \mathcal{A}((u - u_h, \lambda - \lambda_h); (w, \lambda_h - \lambda)) \\
& = \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w, \lambda_h - \lambda)) \\
& \lesssim \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w, \lambda_h - \lambda)) + \underbrace{\mathcal{L}(-w_h, 0) - \mathcal{A}((u_h, \lambda_h); (-w_h, 0))}_{\geq 0} \\
& = \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) + \mathcal{L}(-w_h, 0) \\
& = \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) + \mathcal{L}(-w_h, 0) \\
& \quad - \mathcal{L}(w, \lambda_h - \lambda) + \mathcal{L}(w, \lambda_h - \lambda) \\
& = \underbrace{\mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{L}(w, \lambda_h - \lambda)}_{\leq 0} - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) + \mathcal{L}(w - w_h, 0) \\
& \leq \mathcal{L}(w - w_h, \lambda_h - \lambda) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda))
\end{aligned}$$

□

Lemma 4.13. *The following estimate holds:*

$$\langle \lambda_h - \lambda, u_h - \psi \rangle \leq \|(\psi - u_h)_+\|_2 \|\lambda - \lambda_h\|_{-2} + \langle \lambda_h, (\psi - u_h)_+ \rangle \quad (4.25)$$

Proof. If we take $v_h = 0$ in the discrete problem (4.12) we get:

$$-\langle \mu_h - \lambda_h, u_h \rangle \leq -\langle \mu_h - \lambda_h, \psi_h \rangle \quad (4.26)$$

For the choices $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (4.26) we find that : $\langle u_h - \psi, \lambda_h \rangle = 0$. Hence,

$$\begin{aligned}
\langle \lambda_h - \lambda, u_h - \psi \rangle &= \langle \lambda, \psi - u_h \rangle \\
&\leq \langle \lambda - \lambda_h, (\psi - u_h)_+ \rangle + \langle \lambda_h, (\psi - u_h)_+ \rangle \\
&\leq \|(\psi - u_h)_+\|_2 \|\lambda - \lambda_h\|_{-2} + \langle \lambda_h, (\psi - u_h)_+ \rangle
\end{aligned}$$

□

In the following, we prove that the proposed error indicators are both reliable and efficient.

Theorem 4.14. (*Reliability*)

The following posteriori error estimate holds

$$\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} \lesssim \eta + S \quad (4.27)$$

Proof. From Lemma: 4.12

$$\begin{aligned}
& (\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2})^2 \\
& \lesssim \mathcal{L}(w - w_h, \lambda_h - \lambda) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) \\
& = (f, w - w_h) - \langle \psi, \lambda_h - \lambda \rangle - a(u_h, w - w_h) + \langle \lambda_h, w - w_h \rangle + \langle \lambda_h - \lambda, u_h \rangle
\end{aligned}$$

We have:

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} u'''' v dx &= \int_{x_i}^{x_{i+1}} (u''')' v dx = \left[u''' v \right]_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} u''' v' dx \\
&= \left[u''' v \right]_{x_i}^{x_{i+1}} - \left[u'' v' \right]_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} u'' v'' dx
\end{aligned}$$

We also have from the proprieties Hermite interpolant that $w(x_i) = w_h(x_i)$, $w'(x_i) = w'_h(x_i)$, and we know $u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}^3$.

Therefore

$$a(u_h, w - w_h) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u_h''(w - w_h)'' dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u_h''''(w - w_h) dx = 0$$

Consequently, by using lemma 4.13

$$\begin{aligned} (\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2})^2 &\lesssim (f + \lambda_h, w - w_h) + \langle \lambda_h - \lambda, u_h - \psi \rangle \\ &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} h^2 (f + \lambda_h) h^{-2} (w - w_h) dx + \langle \lambda_h - \lambda, u_h - \psi \rangle \\ &\leq \left(\sum_{i=0}^{n-1} \eta_i^2 \right)^{1/2} \|w - w_h\|_0 + \|(\psi - u_h)_+\|_2 \|\lambda - \lambda_h\|_{-2} \\ &\quad + \langle \lambda_h, (\psi - u_h)_+ \rangle \\ &\lesssim \left(\sum_{i=0}^{n-1} \eta_i^2 \right)^{1/2} \|w\|_2 + \frac{\|(\psi - u_h)_+\|_2^2}{2\varepsilon} + \frac{\varepsilon \|\lambda - \lambda_h\|_{-2}^2}{2} \\ &\quad + \langle \lambda_h, (\psi - u_h)_+ \rangle \end{aligned}$$

Recall estimates of the Hermite interpolant

$$\|w_h\|_2 \leq C\|w\|_2 \quad (4.28)$$

$$h^{-2} \|w - w_h\|_0 \leq C\|w\|_2 \quad (4.29)$$

and we also have

$$\|w\|_2 \lesssim \|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2}$$

Using this last inequality, we obtain the desired result. $\square$

Theorem 4.15. (efficiency) *The following estimate holds:*

$$\eta \lesssim \|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} + \text{osc}(f) \quad (4.30)$$

Proof. We use the bubble functions $b_i \in H_0^2([x_{i-1}, x_i])$, and we introduce the functions γ_i defined by

$$\gamma_i = h^4 b_i (\lambda_h + f_h) \text{ in } [x_{i-1}, x_i] \quad \text{and} \quad \gamma_i = 0 \quad \text{in } \Omega \setminus [x_{i-1}, x_i]$$

By choosing $v_h = \gamma_i$ in (4.3), we obtain

$$(u'', \gamma_i'') - \langle \lambda, \gamma_i \rangle = (f, \gamma_i) \quad (4.31)$$

Using the above and the characteristics of bubble functions, we obtain

$$\begin{aligned} h^4 \|\lambda_h + f_h\|_{0, [x_{i-1}, x_i]}^2 &\lesssim h^4 \left\| \sqrt{b_i} (\lambda_h + f_h) \right\|_{0, [x_{i-1}, x_i]}^2 \\ &= (\lambda_h + f_h, \gamma_i) \\ &= (\lambda_h, \gamma_i) + (f, \gamma_i) + (f_h - f, \gamma_i) \\ &= (u_h^{(4)} + \lambda_h, \gamma_i) + (u'', \gamma_i'') - \langle \lambda, \gamma_i \rangle + (f_h - f, \gamma_i) \\ &= ((u - u_h)'', \gamma_i'') + \langle \lambda_h - \lambda, \gamma_i \rangle + (f_h - f, \gamma_i) \end{aligned}$$

Let $\gamma = \sum_{i=1}^{n-1} \gamma_i$. Summing over $i = 1, \dots, n-1$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n-1} h^4 \|\mu_h + f_h\|_0^2 &\lesssim \sum_{i=1}^{n-1} \{((u - v_h)'', \gamma_i'') + \langle \lambda_h - \lambda, \gamma_i \rangle + (f_h - f, \gamma_i)\} \\
&= ((u - v_h)'', \gamma'') + \langle \lambda_h - \lambda, \gamma \rangle + (f_h - f, \gamma) \\
&\leq \|u - u_h\|_2 \|\gamma\|_2 + \|\lambda_h - \lambda\|_{-2} \|\gamma\|_2 + \text{osc}(f) \left(\sum_{i=1}^{n-1} h^{-4} \|\gamma\|_{0,[x_{i-1}, x_i]}^2 \right)^{\frac{1}{2}} \\
&\lesssim (\|u - u_h\|_2 + \|\lambda_h - \lambda\|_{-2} + \text{osc}(f)) h^{-2} \|\gamma\|_0
\end{aligned} \tag{4.32}$$

However:

$$\begin{aligned}
h^{-4} \|\gamma\|_0^2 &= h^{-4} \sum_{i=1}^{n-1} \|\gamma_i\|_0^2 \\
&= h^4 \sum_{i=1}^{n-1} \|b_i(\lambda_h + f_h)\|_0^2 \\
&\lesssim \sum_{i=1}^{n-1} h^4 \|\lambda_h + f_h\|_0^2
\end{aligned} \tag{4.33}$$

Thus we conclude

$$\eta \lesssim \|u - u_h\|_2 + \|\lambda_h - \lambda\|_{-2} + \text{osc}(f)$$

□

5. VARIATIONAL CONSISTENT SCHEME

In this section, we introduce a new variationally consistent formulation in the spirit of [18]. We apply a similar idea used to derive a bi-orthogonal basis for Lagrange elements to construct a bi-orthogonal basis for the $\mathbb{P}_3$ Hermite elements. This significantly reduces the algorithmic complexity for the implementation of variational inequalities. Let us motivate why bi-orthogonal bases are so important. The mixed formulation involves two spaces, V and Q , which must be discretized carefully. For our fourth-order contact problem, these are typically H^2 and H^{-2} . Stability requires that these spaces satisfy an inf-sup condition; otherwise, spurious oscillations or nonexistence of solutions may occur. The resulting algebraic system includes inequality constraints involving a mass matrix. In large-scale problems, this matrix can be very large. Although it is sparse and symmetric positive definite, its inverse is generally dense. To allow pointwise treatment of inequalities, the mass matrix must be diagonal. This can be achieved through mass lumping or by using bi-orthogonal basis functions.

Let V_h be the $\mathbb{P}_3$ -Hermite finite element space spanned by:

$$\mathcal{B} = \{\Phi_i, \Psi_i : i = 1 \dots n-1\},$$

and let Q_h the dual space spanned by

$$\mathcal{B} = \{\tilde{\Phi}_i, \tilde{\Psi}_i : i = 1 \dots n-1\}.$$

The basis is required to satisfy the following properties :

i) Locality of the support

$$\begin{cases} \text{supp } \tilde{\Phi}_p = \text{supp } \Phi_p \\ \text{supp } \tilde{\Psi}_p = \text{supp } \Psi_p \end{cases} \quad (5.1)$$

ii) Local biorthogonality relation (here δ_{ij} is the usual Kronecker delta)

$$\begin{cases} \int \Phi_i \tilde{\Phi}_j = \delta_{ij} \\ \int \Psi_i \tilde{\Psi}_j = \delta_{ij} \\ \int \Phi_i \tilde{\Psi}_j = 0 \\ \int \Psi_i \tilde{\Phi}_j = 0 \end{cases} \quad (5.2)$$

iii) Best approximation property:

$$\inf_{\mu_h \in Q_h} \|\mu - \mu_h\| \lesssim h^2 |\mu|_2 \quad (5.3)$$

iv) Uniform inf-sup condition

$$\sup_{v_h \in V_h} \frac{b(v_h, \mu_h)}{\|v_h\|_2} \gtrsim \|\mu_h\|_{-2} \quad (5.4)$$

The key idea is to express $\tilde{\Phi}_i$ as a linear combination of the basis functions in $\mathcal{B}$. Since we require $\text{supp } \tilde{\Phi}_i = \text{supp } \Phi_i$, we seek a representation of the form

$$\tilde{\Phi}_i = \begin{cases} a_1 \Phi_i + a_2 \Psi_i + a_3 \Phi_{i-1} + a_4 \Psi_{i-1} + a_5 \Phi_{i+1} + a_6 \Psi_{i+1}, & \text{in } \text{supp } \Phi_i, \\ 0, & \text{otherwise,} \end{cases} \quad (5.5)$$

where a_j , $j = 1, \dots, 6$, are real constants.

Taking (5.2) into account, we obtain the following matrix equation arising from the inner products of $\tilde{\Phi}_i$ with all the elements of $\mathcal{B}$ that do not vanish on $\text{supp } \tilde{\Phi}_i$:

$$\mathcal{M} \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (5.6)$$

Where $\mathcal{M}$ is 6×6 matrix given by:

$$\mathcal{M} = h \begin{pmatrix} \frac{26}{35} & 0 & \frac{9}{70} & \frac{13h}{420} & \frac{9}{70} & \frac{-13h}{420} \\ 0 & \frac{2h^2}{105} & \frac{-13h}{420} & \frac{-h^2}{420} & \frac{13h}{420} & \frac{-h^2}{140} \\ \frac{9}{70} & \frac{-13h}{420} & \frac{13}{210} & \frac{11h}{105} & 0 & 0 \\ \frac{13h}{420} & \frac{-h^2}{140} & \frac{35}{210} & \frac{210}{105} & 0 & 0 \\ \frac{9}{70} & \frac{-13h}{420} & 0 & 0 & \frac{13}{210} & \frac{-11h}{105} \\ \frac{-13h}{420} & \frac{-h^2}{140} & 0 & 0 & \frac{-11h}{210} & \frac{h^2}{105} \end{pmatrix} \quad (5.7)$$

Finally, the coordinates of $\tilde{\Phi}_i$ are

$$\begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h^{-1} \begin{pmatrix} 2 \\ 0 \\ 1 \\ -\frac{12}{h} \\ 1 \\ \frac{12}{h} \end{pmatrix} \quad (5.8)$$

Similarly, we look for,

$$\tilde{\Psi}_i = \begin{cases} (b_1\Phi_i + b_2\Psi_i + b_3\Phi_{i-1} + b_4\Psi_{i-1} + b_5\Phi_{i+1} + b_6\Psi_{i+1}) & \text{in supp } \Psi_i, \\ 0 & \text{Otherwise.} \end{cases} \quad (5.9)$$

Therefore the coordinates of $\tilde{\Psi}_i$ are

$$\begin{bmatrix} b_1 \\ \vdots \\ b_6 \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h^{-2} \begin{pmatrix} 0 \\ \frac{150}{h} \\ -15 \\ \frac{195}{h} \\ 15 \\ \frac{195}{h} \end{pmatrix} \quad (5.10)$$

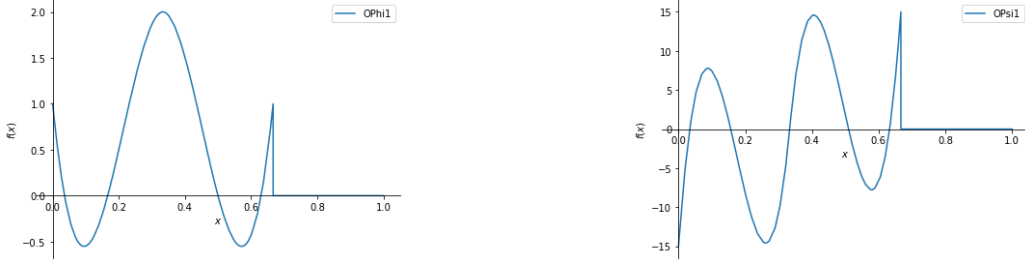


FIGURE 2. Reference biorthogonal basis functions associated with Hermite elements

This construction of the Q_h basis clearly satisfies the conditions (5.1) and (5.2), as well as the condition (5.3). Indeed, since $\tilde{\Phi}$ and $\tilde{\Psi}$ are local linear combinations of Hermite finite element basis we deduce:

$$\inf_{\mu \in Q_h} \|\mu - \mu_h\| \leq \|\mu - \mathcal{I}_h^H \mu\| \lesssim h^2 |\mu|_2 \quad (5.11)$$

For the inf-sup condition, we follow a method analogous to that used in the previous section to prove the stability of the pair $(\mathcal{P}_3^{\text{Hermite}}, \mathbb{P}_0)$. We introduce the following negative discrete norm:

$$\|\xi_h\|_{-2,h}^2 = \sum_{i=1}^n h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 = (h^2 \|\xi_h\|_0)^2, \quad \forall \xi_h \in Q_h \quad (5.12)$$

Now we prove the following:

Lemma 5.1. *There exists a constant $C_4 > 0$ such that*

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C_4 \|\xi_h\|_{-2,h} \quad (5.13)$$

Proof. Let $\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi}$ the reference basis on $[-1, 1]$ of $\Phi_i, \Psi_i, \tilde{\Phi}_i, \tilde{\Psi}_i$, then it is straightforward to verify that:

$$\begin{aligned} \|\Phi_i\|_0^2 &= h \|\Phi\|_0^2, & \|\Psi_i\|_0^2 &= h^2 \|\Psi\|_0^2 \\ \|\tilde{\Phi}_i\|_0^2 &= h^{-1} \|\tilde{\Phi}\|_0^2, & \|\tilde{\Psi}_i\|_0^2 &= h^{-2} \|\tilde{\Psi}\|_0^2 \end{aligned}$$

Let $\xi_h = \sum_{i=1}^{n-1} \alpha_i \tilde{\Phi}_i + \beta_i \tilde{\Psi}_i \in Q_h$, we take:

$$v_h = \sum_{i=1}^{n-1} \frac{\alpha_i}{h} \Phi_i + \frac{\beta_i}{h^2} \Psi_i$$

Then it follows immediately that:

$$\langle \xi_h, v_h \rangle = h^4 \sum_{i=1}^{n-1} \frac{\alpha_i^2}{h} + \frac{\beta_i^2}{h^2}$$

We now proceed to estimate the negative discrete norm of ξ_h . For convenience we assume $\alpha_0 = \beta_0 = 0$ and $\tilde{\Phi}_0 = \tilde{\Psi}_0 = 0$.

$$\begin{aligned} \|\xi_h\|_{-2,h}^2 &= \sum_{i=1}^n h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 \\ &= \sum_{i=1}^n h^4 \int_{x_{i-1}}^{x_i} \left(\alpha_{i-1} \tilde{\Phi}_{i-1} + \beta_{i-1} \tilde{\Psi}_{i-1} + \alpha_i \tilde{\Phi}_i + \beta_i \tilde{\Psi}_i \right)^2 \\ &\leq 4h^4 \sum_{i=1}^{n-1} \int_{x_{i-1}}^{x_i} \left(\alpha_{i-1} \tilde{\Phi}_{i-1} \right)^2 + \left(\beta_{i-1} \tilde{\Psi}_{i-1} \right)^2 + \left(\alpha_i \tilde{\Phi}_i \right)^2 + \left(\beta_i \tilde{\Psi}_i \right)^2 \\ &= 4h^4 \sum \alpha_i^2 \|\tilde{\Phi}_i\|^2 + \beta_i^2 \|\tilde{\Psi}_i\|^2 \\ &= 4h^4 \sum \frac{\alpha_i^2}{h} \|\tilde{\Phi}\|^2 + \frac{\beta_i^2}{h^2} \|\tilde{\Psi}\|^2 \\ &\lesssim h^4 \sum_{i=1}^{n-1} \frac{\alpha_i^2}{h} + \frac{\beta_i^2}{h^2} \\ &= \langle \xi_h, v_h \rangle. \end{aligned}$$

Similarly, we obtain

$$\|v_h\|_0^2 \lesssim h^4 \langle \xi_h, v_h \rangle,$$

and, by applying the inverse inequality (4.15), we obtain

$$\|v_h\|_2^2 \leq Ch^{-4} \|v_h\|_0^2 \lesssim (\langle \xi_h, v_h \rangle)^2.$$

Therefore we conclude

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C_4 \|\xi_h\|_{-2,h}$$

□

For the rest, we follow a proof analogous to one of (4.13). Hence, by introducing the following spaces and sets:

$$\begin{cases} Q_h := \text{span}\{\mathcal{B}\}, \\ \Lambda_h := \left\{ \mu_h = \sum_{i=1}^{n-1} \alpha_i \tilde{\Phi}_i + \beta_i \tilde{\Psi}_i \in Q_h : \alpha_i, \beta_i \geq 0, \forall i = 1 \dots n-1 \right\}. \end{cases} \quad (5.14)$$

Then we have the following result.

Theorem 5.2. *The finite element space pair (V_h, Q_h) in (5.14) is stable and satisfy inf-sup condition*

$$\sup_{v_h \in V_h} \frac{\langle \xi_h, v_h \rangle}{\|v_h\|_2} \geq C \|\xi_h\|_{-2}.$$

Similarly, all results about existence, error estimates from the previous section, remain valid (albeit with possibly different constants) since the proofs are independent of the discrete finite element spaces chosen, provided these spaces are stable and satisfy the inf-sup condition.

6. NUMERICAL APPROXIMATION ALGORITHMS

In this section, we provide a brief overview of two methods to solve variational inequalities: the Uzawa Method and the Primal Dual Active Set (PDAS). Later, we implement these methods for the Euler-Bernoulli obstacle problems using the open-source software Freefem++ [11] and numerically analyze the results.

6.1. Uzawa's algorithm. In this subsection, we briefly describe the classical Uzawa algorithm for solving variational inequalities of the form: Find $x \in \mathbb{R}^N$ and $\lambda \in \Lambda$ such that

$$Ax + B^T \lambda = f, \tag{6.1}$$

$$(\mu - \lambda)^T (Bx - g) \leq 0 \quad \forall \mu \in \Lambda, \tag{6.2}$$

where

$$\Lambda = \{\mu \in \mathbb{R}^L : \mu_k \geq 0\}.$$

Below we outline the algorithm for this method. See [1] for more details about the algorithm for the saddle-point problems and [13] for the variational inequality case.

Algorithm 1 Uzawa Algorithm

- 1: Choose an initial value $\lambda^{(0)}$
- 2: Set $k = 0$
- 3: **repeat**
- 4: Compute the intermediate iterate

$$\lambda_*^{(k+1)} = \lambda^{(k)} - \alpha M^{-1} (S \lambda^{(k)} - h)$$

- 5: Project onto the admissible set Λ :

$$\lambda^{(k+1)} = P_\Lambda (\lambda_*^{(k+1)})$$

- 6: $k \leftarrow k + 1$
- 7: **until** $\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq \varepsilon \|\lambda^{(k+1)}\|$
- 8: Compute the primal variable:

$$u_h = -A^{-1} (B^T \lambda^{(k+1)} - f)$$

The matrix M is the lumped mass matrix of Q_h , and $S = BA^{-1}B^T$ represents the Schur complement of the system

The convergence of the Uzawa algorithm depends on the choice of the parameter α , as stated in the following theorem.

Theorem 6.1. *Let (u, λ) be a solution to the system (6.1, 6.2). Let s_1, s_2 denote the smallest and the largest eigenvalues of $M^{-1}S$, and let $(u^{(k)}, \lambda^{(k)})$ be the iterates produced by Uzawa's algorithm. Then there exists a positive constant $\bar{\alpha} = \frac{2}{s_2} > 0$ such that for any choice $\alpha \in (0, \bar{\alpha})$ we have*

$$u^{(k)} \rightarrow u, \quad \lambda^{(k)} \rightarrow \lambda \text{ as } k \rightarrow \infty.$$

Proof. see [1]. □

6.2. Primal Dual Active Set Method. In this subsection, we discuss the primal-dual active set (PDAS) strategy [12] for the obstacle problem. Starting from the mixed formulation (4.3), we introduce the modified bilinear form

$$b(v, \mu) = -\langle \mu, v \rangle.$$

so that the problem can be written as

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda \\ a(u, v) + b(v, \lambda) = f(v), & \forall v \in X \\ b(u, \mu - \lambda) \leq g(\mu - \lambda), & \forall \mu \in \Lambda \end{cases} \quad (6.3)$$

where here, $\Lambda = \{\mu \in V' : \langle v, \mu \rangle \leq 0, \quad \forall v \in V, v \geq 0\}$.

Let $\{\phi_p\}_p$ be a basis for V_h and $\{\Psi_p\}_p$ a basis for Q_h . Then,

$$u_h = \sum_p \alpha_p \phi_p, \quad \lambda_h = \sum_p \beta_p \Psi_p, \quad \psi_h = \sum_p g_p \phi_p.$$

Then the discrete version of the Karush-Kuhn-Tucker conditions (KKT) reads:

$$\alpha_p \geq g_p, \quad \beta_p \leq 0, \quad (\alpha_p - g_p)\beta_p = 0. \quad (6.4)$$

Which are equivalent to $C(\alpha_p, \beta_p) = 0$ with

$$C(x, y) = y - \min(0, y + c(x - g)), \quad c > 0 \text{ fixed.}$$

Consequently, we can rewrite the discrete version of the primal-dual variational inequality as a nonlinear equality formulation,

$$\begin{cases} A_h \alpha_h + B_h \beta_h = f_h \\ C_h(\alpha_h, \beta_h) = 0 \end{cases} \quad (6.5)$$

If the nonlinear function C_h is differentiable, applying Newton's method, we get

$$\begin{pmatrix} \alpha_h^{k+1} \\ \beta_h^{k+1} \end{pmatrix} = \begin{pmatrix} \alpha_h^k \\ \beta_h^k \end{pmatrix} - \begin{pmatrix} A_h & B_h \\ \partial_{\alpha_h} C_h & \partial_{\beta_h} C_h \end{pmatrix}^{-1} \begin{pmatrix} A_h \alpha_h^k + B_h \beta_h^k - f_h \\ C_h(\alpha_h^k, \beta_h^k) \end{pmatrix}$$

Multiplying on both sides of the system by the Jacobian leads to a system of linear equations to be solved in each Newton step.

$$\begin{cases} A_h \alpha_h^{k+1} + B_h \beta_h^{k+1} = f_h \\ \partial_{\alpha_h} C_h \alpha_h^{k+1} + \partial_{\beta_h} C_h \beta_h^{k+1} = \partial_{\alpha_h} C_h \alpha_h^k + \partial_{\beta_h} C_h \beta_h^k - C_h(\alpha_h^k, \beta_h^k) \end{cases} \quad (6.6)$$

Algorithm 2 Primal–Dual Active Set (PDAS) Algorithm1: Let $\mathcal{P}_h = \{x_0, \dots, x_n\}$ 2: Choose an initial active set $\mathcal{A}_h^0 \subset \mathcal{P}_h$ 3: Set $\mathcal{I}_h^0 = \mathcal{P}_h \setminus \mathcal{A}_h^0$ 4: Set $k = 0$ 5: **repeat**

6: Set

$$\alpha_p^k = g_p \quad \text{for } p \in \mathcal{A}_h^k, \quad \beta_p^k = 0 \quad \text{for } p \in \mathcal{I}_h^k$$

7: Solve the linear system

$$A_h \alpha^k + B_h \beta^k = f$$

8: Update the active and inactive sets:

$$\mathcal{A}_h^{k+1} = \{p \in \mathcal{A}_h^k ; \beta_p^k \leq 0\} \cup \{p \in \mathcal{I}_h^k ; \alpha_p^k \leq 0\}$$

$$\mathcal{I}_h^{k+1} = \mathcal{P}_h \setminus \mathcal{A}_h^{k+1}$$

9: $k \leftarrow k + 1$ 10: **until** $\mathcal{A}_h^k = \mathcal{A}_h^{k-1}$

7. NUMERICAL RESULTS

In this section, we solve the obstacle problem numerically using the Uzawa and PDAS methods introduced earlier. The Uzawa method is applied to the stable mixed formulation with the $(\mathbb{P}_3^{\text{Hermite}}, \mathbb{P}_0)$ pair, while the PDAS method is applied to the mixed formulation with a biorthogonal basis. The implementation was developed in FreeFem++, graphical outputs generated in Python.

We present results for several examples. All examples share the following setup

- i) The interval $\Omega = (0, 1)$.
- ii) A uniform discretisation.
- iii) An initial value chosen as the null vector.
- iv) An error stopping criterion of $\epsilon = 10^{-6}$.

Example 7.1. (Closed interval contact zone). For this example, we take

$$f = 0 \quad \text{and} \quad \psi = 4x^3 - 9x^2 + 6x - 1$$

Then, the Euler-Bernoulli obstacle problem reads:

$$\begin{cases} \text{find } u \in K := \{v \in H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \\ a(u, v - u) \geq 0, \quad \forall v \in K \end{cases} \quad (7.1)$$

The exact solution of this problem is

$$u(x) = \begin{cases} -4x^3 + 3x^2 & \text{if } x \in [0, \frac{1}{2}) \\ 4x^3 - 9x^2 + 6x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

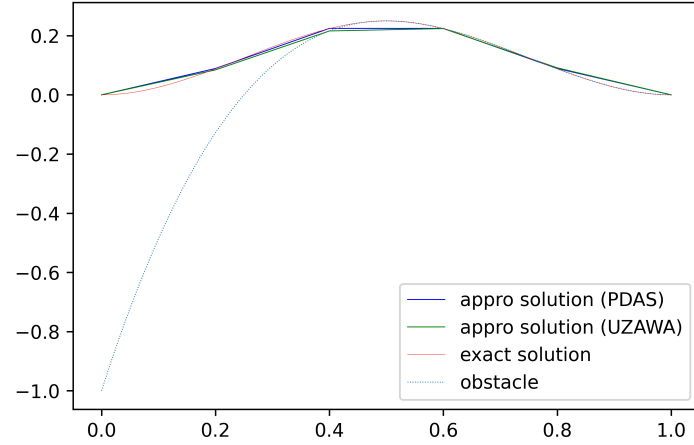
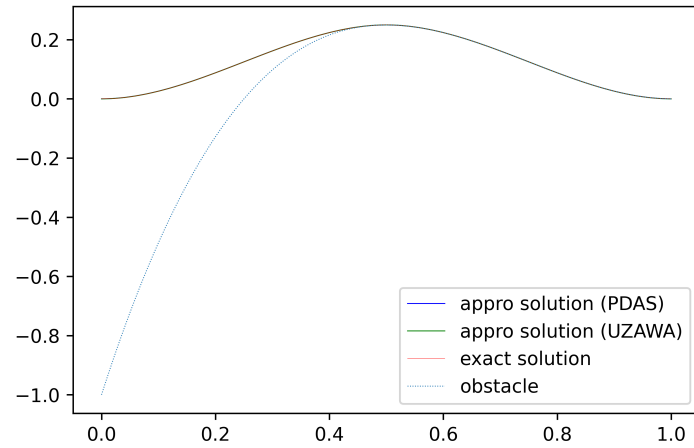
First, we investigate the influence of the Uzawa parameter α . Because in general, it is hard to find the exact values of eigenvalues of a matrix, we ended up testing several candidate values, and the results obtained are reported below. Since similar results were

α	1	10	100	500	1000	1100	1150
Iterations	13585	6965	714	139	63	56	82

TABLE 1. Number of iteration for different values of α in Uzawa algorithm

observed different values of M , we ended up selecting $\alpha = 1100$ as the primary value when testing the Uzawa Algorithm.

N	5	10	50	100	500
Iterations (Uzawa)	32	56	18739	82195	40502
L^2 error (Uzawa)	1.099e-3	9.879e-4	6.670e-05	5.463e-05	4.640e-05
Iterations (PDAS)	4	8	44	71	378
L^2 error (PDAS)	4.524e-3	6.016e-4	6.162e-05	4.325e-05	1.693e-06

TABLE 2. Number of iterations and L^2 Error (Example 7.1)FIGURE 3. Contact zone with $N = 5$ FIGURE 4. Contact zone with $N = 100$

Example 7.2. (single point contact). In this example, we take

$$f = 0 \text{ and } \psi = -3(2x - 1)^2 + 1$$

The exact solution of this problem is given by:

$$u(x) = \begin{cases} -16x^3 + 12x^2 & \text{if } x \in [0, \frac{1}{2}) \\ 16x^3 - 36x^2 + 24x - 4 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

As in the previous example, we perform the Uzawa Method using $\alpha = 1100$.

N	5	10	50	100	500
Iterations (Uzawa)	14	20	6378	12277	5072
L^2 error (Uzawa)	1.992e-03	1.353e-03	2.181e-04	1.867e-04	2.721e-04
Iterations (PDAS)	2	5	18	34	162
L^2 error (PDAS)	1.236e-02	1.978e-15	3.289e-13	8.091e-12	1.1209e-08

TABLE 3. Number of iterations and L^2 Error (Example 7.2).

We observe that the iterations required for the PDAS method to converge to a given tolerance (e.g., machine accuracy) is often several orders of magnitude lower than that for the Uzawa method we have used.

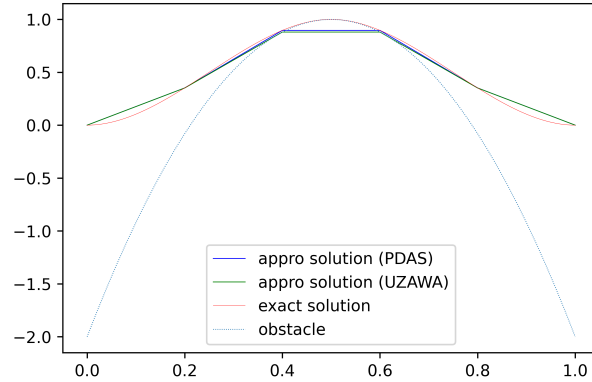


FIGURE 5. Contact zone with $N = 5$

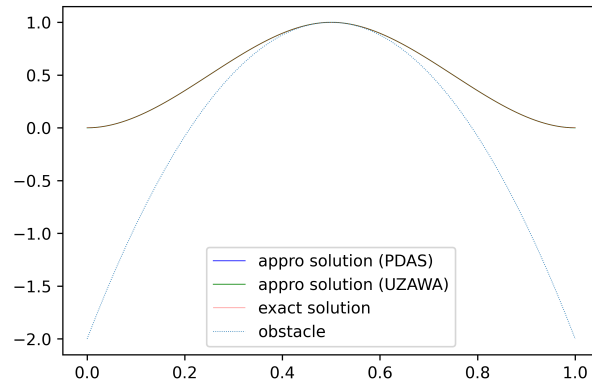


FIGURE 6. Contact zone with $N = 100$

Something peculiar occurs in this example: the fewer the discretisations, the lower the error the PDAS method achieves; however, the error remains small in almost all cases, indicating that PDAS remains effective. Worth the investigation in the future to see whether this affects all the cases of single point contact zone, or it is something particular to this case.

Example 7.3. (Constant obstacle). We take $f = -18432$ and $\psi = -1$

$$\begin{cases} \text{find } u \in K := \{v \in H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \\ a(u, v - u) \geq 0, \quad \forall v \in K \end{cases}$$

The exact solution of this problem is

$$u(x) = \begin{cases} -4x^2(192x^2 - 128x + 24) & \text{if } x \in [0, \frac{1}{4}) \\ -1 & \text{if } x \in [\frac{1}{4}, \frac{3}{4}) \\ -(2x - 2)^2(192x^2 - 256x + 88) & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

In this example, we choose $\alpha = 1000$ because it converged the fastest (when we tried $\alpha = 1100$, as in the previous examples, the method did not converge).

N	5	10	50	100	500
Iterations (Uzawa)	500	3444	2857	3202	6860
L^2 error (Uzawa)	3.867e-02	1.086e-02	1.214e-03	9.038e-04	5.083e-04
Iterations (PDAS)	2	4	14	26	111
L^2 error (PDAS)	8.657e-02	1.326e-02	1.274e-04	7.468e-06	6.082e-08

TABLE 4. Number of iterations and L^2 Error (Example 7.3)

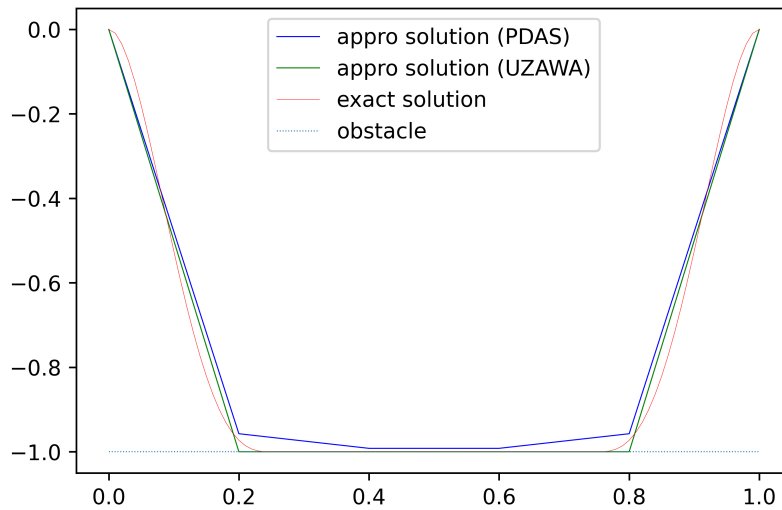
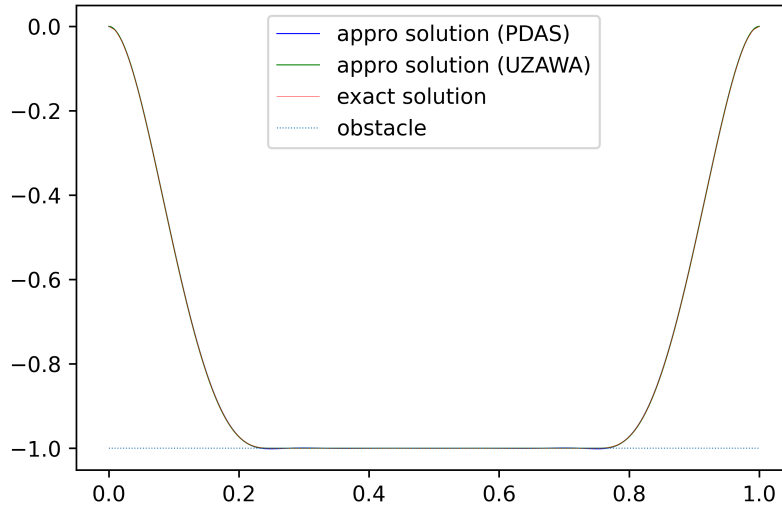


FIGURE 7. Contact zone with $N = 5$

FIGURE 8. Contact zone with $N = 100$

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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