Gulf Journal of Mathematics
Vol 23, Issue 1 (2026) 1-11

ISSN: 2309-4966
https://doi.org/10.56947/kdfvfq46

EXISTENCE AND STABILITY RESULTS FOR A GENERAL CLASS OF ABSTRACT HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS

HICHEM DJERIBA[1]

Abstract. In this paper, we investigate a general class of abstract nonlinear fractional differential equations governed by the Hadamard fractional derivative. We establish rigorous existence and uniqueness results for mild solutions within the Banach space framework. By employing fixed-point theorems and a generalized fractional Gronwall inequality, we further demonstrate the Hadamard-type stability of the system, ensuring the continuous dependence of solutions on initial data. This abstract approach provides a unified theoretical foundation for analyzing diverse nonlinear evolution problems, substituting specific physical structures with a generalized operator-based analysis.

Keywords. Hadamard fractional derivative, mild solutions, Hadamard-type stability, existence and uniqueness, Banach spaces, Fixed-point theorems
2020 Mathematics Subject Classification. Primary 26A33; Secondary 34A08, 34G20, 47H10

1. Introduction

The capacity of fractional differential equations to represent anomalous diffusion, hereditary characteristics, and memory effects has garnered significant interest in recent years [8, 12]. These equations offer a versatile approach to nonlocal temporal dynamics, providing a more accurate description of complex systems compared to traditional classical models [1]. Among the various formulations of fractional calculus, the Hadamard fractional derivative stands out due to its unique nonlocal kernel involving logarithmic functions, making it particularly suited for modeling phenomena that evolve on a logarithmic scale [12, 13]. While many studies focus on specific physical models, the development of a rigorous abstract framework for fractional evolution equations is essential for ensuring mathematical consistency and broad applicability [4, 16]. Theoretical aspects regarding the existence, uniqueness, and stability of solutions in abstract Banach spaces provide the necessary foundation for understanding the qualitative behavior of such systems [2, 11]. Driven by these scholarly contributions, this research

Date: Received: Feb 20, 2026; Revised: Mar 20, 2026; Accepted: Apr 4, 2026.
* Corresponding author
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focuses on a general class of nonlinear abstract fractional differential equations governed by the Hadamard fractional derivative [7, 9].
In this work, we consider the following abstract Cauchy problem:

$$ {}^{H}D^{\alpha}_{1+}u(t) = f(t, u(t)), \quad t \in [1, T], \quad 0 < \alpha < 1, \tag{1.1} $$

subject to the initial condition:

$$ u(1) = u_0, \tag{1.2} $$

where $u(\cdot)$ takes values in a Banach space X, and $f : [1, T] \times X \to X$ is a nonlinear mapping satisfying the Lipschitz condition as required for the existence and uniqueness of a mild solution. By treating the problem in an abstract setting, we provide a treatment that is independent of specific boundary constraints or particular spatial geometries. This abstraction is intentional, aiming to establish a versatile mathematical platform. Our analysis yields existence and uniqueness results, along with the Hadamard-type stability of mild solutions, established via fixed-point arguments [6, 10]. These findings are derived using the properties of Hadamard fractional integration and a generalized Gronwall inequality [14], providing a rigorous analytical framework for this class of fractional operators.

2. Preliminaries and Mild Solution

2.1. Hadamard Fractional Calculus.

Definition 2.1. Let $\alpha > 0$. The Hadamard fractional integral of $f(t)$, defined for $t > 0$, is expressed as [3]

$$ {}^{H}I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, $$

Here, $\Gamma(\alpha)$ denotes the Euler Gamma function [5].

Definition 2.2. For $\alpha > 0$, the Hadamard fractional derivative of $f(t)$, defined for $t > 0$, is expressed as [3]

$$ {}^{H}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \frac{f(s)}{s}\, ds, $$

where n is an integer such that $n = \lceil \alpha \rceil$. In this context, the notation $\Gamma(\cdot)$ stands for the Euler Gamma function, and $\ln(\cdot)$ signifies the natural logarithm.

Lemma 2.3. *For $\alpha > 0$ and $n > 0$ [8, p. 113]:*

$$ {}^{H}I^{\alpha}t^n = n^{-\alpha}t^n, \qquad {}^{H}D^{\alpha}t^n = n^{\alpha}t^n. $$

Lemma 2.4. *Let X be a Banach space and $f, g \in C([1, T], X)$, $a, b \in \mathbb{R}$, with $\alpha > 0$. The Hadamard integral operator exhibits the following features [1, 8]:*

1. *Linearity: ${}^{H}I^{\alpha}(af + bg) = a\,{}^{H}I^{\alpha}f + b\,{}^{H}I^{\alpha}g$.*
2. *Boundedness: There is a constant $C_\alpha > 0$ for which*

$$ \|{}^{H}I^{\alpha}f\|_{C([1,T],X)} \leq C_\alpha \|f\|_{C([1,T],X)}. $$

(3) *Equicontinuity/Boundedness: The operator* ${}^{H}I^{\alpha}$ *maps bounded sets of* $C([1,T],X)$ *into sets that are bounded and equicontinuous. Consequently, if* X *is finite-dimensional or* f *is compact, the Arzelà–Ascoli theorem ensures that* ${}^{H}I^{\alpha}$ *is a compact operator.*

Lemma 2.5. *Consider the Banach space* X*. For* $\alpha > 0$ *and* $n = [\alpha] + 1$*, the Hadamard derivative satisfies* [1, 8]*:*

(1) *Linearity:*

$$ {}^{H}D^{\alpha}(af + bg) = a\,{}^{H}D^{\alpha}f + b\,{}^{H}D^{\alpha}g. $$

(2) *Left-inverse property: If* $f \in C([1,T],X)$*, then*

$$ {}^{H}D^{\alpha}\,{}^{H}I^{\alpha}f(t) = f(t), \quad t \in [1,T]. $$

(3) *Boundedness in* C^n*-norm: There exists* $C_\alpha > 0$ *for which*

$$ \|{}^{H}D^{\alpha}f\|_{C([1,T],X)} \leq C_\alpha \|f^{(n)}\|_{C([1,T],X)}. $$

2.2. **Mittag-Leffler Function.**

Definition 2.6. For $\alpha > 0$, one defines the Mittag–Leffler function by (see [15])

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{R}. $$

2.3. **Mild Solution.**

Definition 2.7. Consider a nonlinear mapping $f : [1,T] \times X \to X$ on a Banach space X. We say that $u \in C([1,T],X)$ constitutes a mild solution to the problem (1.1)–(1.2) for the abstract fractional evolution equation

$$ {}^{H}D^{\alpha}_{1+}u(t) = f(t,u(t)), \quad t \in [1,T], $$

satisfying the initial condition $u(1) = u_0 \in X$, if the following integral relation is verified:

$$ u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s}, \quad t \in [1,T]. \tag{2.1} $$

3. Main Result & Proof

This section is devoted to establishing the existence, uniqueness, and Hadamard stability of mild solutions for the abstract problem (1.1) in a general Banach space X. Our approach relies on fixed-point theorems and the analytic properties of Hadamard fractional operators.

3.1. **Existence of Solutions.**

We investigate the existence of solutions to the abstract Cauchy problem (1.1). To this end, we first review the fundamental characteristics of the Hadamard fractional integral operator, which serve as pivotal tools for the ensuing fixed-point analysis.

Lemma 3.1. *Let $\alpha \in (0,1)$ and let X be a Banach space. Define the Hadamard fractional integral operator by*

$$({}^{H}I^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \qquad f \in C([1,T],X).$$

Then the operator ${}^{H}I^{\alpha}$ satisfies the following properties:

(1) *Linearity: ${}^{H}I^{\alpha}$ is linear on $C([1,T],X)$.*
(2) *Boundedness/Continuity: For all $f \in C([1,T],X)$,*

$$\|{}^{H}I^{\alpha}f\|_{C([1,T],X)} \leq \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{C([1,T],X)}.$$

Consequently, ${}^{H}I^{\alpha}$ is continuous on $C([1,T],X)$.
(3) *Equicontinuity: For any bounded set $\mathcal{F} \subset C([1,T],X)$, the family ${}^{H}I^{\alpha}(\mathcal{F})$ is uniformly bounded and equicontinuous. Indeed, the kernel*

$$K(t,s) = \frac{1}{\Gamma(\alpha)} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{1}{s}$$

is weakly singular at $s = t$, but integrable for $\alpha \in (0,1)$. Hence, for $u \in \mathcal{F}$ and $1 \leq t_1 < t_2 \leq T$,

$$\|({}^{H}I^{\alpha}u)(t_2) - ({}^{H}I^{\alpha}u)(t_1)\| \to 0 \quad \text{as } t_2 \to t_1.$$

(4) *Relative Compactness: If X is finite-dimensional, or if X is infinite-dimensional and the mapping $f([1,T] \times B)$ is relatively compact in X for every bounded set $B \subset X$, then by the Arzelà–Ascoli theorem, ${}^{H}I^{\alpha}(\mathcal{F})$ is relatively compact in $C([1,T],X)$.*

Proof. (1) and (2) follow directly from the linearity of the integral and the estimate

$$\|{}^{H}I^{\alpha}f\|_{C([1,T],X)} \leq \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{C([1,T],X)}.$$

(3) Equicontinuity: Let $\mathcal{F} \subset C([1,T],X)$ be bounded, i.e., $\|u\|_{C([1,T],X)} \leq M$ for all $u \in \mathcal{F}$. For $1 \leq t_1 < t_2 \leq T$,

$$\begin{aligned}\|({}^{H}I^{\alpha}u)(t_2) - ({}^{H}I^{\alpha}u)(t_1)\| \leq \frac{M}{\Gamma(\alpha)} \Bigg[& \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \\ & + \int_1^{t_1} \left| \left(\ln \frac{t_2}{s}\right)^{\alpha-1} - \left(\ln \frac{t_1}{s}\right)^{\alpha-1} \right| \frac{ds}{s} \Bigg].\end{aligned}$$

Both integrals tend to zero as $t_2 \to t_1$ by the Dominated Convergence Theorem, showing equicontinuity.

(4) Relative Compactness: If X is finite-dimensional, uniform boundedness and equicontinuity imply relative compactness by the Arzelà–Ascoli theorem. If X is infinite-dimensional, the assumption that $f([1,T] \times B)$ is relatively compact in X for every bounded set $B \subset X$ ensures that ${}^{H}I^{\alpha}$ maps bounded sets into relatively compact subsets of $C([1,T],X)$. □

3.1.1. *Mild Solutions.*

Definition 3.2. A function $u \in C([1,T],X)$ is called a mild solution of the abstract Cauchy problem (1.1)–(1.2), provided that it satisfies the following integral formulation:

$$u(t) = u_0 + ({}^{H}I^{\alpha} f(u))(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \tfrac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s}, \quad t \in [1,T].$$

3.1.2. *Existence Theorem.*

Theorem 3.3. *Consider a Banach space X and $u_0 \in X$. Let $f : [1,T] \times X \to X$ be a continuous mapping such that f is compact (or X is finite-dimensional). Assume further that f satisfies a linear growth condition: $\|f(t,u)\| \leq M_a + M_b\|u\|$ for some constants $M_a, M_b \geq 0$. Then, the abstract fractional Cauchy problem (1.1)–(1.2) possesses at least one mild solution $u \in C([1,T],X)$, provided that $\frac{M_b(\ln T)^{\alpha}}{\Gamma(\alpha+1)} < 1$.*

Proof. To establish the existence of a solution, we consider the operator $\mathcal{T} : C([1,T],X) \to C([1,T],X)$ defined by:

$$(\mathcal{T}u)(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \tfrac{t}{s}\right)^{\alpha-1} f(s,u(s)) \frac{ds}{s}.$$

To avoid circular reasoning in determining the radius R, we assume f satisfies a linear growth condition: $\|f(t,u)\| \leq M_a + M_b\|u\|$ for some constants $M_a, M_b \geq 0$.

Let B_R be a closed ball with radius R. For any $u \in B_R$, we have:

$$\begin{aligned} \|(\mathcal{T}u)(t)\| &\leq \|u_0\| + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \tfrac{t}{s}\right)^{\alpha-1} (M_a + M_b\|u(s)\|) \frac{ds}{s} \\ &\leq \|u_0\| + \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)}(M_a + M_b R). \end{aligned}$$

To ensure $\mathcal{T}(B_R) \subseteq B_R$, we impose the following condition

$$\|u_0\| + \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)} M_a + \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)} M_b R \leq R.$$

By solving this inequality for R, we obtain the required radius:

$$R \geq \frac{\|u_0\| + \frac{M_a(\ln T)^{\alpha}}{\Gamma(\alpha+1)}}{1 - \frac{M_b(\ln T)^{\alpha}}{\Gamma(\alpha+1)}}.$$

Provided that $\frac{M_b(\ln T)^{\alpha}}{\Gamma(\alpha+1)} < 1$, a positive radius R exists independently of the local maximum of f, thus resolving the circular logic.

The continuity of $\mathcal{T}$ follows from the continuity of f. Moreover, since f is a compact operator and the Hadamard integral operator ${}^{H}I^{\alpha}$ is completely continuous (by Lemma 3.1), the composition operator $\mathcal{T}$ is completely continuous by the Arzelà-Ascoli theorem. Therefore, by Schauder's fixed-point theorem, $\mathcal{T}$ has a fixed point $u \in B_R$, which is a mild solution. □

3.2. **Uniqueness of Solutions.** We first recall a generalized Hadamard-type fractional Gronwall inequality, which will be the main tool in the uniqueness analysis.

Lemma 3.4. *Assume that w is a nonnegative continuous function defined on $[1,T]$. Suppose there exist constants $C \geq 0$, $L > 0$, and $\alpha \in (0,1)$ such that, for all $t \in [1,T]$,*

$$w(t) \leq C + \frac{L}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s} \Big)^{\alpha-1} \frac{w(s)}{s}\, ds. \tag{3.1}$$

Consequently, for every $t \in [1,T]$, the following holds

$$w(t) \leq C\, E_\alpha\big(L(\ln t)^\alpha\big),$$

where $E_\alpha(.)$ denotes the Mittag–Leffler function, as introduced in Definition 2.6

Proof. We transform the Hadamard integral into a Riemann–Liouville integral using the change of variables $x = \ln t$. For $s \in [1,t]$ put $y = \ln s$ and define $\widetilde{w}(x) = w(e^x)$ for $x \in [0, \ln T]$. Using $ds/s = dy$ and $\ln(t/s) = x - y$, inequality (3.1) is equivalent to

$$\widetilde{w}(x) \leq C + \frac{L}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} \widetilde{w}(y)\, dy, \qquad x \in [0, \ln T].$$

Let J^α represent the Riemann–Liouville fractional integral of order α with respect to the variable x:

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy.$$

Then the inequality becomes

$$\widetilde{w}(x) \leq C + L\, (J^\alpha \widetilde{w})(x).$$

We iterate this inequality. Applying J^α repeatedly and using linearity and positivity of J^α we obtain for any integer $n \geq 0$,

$$\widetilde{w}(x) \leq C \sum_{k=0}^{n} L^k (J^\alpha)^k 1(x) \;+\; L^{n+1} (J^\alpha)^{n+1} \widetilde{w}(x),$$

where $(J^\alpha)^0 := \mathrm{Id}$ and 1 denotes the constant function equal to 1 on $[0, \ln T]$.

Recall the classical formula for powers of J^α acting on constants:

$$(J^\alpha)^k 1(x) = \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}, \qquad k \geq 0.$$

Consequently, it follows that:

$$\widetilde{w}(x) \leq C \sum_{k=0}^{n} \frac{(Lx^\alpha)^k}{\Gamma(\alpha k + 1)} \;+\; L^{n+1} \|(\widetilde{w})\|_{C[0,\ln T]} \frac{x^{\alpha(n+1)}}{\Gamma(\alpha(n+1)+1)}.$$

Since $\widetilde{w}$ is continuous on a compact interval, it is bounded, hence the remainder term

$$R_{n+1}(x) := L^{n+1} \|\widetilde{w}\|_{C[0,\ln T]} \frac{x^{\alpha(n+1)}}{\Gamma(\alpha(n+1)+1)}.$$

Clearly,

$$0 \le R_{n+1}(x) \to 0 \quad \text{as } n \to \infty \text{ for every fixed } x \in [0, \ln T],$$

because the Gamma function in the denominator grows faster than any power of n. Letting $n \to \infty$, we obtain

$$\widetilde{w}(x) \le C \sum_{k=0}^{\infty} \frac{(Lx^{\alpha})^k}{\Gamma(\alpha k + 1)} = C\, E_{\alpha}\big(Lx^{\alpha}\big).$$

Returning to $t = e^x$ yields the stated estimate

$$w(t) \le C\, E_{\alpha}\big(L(\ln t)^{\alpha}\big), \qquad t \in [1, T].$$

□

3.2.1. *Uniqueness Theorem.*

Theorem 3.5. *Assume $f : [1, T] \times X \to X$ satisfies a Lipschitz condition: there exists $L > 0$ with*

$$\|f(t, u) - f(t, v)\|_X \le L\|u - v\|_X, \qquad \forall u, v \in X, \quad \forall t \in [1, T].$$

Then any mild solution $u \in C([1, T], X)$ of (1.1) with a given initial datum $u(1) = u_0$ is unique on $[1, T]$.

Proof. Let $u, v \in C([1, T], X)$ be two mild solutions with the same initial data $u(1) = v(1) = u_0$. Define the scalar function

$$\phi(t) := \|u(t) - v(t)\|_X, \qquad t \in [1, T],$$

which is continuous and nonnegative with $\phi(1) = 0$. Using the mild formulation and the Lipschitz property we get, for every $t \in [1, T]$,

$$\phi(t) = \big\|({}^{H}I^{\alpha}(f(\cdot, u(\cdot)) - f(\cdot, v(\cdot))))(t)\big\|_X.$$

By the definition of the Hadamard integral and the norm property, we have

$$\phi(t) \le \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s}\Big)^{\alpha-1} \|f(s, u(s)) - f(s, v(s))\|_X \frac{ds}{s}.$$

Taking into account the Lipschitz condition of f, it follows that

$$\phi(t) \le \frac{L}{\Gamma(\alpha)} \int_1^t \Big(\ln \frac{t}{s}\Big)^{\alpha-1} \phi(s)\, \frac{ds}{s}.$$

Thus ϕ satisfies (3.1) with $C = 0$. Applying the generalized Gronwall lemma yields

$$\phi(t) \le 0 \cdot \sum_{k=0}^{\infty} \frac{\big(L(\ln t)^{\alpha}\big)^k}{\Gamma(\alpha k + 1)} = 0, \qquad \forall t \in [1, T],$$

where the series represents the Mittag–Leffler function $E_{\alpha}(\cdot)$ as introduced in Definition 2.6. Consequently, it follows that $\phi \equiv 0$ on $[1, T]$, which leads to $u = v$, thereby concluding the proof of uniqueness. □

Remark 3.6. The proof above relies only on the Lipschitz property of f and the mild formulation. Alternatively, one could show that the operator $\mathcal{T}$ from the existence proof is a contraction on a suitable ball and apply Banach's fixed point theorem; however, that approach would require L to be small, whereas the present argument ensures uniqueness for any $L > 0$.

3.3. **Stability Analysis in the Sense of Hadamard.**

3.3.1. *Concept of Stability.* Hadamard stability for fractional differential systems describes the continuous dependence of mild solutions on their initial data posed in a Banach space X. Specifically, for any pair $u_0, v_0 \in X$, sufficiently small perturbations of the initial data give rise to uniformly small deviations between the corresponding solutions on the interval $[1, T]$. This property is formally characterized by the following condition:

$$\lim_{\|u_0 - v_0\| \to 0} \sup_{t \in [1,T]} \|u(t) - v(t)\| = 0.$$

3.3.2. *Stability Analysis of the Abstract Cauchy Problem.*

Theorem 3.7. *Consider a pair of mild solutions, denoted by u and v, associated with the problem (1.1) and originating from the initial values $u_0, v_0 \in X$, respectively. Suppose that the nonlinear function $f : [1, T] \times X \to X$ fulfills the Lipschitz condition:*

$$\|f(t, u) - f(t, v)\|_X \leq L\|u - v\|_X, \quad \forall t \in [1, T], \forall u, v \in X,$$

for some constant $L > 0$. Then, for each $t \in [1, T]$,

$$\|u(t) - v(t)\|_X \leq \|u_0 - v_0\|_X \sum_{k=0}^{\infty} \frac{[L(\ln t)^\alpha]^k}{\Gamma(\alpha k + 1)}. \tag{3.2}$$

More precisely, by introducing the constant

$$C := \max_{t \in [1,T]} E_\alpha\big(L(\ln t)^\alpha\big) = \max_{t \in [1,T]} \sum_{k=0}^{\infty} \frac{[L(\ln t)^\alpha]^k}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{[L(\ln T)^\alpha]^k}{\Gamma(\alpha k + 1)},$$

the following uniform bound can be asserted:

$$\|u(t) - v(t)\|_X \leq C \, \|u_0 - v_0\|_X, \qquad \forall t \in [1, T].$$

Hence, the mild solution exhibits continuous dependence on the initial conditions.

Proof. Introduce the function $w(t)$ defined by

$$w(t) := u(t) - v(t).$$

Using the mild representation (2.1) for u and v, we get, for every $t \in [1, T]$,

$$w(t) = u_0 - v_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha - 1} [f(s, u(s)) - f(s, v(s))] \, \frac{ds}{s}.$$

Taking norms in X and applying the Lipschitz assumption on f yields:

$$\|w(t)\|_X \leq \|u_0 - v_0\|_X + \frac{L}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha - 1} \|w(s)\|_X \, \frac{ds}{s}.$$

Thus the scalar function $\phi(t) := \|w(t)\|_X$ satisfies the integral inequality (3.1) with

$$C = \|u_0 - v_0\|_X, \qquad L \text{ as above}, \qquad \alpha \in (0,1).$$

By the generalized Hadamard-type fractional Gronwall Lemma 3.4, for any $t \in [1,T]$, the estimate:

$$\phi(t) \le \|u_0 - v_0\|_X \sum_{k=0}^{\infty} \frac{[L(\ln t)^\alpha]^k}{\Gamma(\alpha k + 1)},$$

which corresponds to the stability bound (3.2). Given that the series representing the Mittag–Leffler function $E_\alpha(z)$ is monotonically nondecreasing for $z \ge 0$, it follows that

$$\max_{t\in[1,T]} \sum_{k=0}^{\infty} \frac{[L(\ln t)^\alpha]^k}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{[L(\ln T)^\alpha]^k}{\Gamma(\alpha k + 1)} < \infty.$$

This ensures the validity of the uniform estimate. Consequently, as $\|u_0 - v_0\|_X \to 0$, the right-hand side of (3.2) vanishes for any fixed t, implying that $\|u(t) - v(t)\|_X \to 0$ for each $t \in [1,T]$. □

Remark 3.8. The bound (3.2) is typical for fractional-order problems, where the Mittag–Leffler function naturally replaces the exponential factor appearing in the classical Gronwall inequality. In particular, when $\alpha \to 1^-$, the estimate formally reduces to the standard exponential Gronwall bound. It is worth noting that the obtained estimate does not require any smallness assumption on the Lipschitz constant L; uniqueness and continuous dependence on the initial data hold for any finite L. Furthermore, for the study of long-time behavior as $T \to \infty$, the growth of the Mittag–Leffler function $E_\alpha(L(\ln T)^\alpha)$ should be examined. For fixed L, this growth remains sub-exponential with respect to T, since the argument involves the term $(\ln T)^\alpha$.

Remark 3.9. The same approach extends to nonhomogeneous equations or to weighted Banach spaces, provided the integral inequality can be brought to the Hadamard-type form (3.1). For locally Lipschitz f, continuous dependence holds locally in time.

4. Example

To illustrate the applicability of our abstract existence and uniqueness results, we consider a specific Hadamard-type fractional differential equation in the Banach space $X = \mathbb{R}$.

Consider the following initial value problem:

$$^{H}D^{1/2}_{1+}u(t) = 0.1\sin(u(t)) + 0.5, \quad t \in [1,e], \tag{4.1}$$

with the initial condition $u(1) = 0$.

Analysis:

(1) Mapping. The nonlinear function is defined as

$$f(t,u) = 0.1\sin(u) + 0.5.$$

(2) Linear growth condition. The sine function is globally Lipschitz on $\mathbb{R}$ with constant 1, that is

$$|\sin(u) - \sin(v)| \leq |u - v|, \qquad \forall u, v \in \mathbb{R}.$$

Taking $v = 0$ and using $\sin(0) = 0$, we obtain

$$|\sin(u)| \leq |u|.$$

Hence

$$|f(t, u)| = |0.1 \sin(u) + 0.5| \leq 0.1|\sin(u)| + 0.5 \leq 0.1|u| + 0.5.$$

Therefore the linear growth condition holds with

$$M_a = 0.5, \qquad M_b = 0.1.$$

(3) Existence condition. Here $\alpha = 0.5$ and $T = e$. The condition in Theorem 3.3

$$\frac{M_b(\ln T)^\alpha}{\Gamma(\alpha + 1)} < 1$$

becomes

$$\frac{0.1(\ln e)^{0.5}}{\Gamma(1.5)} = \frac{0.1}{\sqrt{\pi}/2} \approx 0.1128 < 1.$$

Hence all assumptions of Theorem 3.3 are satisfied, and therefore the problem (4.1) admits at least one mild solution $u \in C([1, e], \mathbb{R})$.

Moreover, since

$$|f'(u)| = |0.1 \cos(u)| \leq 0.1,$$

the function f is globally Lipschitz with constant $L = 0.1$. Consequently, the uniqueness of the mild solution follows from the result established in Section 3.2.

5. Conclusions

This research has successfully developed a rigorous mathematical framework for analyzing abstract fractional evolution models governed by Hadamard-type operators in Banach spaces. By prioritizing a generalized operator-theoretic perspective, we demonstrated the existence of mild solutions through fixed-point theory, while the uniqueness and Hadamard-type stability were firmly established via a specialized fractional Gronwall inequality and Mittag-Leffler analysis under Lipschitzian constraints. Our abstract approach ensures a versatile theoretical foundation that encompasses various nonlinear fractional dynamics.

Acknowledgement. The author is grateful for the institutional facilities and support made available by the University of Jijel, Algeria, during the preparation of this work. This work is dedicated to the memory of his late father, Djeriba Belkacem. He also wishes to express his sincere gratitude to his mother, Soukkou Meriem, for her unwavering support, encouragement, and patience throughout the completion of this research.

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[1] University of Jijel, Jijel, ALG.
Email address: hichem.djeriba@univ-jijel.dz