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POISSON-LIE STRUCTURES ON THE FOUR-DIMENSIONAL OSCILLATOR LIE GROUPS

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Abstract. This paper studies local normal forms of Poisson–Lie structures on four-dimensional Lie groups, which play a fundamental role in mechanics and physics. A central problem is to determine local coordinates under diffeomorphisms that yield the simplest possible expressions of these structures. We compute all Poisson–Lie structures via their correspondence with Lie bialgebra structures, using the fact that any connected Poisson–Lie group arises from a Lie bialgebra. We then prove their local linearizability near the identity by constructing explicit normal forms, obtained through solving an associated system of differential equations.

Keywords. Poisson-Lie groups, Lie bialgebras, Oscillator groups
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1. Introduction

Let $n \in \mathbb{N}^*$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n$.

The λ-oscillator algebra, denoted by $\mathcal{G}_n(\lambda)$, is a real solvable Lie algebra of dimension $2n+2$. With respect to the basis $(e_{-1}, e_0, e_i, \tilde{e}_i)_{1\leq i\leq n}$, its non-vanishing Lie brackets are

$$[e_{-1}, e_i] = \lambda_i \tilde{e}_i, \qquad [e_{-1}, \tilde{e}_i] = -\lambda_i e_i, \qquad [e_i, \tilde{e}_i] = e_0.$$

Moreover, $\mathcal{G}_n(\lambda)$ admits a decomposition as a semidirect product of the Heisenberg Lie algebra with a one-dimensional abelian Lie algebra.

The associated λ-oscillator Lie group $\mathcal{G}_n(\lambda)$ is realized on the manifold $\mathbb{R}^{2n+2} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ equipped with the multiplication

$$(t, s, z).(t', s', z') = \left(t + t', s + s' + \tfrac{1}{2}\sum_{j=1}^{n} Im(\overline{z_j} z_j' e^{it\lambda_j}), ..., z_j + z_j' e^{it\lambda_j}, ...\right).$$

The groups $G_n(\lambda)$ constitute the unique family of non-abelian, connected, simply connected solvable Lie groups that admit a bi-invariant Lorentzian metric [14]. It is well established that $G_n(\lambda)$ play a significant role in the construction of homogeneous compact Lorentzian manifolds introduced by Medina and Revoy

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[15]. Such manifolds could be seen as quotients of oscillator groups by discrete subgroups. We emphasize that oscillator groups have attracted considerable attention in differential geometry and mathematical physics [1, 2, 5, 9, 12].

Up to isomorphism, there exists a unique four-dimensional oscillator Lie algebra, which we denote by $\mathcal{G}_1(\lambda)$. In this work, our focus is the four-dimensional oscillator group $G_1(\lambda)$. We begin by determining all Poisson–Lie structures on $G_1(\lambda)$ via their correspondence with Lie bialgebra structures on the associated Lie algebra $\mathcal{G}_1(\lambda)$.

Next, we prove that all these Poisson–Lie structures are locally linearizable around the identity element by explicitly constructing suitable linearizing coordinate systems. Such coordinates are of particular importance for Poisson–Lie structures, as they clarify the local geometry of symplectic leaves and facilitate the analysis of singularities of the Poisson tensor. Moreover, in the context of Hamiltonian dynamical systems, they simplify the equations of motion and enable a more direct identification of first integrals (Casimirs). Linearizing coordinates also streamline the computation of Poisson cohomology, as they frequently reduce the problem to the cohomology of the associated Lie algebra; however, this simplification is typically local in nature [11].

The article is structured as follows. Section 2 establishes the notation and terminology, and formulates the linearization problem for Poisson–Lie structures addressed in this work. Section 3 contains the main results, where we demonstrate the existence of coordinate systems in which the Poisson–Lie structures on the four-dimensional oscillator group take linear form.

2. Preliminaries

The concept of a Poisson–Lie group was first proposed by Drinfeld [7] as a semi-classical analogs of quantum groups. A Poisson–Lie structure on a Lie group G is defined by a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(G)$ for which the group multiplication map $\mu : G \times G \to G$ is Poisson, that is,

$$\{f \circ \mu,\, g \circ \mu\}_{C^\infty(G\times G)}(x,y) = \{f,g\}_{C^\infty(G)}(\mu(x,y)), \quad x,y \in G,\ f,g \in C^\infty(G),$$

where $G \times G$ is endowed with the product Poisson structure.

According to Drinfel'd [6, 7], this formulation is equivalent to specifying a skew-symmetric contravariant 2-tensor field P on G that satisfies the Jacobi identity together with

$$P(xy) = L_{x*}P_y + R_{y*}P_x, \qquad \forall x,y \in G,$$

where L_{x*} (resp. R_{y*}) denotes the differential of left (resp. right) translation. In this setting, the bivector field P is called multiplicative.

Let $(\mathcal{G}, [\,,\,])$ be the Lie algebra of G and let $\mathcal{G}^*$ denote its dual. The differential of P at the identity element $e \in G$ induces a skew-symmetric cocommutator $\delta : \mathcal{G} \to \bigwedge^2 \mathcal{G}$ fulfill the following conditions

(i) The map δ is a 1-cocycle; namely,

$$\delta([X,Y]) = \mathrm{ad}_X\delta(Y) - \mathrm{ad}_Y\delta(X), \qquad \forall X,Y \in \mathcal{G},$$

where ad denotes the adjoint representation extended to $\bigwedge^2 \mathcal{G}$.

(ii) The dual map $\delta^* : \bigwedge^2 \mathcal{G}^* \to \mathcal{G}^*$ defines a Lie bracket on $\mathcal{G}^*$.

The cocommutator δ is therefore called the Lie bialgebra structure associated with the bivector field P. Reciprocally, Lu and Weinstein [13] established that if G is simply connected, then every Lie bialgebra structure $\delta : \mathcal{G} \to \bigwedge^2 \mathcal{G}$ on $\mathcal{G} = \mathrm{Lie}(G)$ integrates uniquely to a Poisson–Lie structure P on G satisfying

$$(d_e P)(S) = \delta(S), \qquad \forall S \in \mathcal{G}.$$

The preceding relation can be written as

$$L_{\tilde{S}} P(x) = l_{x*}\, \delta(S), \qquad \forall S \in \mathcal{G},\ \forall x \in G, \tag{2.1}$$

where $\tilde{S}$ is a vector field on G such that $\tilde{S}(e) = S$, and $L_{\tilde{S}} P$ represents the Lie derivative of P along $\tilde{S}$.

Remark 2.1. A Lie bialgebra $(\mathcal{G}, \delta)$ is said to be coboundary if there exists an element $r \in \bigwedge^2 \mathcal{G}$ such that $\delta(u) = \mathrm{ad}_u\, r$ for any $u \in \mathcal{G}$. In this case, condition (i) holds automatically, while condition (ii) is equivalent to requiring that r fulfills the generalized classical Yang–Baxter equation:

$$\mathrm{ad}_u [r, r] = 0, \qquad \text{for all } u \in \mathcal{G},$$

where $[r, r] \in \bigwedge^3 \mathcal{G}$ denotes the Schouten bracket[16].

At the identity element e of G, the Poisson–Lie structure P necessarily has rank zero. In a neighborhood U of e, one can introduce local coordinates $(x_1, x_2, \ldots, x_n)$; in terms of these coordinates P takes the form

$$P(x) = \sum_{1 \le i < j \le n} P_{ij}(x)\, \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \qquad \forall x \in U,$$

where $P_{ij} \in C^\infty(G)$ and $P_{ij}(e) = 0$. Equivalently, the Poisson bracket of the coordinate functions is given by

$$\{x_i, x_j\}(x) = \langle P, dx_i \wedge dx_j \rangle = P_{ij}(x), \qquad \forall x \in U.$$

Expanding P_{ij} as a Taylor series around e yields

$$P_{ij}(x) = \sum_{1 \le k \le n} c_{ij}^k\, x_k \ + \ \theta_{ij}(x),$$

where θ_{ij} is a smooth function whose Taylor expansion begins with terms of order at least two, and the coefficients are $c_{ij}^k = \frac{\partial P_{ij}}{\partial x_k}(e)$.

The coefficients c_{ij}^k arising from this expansion have a fundamental algebraic interpretation. Because P satisfies the Jacobi identity, the collection $\{c_{ij}^k\}_{\substack{1 \le i < j \le n \\ 1 \le k \le n}}$ constitutes a set of structure constants for the Lie algebra $(\mathcal{G}^*, \delta^*)$ which is the Lie algebra dual to $(\mathcal{G}, [\,,\,])$. Thus $\mathcal{G}^*$ may be viewed as the linearizing Lie algebra associated with the Poisson–Lie structure P.

The linear terms in the expansion define a linear Poisson structure, known as the linear part of P and denoted P_0. In the local coordinates $(x_1, x_2, \ldots, x_n)$ it

is expressed as

$$P_0(x) = \sum_{\substack{1\leq i<j\leq n\\ 1\leq k\leq n}} C_{ij}^k\, x_k\, \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{2.2}$$

A fundamental question in the theory of Poisson structures, which was first conjectured by Weinstein [17] and subsequently investigated by several authors [3, 4, 8, 10], concerns the local linearizability of Poisson structures. More precisely, the problem can be stated as follows:

Does there exist a new system $(y_1, \ldots, y_n)$ of coordinates defined near the identity e where the functions θ_{ij}^k vanish identically?
So that in terms of this system, the expression of the Poisson structure P is linear; that is

$$P(x) = \sum_{\substack{1\leq i<j\leq n\\ 1\leq k\leq n}} C_{ij}^k y_k \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$$

3. Main results

The aim of this section is twofold. First, we determine all Poisson–Lie structures on four-dimensional Lie groups by examining the corresponding Lie bialgebra structures on their Lie algebras. Second, we prove that these Poisson–Lie structures are locally linearizable near the identity element by giving an explicit local normal form.

3.1. **Four-dimensional oscillator Lie algebra and group.**

The four-dimensional oscillator Lie algebra $\mathcal{G}_1(\lambda)$ can be represented as a matrix Lie algebra defined by

$$\mathcal{G}_1(\lambda) = \left\{ A(t,x,y,z) = \begin{pmatrix} 0 & -y & z & -2x \\ 0 & 0 & t & z \\ 0 & -t & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix} : (t,z,x,y) \in \mathbb{R}^4 \right\}.$$

The standard basis for $\mathcal{G}_1(\lambda)$ is given by

$$e_{-1} = A(1,0,0,0), \quad e_0 = A(0,1,0,0), \quad e_1 = A(0,0,1,0), \quad \widetilde{e}_1 = A(0,0,0,1).$$

The only non-vanishing commutation relations between these bases elements are

$$[e_{-1}, e_1] = \widetilde{e}_1 \quad [e_{-1}, \widetilde{e}_1] = -e_1 \quad [e_1, \widetilde{e}_1] = e_0.$$

The associated connected and simply connected Lie group corresponding to this Lie algebra is denoted by $G_1(\lambda)$, and it is given by

$$G_1(\lambda) = \left\{ \begin{pmatrix} 0 & -y\cos(t) - z\sin(t) & z\cos(t) - y\sin(t) & -2x \\ 0 & \cos(t) & \sin(t) & z \\ 0 & -\sin(t) & \cos(t) & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : (t,z,x,y) \in \mathbb{R}^4 \right\}.$$

Therewith, $G_1(\lambda)$ can be identified with $\mathbb{R}^4$, corresponding to the group multiplication law

$$(t,z,x,y).(t',z',x',y') = \Big(t+t', z+z'+\frac{1}{2}((xy'-yx')\cos t+(xx'+yy')\sin t) \\ , x+x'\cos t-y'\sin t, y+y'\cos t+x'\sin t\Big)$$

Denote by $(E_{-1}, E_0, E_1, \widetilde{E}_1)$ the basis of the left-invariant vector fields on $G_1(\lambda)$ that coincide with $(e_{-1}, e_0, e_1, \widetilde{e}_1)$ at the identity. A direct computation yields explicit expressions for these vector fields in terms of local coordinates; they are given by

$$\begin{aligned} E_{-1} &= \frac{\partial}{\partial t}, & E_1 &= \frac{1}{2}(x\sin t - y\cos t)\frac{\partial}{\partial z} + \cos t\frac{\partial}{\partial x} + \sin t\frac{\partial}{\partial y}, \\ E_0 &= \frac{\partial}{\partial z}, & \widetilde{E}_1 &= \frac{1}{2}(x\cos t + y\sin t)\frac{\partial}{\partial z} - \sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y}. \end{aligned}$$

3.2. **Lie Bialgebra and Poisson-Lie structures on the four-dimensional oscillator group.** We now turn to the description of Lie bialgebra structures on the four-dimensional oscillator Lie algebras. According to the classification established in [1], each such structure is determined by a linear map $\delta : \mathcal{G}_1(\lambda) \to \bigwedge^2 \mathcal{G}_1(\lambda)$. In terms of the basis vectors $(e_{-1}, e_0, e_1, \widetilde{e}_1)$, the cocommutators are given explicitly as follows

$$\begin{aligned} \delta(e_{-1}) &= ce_0\wedge e_1 - be_0\wedge\widetilde{e}_1 \\ \delta(e_0) &= 0 \\ \delta(e_1) &= -\alpha e_0\wedge e_1 + ae_0\wedge\widetilde{e}_1 \\ \delta(\widetilde{e}_1) &= -ae_0\wedge e_1 - \alpha e_0\wedge\widetilde{e}_1, \end{aligned}$$

where a, b, c and $\alpha \in \mathbb{R}$.

Remark 3.1. The Lie bialgebra on four-dimensional oscillator Lie algebras $(\mathcal{G}_1(\lambda), \delta)$ is always coboundary. Indeed, for

$$r = ae_{-1}\wedge e_0 - be_0\wedge e_1 - ce_0\wedge\widetilde{e}_1 + \alpha e_1\wedge\widetilde{e}_1,$$

we have $\delta(u) = \mathrm{ad}_u\, r$, for any $u \in \mathcal{G}_1(\lambda)$. In particular

$$r = ae_{-1}\wedge e_0 - be_0\wedge e_1 - ce_0\wedge\widetilde{e}_1$$

satisfies the classical Yang–Baxter equation, that is, $[r, r] = 0$.

Let P denote the Poisson–Lie structures on the four-dimensional oscillator group $G_1(\lambda)$ associated with the Lie bialgebra structures δ. We introduce the following notation:

$$P = P_{-10}E_{-1}\wedge E_0 + P_{-11}E_{-1}\wedge E_1 + \widetilde{P}_{-11}E_{-1}\wedge\widetilde{E}_1 + P_{01}E_0\wedge E_1 + \widetilde{P}_{01}E_0\wedge\widetilde{E}_1 + \widetilde{P}_{11}E_1\wedge\widetilde{E}_1.$$

The $(P_{-10}, P_{-11}, \widetilde{P}_{-11}, P_{01}, \widetilde{P}_{01}, \widetilde{P}_{11})$ are the components of the Poisson-Lie structure P in the basis

$$(E_{-1} \wedge E_0, E_{-1} \wedge E_1, E_{-1} \wedge \widetilde{E}_1, E_0 \wedge E_1, E_0 \wedge \widetilde{E}_1, E_1 \wedge \widetilde{E}_1)$$

of the bivector fields on $G_1(\lambda)$.

By solving equation (2.1), which characterizes the correspondence between P and δ, a direct computation shows that the only non-vanishing components of P are

$$\begin{aligned} P_{01}(t,z,x,y) &= (-\alpha x - ay + b)\cos(t) + (ax - \alpha y + c)\sin(t) - b \\ \widetilde{P}_{01}(t,z,x,y) &= (ax - \alpha y + c)\cos(t) + (\alpha x + ay - b)\sin(t) - c. \end{aligned}$$

Since,

$E_0 \wedge E_1 = \cos(t)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \sin(t)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial y}$ and $E_0 \wedge \widetilde{E}_1 = -\sin(t)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + \cos(t)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial y}$, we obtain:

Proposition 3.2. *In terms of the local coordinates (t,z,x,y), the Poisson-Lie structures on the four-dimensional oscillator group are given explicitly by*

$$P = (-\alpha x - ay - b\cos t + c\sin t + b)\,\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} + (ax - \alpha y - b\sin t - c\cos t + c)\,\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial y},$$

where a, b, c and α are real parameters.

We now turn to the linearization problem for these Poisson-Lie structures. Observe that a Lie bialgebra structure δ associated with a Poisson–Lie structure P induces a linear Poisson structure on the additive Lie group $\mathcal{G}$ ($\mathcal{G} \simeq \mathbb{R}^n$), which can be written as

$$\delta(a) = \sum_{\substack{1\le i<j\le n \\ 1\le k\le n}} c_{ij}^k\, a_k\, \partial_i \wedge \partial_j, \qquad a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \tag{3.1}$$

where $(\partial_1, \ldots, \partial_n)$ denotes the canonical basis of $\mathbb{R}^n$.

Expression (3.1) coincides with the linear component P_0 given in (2.2). Consequently, the linearization problem for a Poisson-Lie structure can be formulated as follows:

Does there exist a local Poisson diffeomorphism $\varphi : G \to \mathcal{G}$ mapping a neighborhood of the identity element $e \in G$ into a neighborhood of $0 \in \mathcal{G}$ such that $\varphi(e) = 0$?

If $(x'_1, \ldots, x'_n)$ denote the component functions of φ, then φ satisfies the system

$$\{x'_i, x'_j\} = \sum_{1\le k\le n} c_{ij}^k\, x'_k, \qquad 1 \le i < j \le n. \tag{3.2}$$

Specializing this question to the four-dimensional oscillator Poisson-Lie group, we obtain the following main result.

Theorem 3.3. *There exists a new system (t', z', x', y') of coordinates near the identity, in which the expression of the Poisson-Lie structure P on the four-dimensional oscillator Poisson-Lie group is linear; that is,*

$$P = (-\alpha x' - ay' + ct')\frac{\partial}{\partial z'} \wedge \frac{\partial}{\partial x'} + (ax' - \alpha y' - bt')\frac{\partial}{\partial z'} \wedge \frac{\partial}{\partial y'}.$$

Proof. Note that the tensor P given in Proposition 3.2 can be written as

$$P = \frac{\partial}{\partial z} \wedge i_{dz}P,$$

where

$$i_{dz}P = (-\alpha x - ay - b\cos(t) + c\sin(t) + b)\frac{\partial}{\partial x} + (ax - \alpha y - b\sin(t) - c\cos(t) + c)\frac{\partial}{\partial y},$$

is the vector field that arises from the interior product of the tensor P and the 1-form dz. This vector field is independent of the coordinate z, and its linear part is written as

$$(i_{dz}P)_0 = (-\alpha x - ay + ct)\frac{\partial}{\partial x} + (ax - \alpha y - bt)\frac{\partial}{\partial y}.$$

So, if there exists a new system $\varphi = (t', z', x', y')$ of coordinates defined near 0, in terms of which the expression of $i_{dz}P$ is linear; that is,

$$\varphi_* i_{dz}P = (i_{dz}P)_0,$$

then the Poisson tensor P is linear in these coordinates. Indeed, it holds that the operator $\frac{\partial}{\partial z}$ is invariant under φ, and

$$\varphi_* P = \varphi_*(\frac{\partial}{\partial z} \wedge i_{dz}P) = \varphi_*\frac{\partial}{\partial z} \wedge \varphi_* i_{dz}P = \frac{\partial}{\partial z} \wedge (i_{dz}P)_0 = P_0.$$

Consequently, we put $z' = z$, and we look for t', x', and y' that satisfy the system of equations (3.2).

In this case, the system of equations (3.2) will be

$$\begin{cases} f\dfrac{\partial x'}{\partial x} + g\dfrac{\partial x'}{\partial y} = -\alpha x' - ay' + ct', \\ f\dfrac{\partial y'}{\partial x} + g\dfrac{\partial y'}{\partial y} = ax' - \alpha y' - bt', \\ f\dfrac{\partial t'}{\partial x} + g\dfrac{\partial t'}{\partial y} = 0. \end{cases} \tag{3.3}$$

where, $f = -\alpha x - ay - b\cos(t) + c\sin(t) + b$ and $g = ax - \alpha y - b\sin(t) - c\cos(t) + c$. In particular, we show that the coordinates

$$(t', z', x', y') = (2\tan(\frac{t}{2}), z, x + y\tan(\frac{t}{2}), y - x\tan(\frac{t}{2})),$$

are solutions to the system of equations (3.3). Thus, we obtain the desired coordinates. □

References

1. Boucetta, M. & Medina, A. (2011). Solutions of the Yang–Baxter equations on quadratic Lie groups: The case of oscillator groups. J. Geom. Phys. 61, 2309-2320. https://doi.org/10.1016/j.geomphys.2011.07.004
2. Bromberg, S. & Medina, A. (2004). Geometry of oscillator groups and locally symmetric manifold. Geom. Dedic. 106, 97–111. https://doi.org/10.1023/B:GEOM.0000033845.70512.13
3. Chloup-Arnould, V. (1997). Linearization of Some Poisson-Lie Tensor. Journal of Geometry and Physics. 24, 145−195. https://doi.org/10.1016/S0393-0440(97)00004-1
4. Conn, J. (1985). Normal Forms for Smooth Poisson Structures. Annals of Mathematics. 121, 565-593. http://dx.doi.org/10.2307/1971210
5. Díaz, R.D., Gadea,P.M. & Oubiña, J.A. (1999). Reductive decompositions and Einstein–Yang–Mills equations associated to the oscillator group. J. Math. Phys. 40(7), 3490–3498. https://doi.org/10.1063/1.532902
6. Drinfeld, V.G. (1983). Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations. Soviet Math. Dokl. 27, 68-71. https://doi.org/10.1142/9789812798336_0009
7. Drinfeld, V.G. (1988). Quantum Groups. J Math Sci 41, 898–915. https://doi.org/10.1007/BF01247086
8. Dufour, J.P. (1990). Linéarisation de certaines structures de Poisson. Journal of Differential Geometry. 32, 415- 428. https://doi.org/10.4310/jdg/1214445313
9. Gadea, P.M. & Oubiña, J.A. (1999). Homogeneous Lorentzian structures on the oscillator groups. Arch. Math. (Basel). 73(4), 311–320. https://doi.org/10.1007/s000130050403
10. Ganbouri, B. & Mansouri, M.W. (2021). Linearization of Poisson-Lie structures on the 2D Euclidean and (1+1) Poincaré groups. Ural Mathematical Journal. 7(2), 33-42. http://dx.doi.org/10.15826/umj.2021.2.002
11. Iskamle, B., Tcheka, C. & Dongho, J. (2024). On the Lichnerowicz-Poisson cohomology complex and quasi-Poisson algebras. Gulf Journal of Mathematics, 18(2), 47-62. https://doi.org/10.56947/gjom.v18i2.2228
12. Levichev, A.V. (1986). Chronogeometry of an electromagnetic wave given by a biinvariant metric on the oscillator groups. Sib. Math. J. 27(2), 237–245. https://doi.org/10.1007/bf00969391
13. Lu, J.H. & Weinstein, A. (1990). Poisson-Lie Group, Dressing Transformations and Bruhat Decomposition. Journal of Differential Geometry. 31, 301-599. https://doi.org/10.4310/jdg/1214444324
14. Medina, A. (1985). Groupes de Lie munis de métriques bi-invariantes. Tohoku Math. J. 37(4), 405-421. https://doi.org/10.2748/tmj/1178228586
15. Medina, A. & Revoy, P. (1985). Les groupes oscillateurs et leurs réseaux. Manuscr. Math. 52(1–3), 81–95. https://doi.org/10.1007/bf01171487
16. Moukala, N. M. & Bossoto, B. G. R. (2023). On Lie-Rinehart-Poisson algebras structures. Gulf Journal of Mathematics, 15(1), 131-145. https://doi.org/10.56947/gjom.v15i1.1384
17. Weinstein, A. (1983). The local structures of Poisson manifolds. J. Diff. Geom. 18, 523-557. https://doi.org/10.4310/jdg/1214437787

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