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PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO NON AUTONOMOUS THIRD ORDER DIFFERENTIAL EQUATION WITH MULTIPLE DEVIATING ARGUMENTS

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Abstract. The pseudo almost automorphic solutions of non-autonomous third-order differential equation with multiple delays is investigated. By some sufficient conditions, we obtain the uniformly bounded of the solutions. Moreover, by Banach fixed point theorem we obtain the existence and uniqueness of the solutions. In addition, by suitable Laypunov functions, we give the globally exponential stability of the pseudo almost automorphic solutions. Finally, a numerical example is given to illustrate our results

1. Introduction

Almost automorphic functions are part of a hierarchy of functions which starts with periodic functions. These functions were first introduced in the literature by S. Bochner [20] as a natural generalization of the classical concept of almost periodic function. In [24], Xiao et al introduced a new class of functions, which is the pseudo almost automorphic and which is the most important generalization of the almost automorphic functions. These functions were used in many fields, and are the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mathematical biology, control theory, and other related fields (one can see [4, 11, 12, 13, 21]).

Recently, many authors studied the nonlinear third-order differential equation [1, 2]. These studies are interested mainly in stability and boundedness of the solutions [1, 5, 6, 8, 10, 18]. It should be mentioned that the proofs are based chiefly on the Lyapunov function second method.

The boundedness and existence of periodic solutions are very important in the theory and applications of differential equations. To the best of our knowledge, few authors have discussed the existence and uniqueness of a periodic solution to delay third-order differential equations. For instance, in 1978, Yoshizawa [25] and Chukwu [10] found certain sufficient conditions that guarantee the existence of a periodic solution to nonlinear differential of the third-order with the constant

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deviating argument:

$$\begin{aligned} x^{(3)} &+ f(t, x^{(1)}, x^{(2)})x^{(2)} + g(x(t-h), x^{(1)}(t-h)) + i(x(t-h)) \\ &= p(t, x, x^{(1)}, x(t-h), x^{(1)}(t-h), x^{(2)}). \end{aligned}$$

Also, Tunç [7] investigated and discussed the existence and uniqueness of a periodic solutions of non linear differential equation of the third-order with multiple deviating arguments $\tau_i (i = 1, 2, \dots, n)$:

$$\begin{aligned} x^{(3)} &+ \psi(x^{(1)})x^{(2)} + \sum_{i=1}^{n} g_i(x^{(1)}(t-\tau_i)) + f(x) \\ &= p(t, x, x(t-\tau_1), \dots, x^{(1)}, x(t-\tau_1), \dots, x^{(1)}(t-\tau_n), \dots, x). \end{aligned}$$

Recently, Tunç and Gözen [8] studied the stability and uniform boundedness in the multidelay differential equations of the third-order:

$$x^{(3)}(t) + a(t)x^{(2)}(t) + nb(t)g(x^{(1)}(t)) + c(t)\sum_{i=1}^{n} k_i(x(t-r_i)) = p(t), \quad (1.1)$$

where r_i are certain positive constants, and $a(.), b(.), c(.), g(x^{(1)}(.)), k_i(x(.))$, and $p(.)$ are real valued and continuous functions with $g(0) = k_i(0) = 0$.

Recently, Omeike [18] investigated the stability and boundedness of nonlinear differential equation of third-order with variable delay $r(t)$:

$$x^{(3)}(t) + Ax^{(2)}(t) + Bx^{(1)}(t) + H(x(t-r(t))) = p(t).$$

The existence of almost periodic, almost automorphic, and pseudo almost automorphic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory, and others(see [4, 11, 16, 21, 24]). Motivated by the above discussion, the aim of this paper is to study the oscillations and the stability of the pseudo almost automorphic solutions of the equation:

$$x^{(3)}(t) + a(t)x^{(2)}(t) + b(t)x^{(1)}(t) + \sum_{i=1}^{n} g_i(t, x(t-r_i(t))) = p(t). \quad (1.2)$$

Hence, we derive sufficient conditions which assure the exponential stability and the uniformly bounded by using new methods, and by Banach's fixed point Theorem we obtain the existence and uniqueness of the pseudo almost automorphic solution. At the best knowledge of the authors, this is the first paper dealing with the pseudo almost automorphic solutions of equation (1.2).

This article is organized as follows: In section 2, we recall some basic definitions of the pseudo almost automorphic functions. Next in Section 3, we shall derive sufficient conditions for checking the existence of the pseudo almost automorphic solution. Section 4 is dedicated to prove uniformly bounded of the solutions. In section 5, is devoted to prove the existence and uniqueness of the pseudo almost automorphic solution. In section 6, we obtain the exponential stability of the pseudo almost automorphic solution. In section 7, an example is given to demonstrate the applicability of our result. In section 8, a comparative study with previous works is discussed. Conclusions are drawn in Section 9.

2. Preliminaries

2.1. **Pseudo almost automorphic functions.** This paragraph recalls some interesting characteristics of pseudo almost automorphic functions which is necessary for the study of the existence and uniqueness of pseudo almost automorphic solutions.

Let $BC(\mathbb{R}, \mathbb{R})$ be the space of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the super norm:

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|.$$

Now let us recall the concept of pseudo almost automorphic functions. For more details, one can see [20, 24].

Definition 2.1. [20] A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost automorphic if for every sequence $(\sigma_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} f(t + s_n) = g(t)$$

exists for all $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t) \text{ for all } t \in \mathbb{R}$$

are well defined for each $t \in \mathbb{R}$.

Let $AA(\mathbb{R}, \mathbb{R})$ denote the collection of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{R}$. It can be easily shown that $(AA(\mathbb{R}, \mathbb{R}), \|.\|_\infty)$ is a banach space.

Example 2.2. Let $k : \mathbb{R} \to \mathbb{R}$ be such that:

$$k(t) = \sin\left(\frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)}\right), t \in \mathbb{R}.$$

Then, k is almost automorphic but it is not uniformly continuous on $\mathbb{R}$. Then, it is not almost periodic.

Definition 2.3. [22] A jointly continuous function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \to F(t, x)$ is almost automorphic for all $x \in K (K \subset \mathbb{R}$ being any bounded subset). Equivalently, for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that:

$$\lim_{n\to\infty} F(t + s_n, x) = G(t, x)$$

is well defined in $t \in \mathbb{R}$ and for each $x \in K$, and

$$\lim_{n\to\infty} G(t - s_n, x) = F(t, x) \text{ for all } t \in \mathbb{R}.$$

for all $t \in \mathbb{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Definition 2.4. [19] A continuous function $f : \mathbb{R} \times X \to Y$ is said to be almost automorphic in t uniformly with respect to x in X if the two following conditions hold:

(i): $\forall x \in X,\ f(.,x) \in AA(\mathbb{R}, Y)$.

(ii): f is uniformly continuous on each compact K of X with respect to the second variable x.

Denoted by $AA_U(\mathbb{R} \times X, Y)$ the set of all such functions

Theorem 2.5. [14] *Take $f \in AA_U(\mathbb{R} \times X, Y)$. Then the superposition operator N_f defined by:*

$$N_f : AA(\mathbb{R}, Y) \to AA(\mathbb{R}, Y), N_f(u) := [t \mapsto f(u(t), t)],$$

is well defined continuous from $AA(\mathbb{R}, Y)$ in $AA(\mathbb{R}, Y)$.

Theorem 2.6. [19] *Let $f \in AA_U(\mathbb{R} \times X, Y)$. If $x \in AA(\mathbb{R}, X)$, then $[t \mapsto f(t, x(t))] \in AA(\mathbb{R}, Y)$.*

The class of functions $PAP_0(\mathbb{R}, \mathbb{R})$ defined as follows:

$$PAP_0(\mathbb{R}, \mathbb{R}) = \left\{ g \in BC(\mathbb{R}, \mathbb{R}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |g(t)| dt = 0 \right\}.$$

Similarly, $PAP_0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |F(t, x)| dt = 0$$

uniformly in $x \in K$, where $K \subset \mathbb{R}$ is any bounded subset.

Definition 2.7. [24] A function $f \in BC(\mathbb{R}, \mathbb{R})$ is called pseudo almost automorphic if it can be expressed as $f = g + h$, where $g \in AA(\mathbb{R}, \mathbb{R})$ and $h \in PAP_0(\mathbb{R}, \mathbb{R})$. The collection of such functions will be denoted by $PAA(\mathbb{R}, \mathbb{R})$.

The functions g and h appearing in Definition (2.7), are respectively called the almost automorphic and the ergodic perturbation components of f. Moreover, the decomposition given in Definition (2.7) is unique since $PAA(\mathbb{R}, \mathbb{R}) = AA(\mathbb{R}, \mathbb{R}) \oplus PAP_0(\mathbb{R}, \mathbb{R})$.

Theorem 2.8. [22] *The space $PAA(\mathbb{R}, \mathbb{R})$ equipped with the supnorm $\|.\|_\infty$ is a Banach space.*

Remark 2.9. Clearly we have the following hierarchy with strict inclusions:

$$AP(\mathbb{R}, \mathbb{R}) \subset AA(\mathbb{R}, \mathbb{R}) \subset PAA(\mathbb{R}, \mathbb{R}).$$

Lemma 2.10. [15] *Let f be a nonnegative function defined on $[0, \infty)$ such that f is integrable on $[0, \infty)$ and is uniformly continuous on $[0, \infty)$. Then,* $\lim_{t \to \infty} f(t) = 0$.

3. Main assumptions

In this paper, we consider the nonautonomous differential equation of the third-order with multiple deviating arguments:

$$x^{(3)}(t) + a(t)x^{(2)}(t) + b(t)x^{(1)}(t) + \sum_{i=1}^{n} g_i(t, x(t - r_i(t))) = p(t),$$

where $a(.)$ almost automorphic, $r(.)$ and $p(.)$ are pseudo almost automorphic functions, and for $i = 1, 2, \ldots, n$, $g_i(., x(. - r_i(.)))$, $b(.)$ are real valued and continuous, bounded functions, with $g_i(0, 0) = 0$.

Let us consider

$$x'(t) = y(t) - \alpha x(t), y'(t) = z(t) - \beta y(t)$$

where $\alpha, \beta > 1$ are constant.

Then the equation (1.2) can be written as follows:

$$\begin{aligned} x'(t) &= y(t) - \alpha x(t) \\ y'(t) &= z(t) - \beta y(t) \\ z'(t) &= -(a(t) - \alpha - \beta)z(t) + [(\alpha + \beta)(a(t) - \alpha) - b(t) - \beta^2]y(t) \\ &\quad -[\alpha^2(a(t) - \alpha) - \alpha b(t)]x(t) - \sum_{i=1}^{n} g(t, x(t - r_i(t))) + p(t). \end{aligned} \tag{3.1}$$

Let $\Omega = (PAA(\mathbb{R}, \mathbb{R}))^3$. Then, Ω is a Banach space with the norm defined by:

$$\|\omega\|_\Omega = \sup_{t \in \mathbb{R}} \|\omega(t)\| = \sup_{t \in \mathbb{R}} \max_{1 \le i \le 3} |\omega_i(t)|.$$

In this section, we consider the following conditions:

A4.1: The function g is global Lipschitz continuous, that is, there exists $L_g > 0$ such that for all $u, v \in \mathbb{R}$

$$|g_i(t, u) - g_i(t, v)| \le L_g |u - v|.$$

A4.2: There exists constants $\alpha > 1, \beta > 1$ such that:

$$0 < \frac{\sup_t \Big(|(\alpha + \beta)(a(t) - \alpha) - \beta^2 + \alpha^2(a(t) - \alpha) - b(t)(1 + \alpha) - nL_g| \Big)}{\inf_t\{a(t) - \alpha - \beta\}} < 1.$$

4. Uniformly bounded of the solutions

Theorem 4.1. *Under assumption (A4.2), and*

$$-\alpha^2(\underline{a} - \alpha) \quad + \quad \alpha\bar{b} - nL_g + (\alpha + \beta)(\bar{a} - \alpha) - \beta^2 - \underline{b} > \bar{a} - \alpha - \beta.$$

There exists $N > 0$, such that any solution $X(t) = (x(t), y(t), z(t))$ of system (3.1) *satisfies*

$$|x(t)| \le N, |y(t)| \le N, |z(t)| \le N \text{ for all } t \ge 0.$$

Proof. Let $w(t) = \max\{|x(t)|, |y(t)|, |z(t)|\}$ for all $t \ge 0$. We have to proof that $w(t) \le N$. Suppose that there exists t_1 such that

$$\begin{cases} w(t_1) = N, \\ w(t) < N, 0 \le t < t_1. \end{cases}$$

Next, we prove this contradiction in three cases:

Case1:$\max\{|x(t)|,|y(t)|,|z(t)|\}=|x(t)|$**.:** Now we have to calculate the derivative of $w(t_1)$.

$$\begin{aligned}0\leq w'(t_1) &= -\alpha x(t_1)+y(t_1)\\ &\leq -\alpha N+N=N(1-\alpha)<0,\end{aligned}$$

which is a contradiction. So, $|w(t)|\leq N$.

Case2: $\max\{|x(t)|,|y(t)|,|z(t)|\}=|y(t)|$**.:** Then

$$\begin{aligned}0\leq w'(t_1) &= -\beta y(t_1)+z(t_1)\\ &\leq -\beta N+N=N(1-\beta)<0,\end{aligned}$$

which is a contradiction. Then, $|w(t)|\leq N$.

Case3:$\max\{|x(t)|,|y(t)|,|z(t)|\}=|z(t)|$**:**

$$\begin{aligned}0 &\leq w'(t_1)=(a(t_1)-\alpha-\beta)z(t_1)-[(\alpha+\beta)(a(t_1)-\alpha)-b(t_1)-\beta^2]y(t_1)\\ &\quad +[\alpha^2(a(t_1)-\alpha)-\alpha b(t_1)]x(t_1)+\sum_{i=1}^{n}g(t_1,x(t_1-r_i(t_1)))-p(t_1)\\ &\leq [\bar{a}-\alpha-\beta-[(\alpha+\beta)(a(t_1)-\alpha)-b(t_1)-\beta^2]+\alpha^2(a(t_1)-\alpha)\\ &\quad -\alpha b(t_1)+nL_g]N<0,\end{aligned}$$

which is a contradiction. Consequently, $w(t)\leq N$ for all $t\geq 0$.

□

5. Existence of the pseudo almost automorphic solutions

Lemma 5.1. *If $\iota(.)\in PAA(\mathbb{R},\mathbb{R})$, then the function $\Theta:t\mapsto\int_{-\infty}^{t}\iota(s)e^{-(t-s)\alpha}ds$ is pseudo almost automorphic function.*

Proof. Clearly, $\int_{-\infty}^{t}\iota(s)e^{-(t-s)\alpha}ds$ is well defined and $\Theta(.)\in BC(\mathbb{R},\mathbb{R})$. Let $\iota()\in PAA(\mathbb{R},\mathbb{R})$, then the function ι is written as follows $\iota=\iota_1+\iota_2$ where $\iota_1\in AA(\mathbb{R},\mathbb{R})$ and $\iota_2\in PAP_0(\mathbb{R},\mathbb{R})$. That is

$$\begin{aligned}\Theta(t) &= \int_{-\infty}^{t}\iota_1(s)e^{-(t-s)\alpha}ds+\int_{-\infty}^{t}\iota_2(s)e^{-(t-s)\alpha}ds\\ &= \Theta_1(t)+\Theta_2(t),\end{aligned}$$

where

$$\Theta_1(t)=\int_{-\infty}^{t}\iota_1(s)e^{-(t-s)\alpha}ds,\Theta_2(t)=\int_{-\infty}^{t}\iota_2(s)e^{-(t-s)\alpha}ds.$$

Now, we shall prove that $\Theta_1(.)\in AA(\mathbb{R},\mathbb{R})$. Since $\iota_1(.)\in AA(\mathbb{R},\mathbb{R})$, then for every sequence $(s'_n)_{n\in\mathbb{N}}\in\mathbb{R}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that:

$$\lim_{n\to\infty}\iota_1(s+s_n) = v(s),$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to\infty}v(s-s_n)=\iota_1(s),$$

for each $t \in \mathbb{R}$. Set

$$M(t) = \int_{-\infty}^{t} \upsilon(s)e^{-(t-s)\alpha}ds, N(t) = \int_{-\infty}^{t} \upsilon(s-s_n)e^{-(t-(s-s_n))\alpha}ds.$$

Then for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for $n > n_0$, one has

$$|\iota_1(t+s_n) - \upsilon(t)| \ < \ \varepsilon, \text{ for each } t \in \mathbb{R}.$$

$$\begin{aligned}
|\Theta_1(t+s_n) - M(t)| &= |\int_{-\infty}^{t+s_n} \iota_1(s)e^{-(t+s_n-s)\alpha}ds - \int_{-\infty}^{t} \upsilon(s)e^{-(t-s)\alpha}ds| \\
&= \left| \int_{-\infty}^{t} \iota_1(s+s_n)e^{-(t-s)\alpha}ds - \int_{-\infty}^{t} \upsilon(s)e^{-(t-s)\alpha}ds \right| \\
&= \left| \int_{-\infty}^{t} (\iota_1(s+s_n) - \upsilon(s))e^{-(t-s)\alpha}ds \right| \\
&\leq \varepsilon \int_{-\infty}^{t} e^{-(t-s)\alpha}ds < \varepsilon.
\end{aligned}$$

Hence, one has

$$\lim_{n\to\infty} |\Theta_1(t+s_n) - M(t)| = 0, \text{ for each } t \in \mathbb{R}.$$

Similarly, one can obtain

$$\lim_{n\to\infty} |\Theta_1(t) - N(t-s_n)| = 0.$$

Therefore, $\Theta_1(.) \in AA(\mathbb{R}, \mathbb{R})$. Next, we have to prove that $\Theta_2 \in PAP_0(\mathbb{R}, \mathbb{R})$, i.e. $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | \int_{\infty}^{t} e^{-(t-s)\alpha}\iota_2(s)ds|dt = 0$.

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | \int_{\infty}^{t} e^{-(t-s)\alpha}\iota_2(s)ds|dt &\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{t} e^{-(t-s)\alpha}|\iota_2(s)|dsdt \\
&\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\alpha}|\iota_2(s)|dsdt \\
&\quad + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\alpha}|\iota_2(s)|dsdt \\
&\quad = I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\alpha}|\iota_2(s)|dsdt, I_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\alpha}|\iota_2(s)|dsdt.$$

Now, we have to demonstrate that $I_1 = I_2 = 0$.

$$\begin{aligned} I_1 &\leq \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\infty}^{-T} e^{-(t-s)\alpha} ds dt \\ &\leq \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-t\alpha} \int_{\infty}^{-T} e^{s\alpha} ds dt \\ &= \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T} \frac{e^{-T\alpha}}{\alpha} \int_{-T}^{T} e^{-t\alpha} dt \\ &= \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T\alpha^2} [1 - e^{-2T\alpha}] = 0. \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)\alpha} |\iota_2(s)| ds dt &\leq \frac{1}{2T} \int_{-T}^{T} \int_{0}^{T+u} e^{-u\alpha} |\iota_2(t-u)| du dt \\ &\leq \frac{1}{2T} \int_{0}^{\infty} e^{-\alpha u} (\int_{-T-u}^{T+u} |\iota_2(s)| ds) du \\ &\leq \int_{0}^{\infty} e^{-\alpha u} (\frac{1}{2T} \int_{-T-u}^{T+u} |\iota_2(s)| ds) du. \end{aligned}$$

Since $\iota_2 \in PAP_0(\mathbb{R}, \mathbb{R})$, then $\lim_{T\to\infty} \frac{1}{2T} \int_{-T-u}^{T+u} |\iota_2(s)| ds = 0$, uniformly with respect to u. Finally, by the Lebesgue's dominated convergence Theorem, we obtain $\Theta_2(.) \in PAP_0(\mathbb{R}, \mathbb{R})$. Consequently, Θ is the sum of almost automorphic and ergodic functions, then Θ belongs to $PAA(\mathbb{R}, \mathbb{R})$. □

Lemma 5.2. *Let $a(.) \in AA(\mathbb{R}, \mathbb{R})$ and $\iota(.) \in PAA(\mathbb{R}, \mathbb{R})$, then the function $\Theta : t \mapsto \int_{-\infty}^{t} \iota(s) e^{-\int_s^t (a(u)-\alpha-\beta) du} ds$ is pseudo almost automorphic functions.*

Proof. It is easy to see that $\int_{-\infty}^{t} \iota(s) e^{-\int_s^t (a(u)-\alpha-\beta) du} ds \in BC(\mathbb{R}, \mathbb{R})$. Note that $\psi(t) = a(t) - \alpha - \beta$. Let $\iota() \in PAA(\mathbb{R}, \mathbb{R})$, then we can write $\iota = \iota_1 + \iota_2$ where $\iota_1 \in AA(\mathbb{R}, \mathbb{R})$ and $\iota_2 \in PAP_0(\mathbb{R}, \mathbb{R})$. Then

$$\begin{aligned} \Theta(t) &= \int_{-\infty}^{t} \iota_1(s) e^{-\int_s^t \psi(u) du} ds + \int_{-\infty}^{t} \iota_2(s) e^{-\int_s^t \psi(u) du} ds \\ &= \Theta_1(t) + \Theta_2(t), \end{aligned}$$

where

$$\Theta_1(t) = \int_{-\infty}^{t} \iota_1(s) e^{-\int_s^t \psi(u) du} ds, \Theta_2(t) = \int_{-\infty}^{t} \iota_2(s) e^{-\int_s^t \psi(u) du} ds.$$

Now, we have to show that $\Theta_1(.) \in AA(\mathbb{R}, \mathbb{R})$. Since $a(.), \iota_1(.) \in AA(\mathbb{R}, \mathbb{R})$, then for every sequence $(\tau'_n)_{n\in\mathbb{N}} \in \mathbb{R}$, there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \iota_1(s + \tau_n) = v(s), \lim_{n\to\infty} a(s + \tau_n) = v_1(s),$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} v(s - \tau_n) = \iota_1(s), \lim_{n\to\infty} v_1(s - \tau_n) = a(s)$$

for each $t \in \mathbb{R}$. Set

$$M(t) = \int_{-\infty}^{t} v(s)e^{-\int_s^t (v_1(u)-\alpha-\beta)du}ds, N(t) = \int_{-\infty}^{t} v(s-\tau_n)e^{-\int_{s-\tau_n}^t (v_1(u)-\alpha-\beta)du}ds.$$

Moreover, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for $n > n_0$, one has

$$|\iota_1(s+\tau_n) - v(s)| \;<\; \frac{\varepsilon}{\underline{\psi}}.$$

Then

$$\begin{aligned}
|\Theta_1(t+\tau_n) - M(t)| &= |\int_{-\infty}^{t+\tau_n} \iota_1(s)e^{-\int_s^{t+\tau_n}(a(u)-\alpha-\beta)du}ds - \int_{-\infty}^{t} v(s)e^{-\int_s^t (v_1(u)-\alpha-\beta)du}ds| \\
&\leq \int_{-\infty}^{t} |\iota_1(s+\tau_n) - v(s)|e^{-\int_s^t (a(u+\tau_n)-\alpha-\beta)du}ds \\
&\quad + \int_{-\infty}^{t} |v(s)||e^{-\int_s^t (a(u+\tau_n)-\alpha-\beta)du} - e^{-\int_s^t (v_1(u)-\alpha-\beta)du}|ds \\
&\leq \int_{-\infty}^{t} e^{-(t-s)\underline{\psi}}|\iota_1(s+\tau_n) - v(s)|ds \\
&\quad + \|v\|_\infty \int_{-\infty}^{t} e^{-(t-s)\underline{\psi}}|e^{-\int_s^t (a(u+\tau_n)-v_1(u))du} - 1|ds \\
&< \varepsilon.
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty} |\Theta_1(t+\tau_n) - M(t)| = 0.$$

Reasoning as above, one can prove

$$\lim_{n\to\infty} |N(t-\tau_n) - \Theta_1(t)| = 0.$$

Let us study the ergodicity of $\Theta_2(.)$.

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\int_{\infty}^{t} e^{-\int_s^t \psi(u)du}\iota_2(s)ds|dt &\leq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{\infty}^{t} e^{-(t-s)\underline{\psi}}|\iota_2(s)|dsdt \\
&\leq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{\infty}^{-T} e^{-(t-s)\underline{\psi}}|\iota_2(s)|dsdt \\
&\quad + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t} e^{-(t-s)\underline{\psi}}|\iota_2(s)|dsdt \\
&\leq I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \lim_{T\to\infty}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{\infty}^{-T} e^{-(t-s)\underline{\psi}}|\iota_2(s)|dsdt, I_2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t} e^{-(t-s)\underline{\psi}}|\iota_2(s)|dsdt.$$

Now, we shall prove that $I_1 = I_2 = 0$.

$$\begin{aligned}
I_1 &\leq \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{\infty}^{-T} e^{-(t-s)\underline{\psi}} ds dt \\
&\leq \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-t\underline{\psi}}\int_{\infty}^{-T} e^{s\underline{\psi}} ds dt \\
&= \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T}\frac{e^{-T\underline{\psi}}}{\underline{\psi}}\int_{-T}^{T} e^{-t\underline{\psi}} dt \\
&= \|\iota_2\|_\infty \lim_{T\to\infty} \frac{1}{2T\underline{\psi}^2}[1-e^{-2T\underline{\psi}}] = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\int_{0}^{T+u} e^{-u\underline{\psi}}|\iota_2(t-u)| du dt &\leq \frac{1}{2T}\int_{0}^{\infty} e^{-\underline{\psi}u}\Big(\int_{-T-u}^{T+u}|\iota_2(s)|ds\Big)du \\
&\leq \int_{0}^{\infty} e^{-\underline{\psi}u}\Big(\frac{1}{2T}\int_{-T-u}^{T+u}|\iota_2(s)|ds\Big)du.
\end{aligned}$$

Since $\iota_2 \in PAP_0(\mathbb{R},\mathbb{R})$, then $\lim_{T\to\infty}\frac{1}{2T}\int_{-T-u}^{T+u}|\iota_2(s)|ds = 0$, uniformly with respect to u. Finally, by the Lebesgue's dominated convergence Theorem, we obtain $I_2 = 0$. Consequently, Θ belongs to $PAA(\mathbb{R},\mathbb{R})$. □

Lemma 5.3. *If $x(.) \in PAA(\mathbb{R},\mathbb{R}_+)$ and $\tau(.) \in AA(\mathbb{R},\mathbb{R}_+)$. Then $x(.-\tau(.)) \in PAA(\mathbb{R},\mathbb{R}_+)$.*

Proof. $x(.) \in PAA(\mathbb{R},\mathbb{R}_+)$, by the composition theorem $x(.)$ can be written as, $x = x_1 + x_2$, where $x_1(.) \in AA(\mathbb{R},\mathbb{R}_+)$ and $x_2(.) \in PAP_0(\mathbb{R},\mathbb{R}_+)$.

$$x(t-\tau(t)) = x_1(t-\tau(t)) + x(t-\tau(t)) - x_1(t-\tau(t)).$$

Next, we shall prove that $x_1(.-\tau(.)) \in AA(\mathbb{R},\mathbb{R}_+)$. Let f be a function defined by $f(t,r) = x_1(t-r)$. Let us prove that $f \in AA_U(\mathbb{R}\times\mathbb{R},\mathbb{R}_+)$. We prove this result in two steps:

(1) Since $x_1(.) \in AA(\mathbb{R},\mathbb{R}_+)$ and the space $AA(\mathbb{R},\mathbb{R}_+)$ is invariant by translation then $x_1(.-r) \in AA(\mathbb{R},\mathbb{R}_+)$.
(2) $x_1(t-.)$ is almost automorphic, then $x_1(t-.)$ is continuous and by Heine's theorem, $x_1(t-.)$ is uniformly continuous in each compact K of $\mathbb{R}$.

Hence, by Theorem (2.5), $f \in AA_U(\mathbb{R}\times\mathbb{R},\mathbb{R}_+)$. Let us define the Nemytskii's operator N_f by

$$\begin{aligned}
N_f : AA(\mathbb{R},\mathbb{R}) &\to AA(\mathbb{R},\mathbb{R}) \\
\tau &\mapsto N_f(\tau) := [t \mapsto f(\tau(t),t)].
\end{aligned}$$

Since $f \in AA_U(\mathbb{R}\times\mathbb{R},\mathbb{R}_+)$, then N_f is well defined and continuous. Besides, $\tau(.) \in AA(\mathbb{R},\mathbb{R}_+)$, then by Theorem (2.6), the function $t \mapsto f(t,\tau(t)) = x_1(t-\tau(t))$ is almost automorphic.

Now, we show that $x(.-\tau(.))-x_1(.-\tau(.))\in PAP_0(\mathbb{R},\mathbb{R}_+)$.

$$\begin{aligned} \lim_{T\to\infty} & \quad \frac{1}{2T}\int_{-T}^{T}|x(t-\tau(t))-x_1(t-\tau(t))|dt \\ = & \quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x_2(t-\tau(t))|dt \\ \leq & \quad \lim_{T\to\infty}\frac{T+\bar{\tau}}{T}\frac{1}{2(T+\bar{\tau})}\int_{-(T+\bar{\tau})}^{T+\bar{\tau}}|x_2(t)|dt \\ & \quad =0. \end{aligned}$$

Thus, $x(.-\tau(.))-x_1(.-\tau(.))\in PAP_0(\mathbb{R},\mathbb{R}_+)$. So $x(.-\tau(.))\in PAA(\mathbb{R},\mathbb{R}_+)$. $\square$

Theorem 5.4. *Under assumptions* $(A4.1)-(A4.2)$*, the equation* (3.1) *has a unique pseudo almost automorphic solution* $\tilde{X}\in\Omega$*.*

Proof. Let us consider the operator T defined by, for all $(x,y,z)\in\Lambda=(PAA(\mathbb{R},\mathbb{R}))^3$,

$$T(x,y,z)(t) \quad = \quad \left(\int_{-\infty}^{t}y(s)e^{-(t-s)\alpha}ds,\int_{-\infty}^{t}z(s)e^{-(t-s)\beta}ds,\int_{-\infty}^{t}\varphi(s)e^{-\int_s^t\psi(u)du}ds\right),$$

where

$$\begin{aligned} \varphi(t) \quad = \quad & p(t)+[(\alpha+\beta)(a(t)-\alpha)-\beta^2-b(t)]y(t)-[\alpha^2(a(t)-\alpha)-\alpha b(t)]x(t) \\ & -\sum_{i=1}^{n}g_i(t,x(t-r_i)) \end{aligned}$$

It is clear that by Lemma (5.1), Lemma (5.2), and Lemma (5.3) the operator T is a mapping of Ω into itself. Now, we have to prove that the operator T is a contraction. For $X=(x,y,z),V=(u,v,w)\in\Omega$, we have

$$TV(t) \quad = \quad (\left(\int_{-\infty}^{t}v(s)e^{-(t-s)\alpha}ds,\int_{-\infty}^{t}w(s)e^{-(t-s)\beta}ds,\int_{-\infty}^{t}\phi(s)e^{-\int_s^t\psi(u)du}ds\right),$$

where

$$\begin{aligned} \phi(t) \quad = \quad & p(t)+[(\alpha+\beta)(a(t)-\alpha)-\beta^2-b(t)]v(t)-[\alpha^2(a(t)-\alpha)-\alpha b(t)]u(t) \\ & -\sum_{i=1}^{n}g_i(t,u(t-r_i)), \end{aligned}$$

and

$$\begin{aligned} \varphi(t)-\phi(t) \quad = \quad & [(\alpha+\beta)(a(t)-\alpha)-\beta^2-b(t)](y(t)-v(t)) \\ & +(\alpha^2(a(t)-\alpha)-\alpha b(t)))(u(t)-x(t)) \\ & -\sum_{i=1}^{n}[g_i(t,x(t-r_i(t)))-g_i(t,u(t-r_i(t)))]. \end{aligned}$$

Then

$$\begin{aligned}\|TX-TV\|_\Omega &= \sup_t \max|(\int_{-\infty}^t (y(s)-v(s))e^{-(t-s)\alpha}ds, \int_{-\infty}^t (z(s)-w(s))e^{-(t-s)\beta}ds,\\ &\int_{-\infty}^t (\varphi(s)-\phi(s))e^{-\int_s^t \psi(u)du}ds)|\end{aligned}$$

Next, we have to demonstrate that T is a contraction mapping in three steps.

- **Case 1:** $\|TX-TV\|_\Lambda = \sup_t |\int_{-\infty}^t (y(s)-v(s))e^{-(t-s)\alpha}ds|$

$$\begin{aligned}\|TX-TV\|_\Omega &\leq \sup_t \int_{-\infty}^t |y(s)-v(s)|e^{-(t-s)\alpha}ds\\ &\leq \|X-V\|_\Lambda \sup_t \int_{-\infty}^t e^{-(t-s)\alpha}ds\\ &\leq \frac{1}{\alpha}\|X-V\|_\Lambda.\end{aligned}$$

  Consequently,

$$\|TX-TV\|_\Omega \leq \frac{1}{\alpha}\|X-V\|_\Omega.$$

- **Case 2:** $\|TX-TV\|_\Omega = \sup_t |\int_{-\infty}^t (z(s)-w(s))e^{-(t-s)\beta}ds|$.
  Similarly to case 1, we obtain

$$\|TX-TV\|_\Omega \leq \frac{1}{\beta}\|X-V\|_\Omega.$$

- **Case 3:** $\|TX-TV\|_\Omega = \sup_t |\int_{-\infty}^t (\varphi(s)-\phi(s))e^{-\int_s^t \psi(u)du}ds|$.

$$\begin{aligned}\|TX-TV\|_\Omega &\leq \sup_t \int_{-\infty}^t e^{-\int_s^t \psi(u)du}\Big|[(\alpha+\beta)(a(t)-\alpha)-\beta^2-b(t)](y(t)-v(t))\\ &+(\alpha^2(a(t)-\alpha)-\alpha b(t))(u(t)-x(t))\\ &-\sum_{i=1}^n [g_i(t,x(t-r_i(t)))-g_i(t,u(t-r_i(t)))]\Big|ds\\ &\leq \|X-V\|_\Omega(\sup_\xi \Big|(\alpha+\beta)(a(\xi)-\alpha)\\ &-\beta^2-b(\xi)(1+\alpha)+\alpha^2(a(\xi)-\alpha)-nL_g\Big|)\int_{-\infty}^t e^{-(t-s)\underline{\psi}}ds\\ &\leq \|X-V\|_\Omega \frac{\sup_\xi \Big|(\alpha+\beta)(a(\xi)-\alpha)-\beta^2-b(\xi)(1+\alpha)+\alpha^2(a(\xi)-\alpha)-nL_g\Big|}{\inf_\xi(a(\xi)-\alpha-\beta)},\end{aligned}$$

which proves that T is a contraction. Consequently, T has a unique fixed point $\tilde{X}\in\Omega$. □

6. Exponential stability of the pseudo almost automorphic solutions

Theorem 6.1. *Assume that* $(A4.1) - (A4.2)$ *hold. Then, the pseudo almost automorphic solution* $\tilde{X}(t)$ *of system* (3.1) *is globally exponentially stable.*

Proof. Let's take $X(t) = (x(t), y(t), z(t))$ as another solution of (3.1). Let us consider the following Lyapunov function

$$\begin{aligned} V(t) &= [|x^*(t) - x(t)| + |y^*(t) - y(t)| + |z^*(t) - z(t)|]e^{-t\delta} \\ &+ \int_0^t |(\alpha+\beta)(a(s)-\alpha) - b(s) - \beta^2||y^*(s) - y(s)|ds \\ &+ \int_0^t \alpha^2 |a(s) - \alpha - \alpha b(s)||x^*(s) - x(s)|ds, t \geq 0, \end{aligned}$$

where $\delta > 0$. Note that $\eta = \min(\delta+\alpha, \delta+\beta, \delta+\underline{a}-\alpha-\beta)$.

Calculating the upper right derivative $D^+V(t)$ of $V(t)$, we obtain

$$\begin{aligned} D^+V(t) &\leq [-\delta|x^*(t) - x(t)| - \alpha|x^*(t) - x(t)| - \delta|y^*(t) - y(t)| - \beta|y^*(t) - y(t)| \\ &\quad -(a(t) - \alpha - \beta + \delta)|z^*(t) - z(t)|e^{-t\delta} \\ &\leq -e^{-t\delta}[(\delta+\alpha)|x^*(t) - x(t)| \\ &\quad +(\beta+\delta)|y^*(t) - y(t)| + (\underline{a} - \alpha - \beta + \delta)|z^*(t) - z(t)|] \\ &\leq -\eta e^{-t\delta}[|x^*(t) - x(t)| + |y^*(t) - y(t)| + |z^*(t) - z(t)|] \\ &< 0. \end{aligned} \tag{6.1}$$

It follows that $D^+V(t) \leq 0$. Obviously, equation (3.1) is stable in Lyapunov sense. Simultaneously, an integration of (6.1) over $[\xi, t]$ leads to:

$$\int_\xi^t \eta \quad e^{-s\delta}(|x^*(s) - x(s)| \tag{6.2}$$

$$+|y^*(s) - y(s)| + |z^*(s) - z(s)|)ds \leq V(\xi) - V(t), \tag{6.3}$$

for all $t \geq \xi$. Noting that $V(t) \geq 0$, it follows from (6.2) that

$$\int_\xi^\infty e^{-s\delta}(|x^*(s) - u(s)| \quad + \quad |y^*(s) - v(s)| + |z^*(s) - w(s)|)ds \leq \frac{V(\xi)}{\eta}.$$

Hence by Lemma (2.10), we have

$$\lim_{t\to\infty} e^{-t\delta}|x^*(t) - x(t)| = 0, \lim_{t\to\infty} e^{-t\delta}|y^*(t) - y(t)| = 0, \lim_{t\to\infty} e^{-t\delta}|z^*(t) - z(t)| = 0.$$

Then, equation (3.1) is globally exponentially stable. □

7. Example

In this section, we give an example to illustrate the validity of our results.

$$x^{(3)} + a(t)x^{(1)} + b(t)x^{(2)} + \sum_{i=1}^n g_i(t, x(t - r_i(t))) = p(t). \tag{7.1}$$

For

$$\begin{aligned} a(t) &= 5 + 2\sin(\frac{2}{2+\sin(t)+\sin(\sqrt{2}t)})^2, \\ b(t) &= 7 + 2\sin(t)^2, g(t, x(t-r(t))) = 3|x(t-0.2)| \end{aligned}$$

and

$$\begin{aligned} p(t) &= 13 + \sin(\frac{2}{2+\sin(t)+\sin(\sqrt{2}t)}) + \exp(-(t\cos(t))^2, \\ r(t) &= 0.2, \end{aligned}$$

in addition to that $\alpha = \beta = 1.02$, the equation (7.1) satisfies $(A4.1) - (A4.2)$ and

$$\frac{\sup_t \left(|(\alpha+\beta)(a(t)-\alpha) - \beta^2 + \alpha^2(a(t)-\alpha) - b(t)(1+\alpha) - L_g| \right)}{\inf_t \{a(t) - \alpha - \beta\}} \approx 0.074 < 1.$$

Therefore, using Theorem (5.4), we conclude that system (7.1) has a unique pseudo almost automorphic solution. Besides, this unique solution is globally exponentially stable. Figures 4.5. confirm that the proposed conditions in our theoretical results are effective for this example.

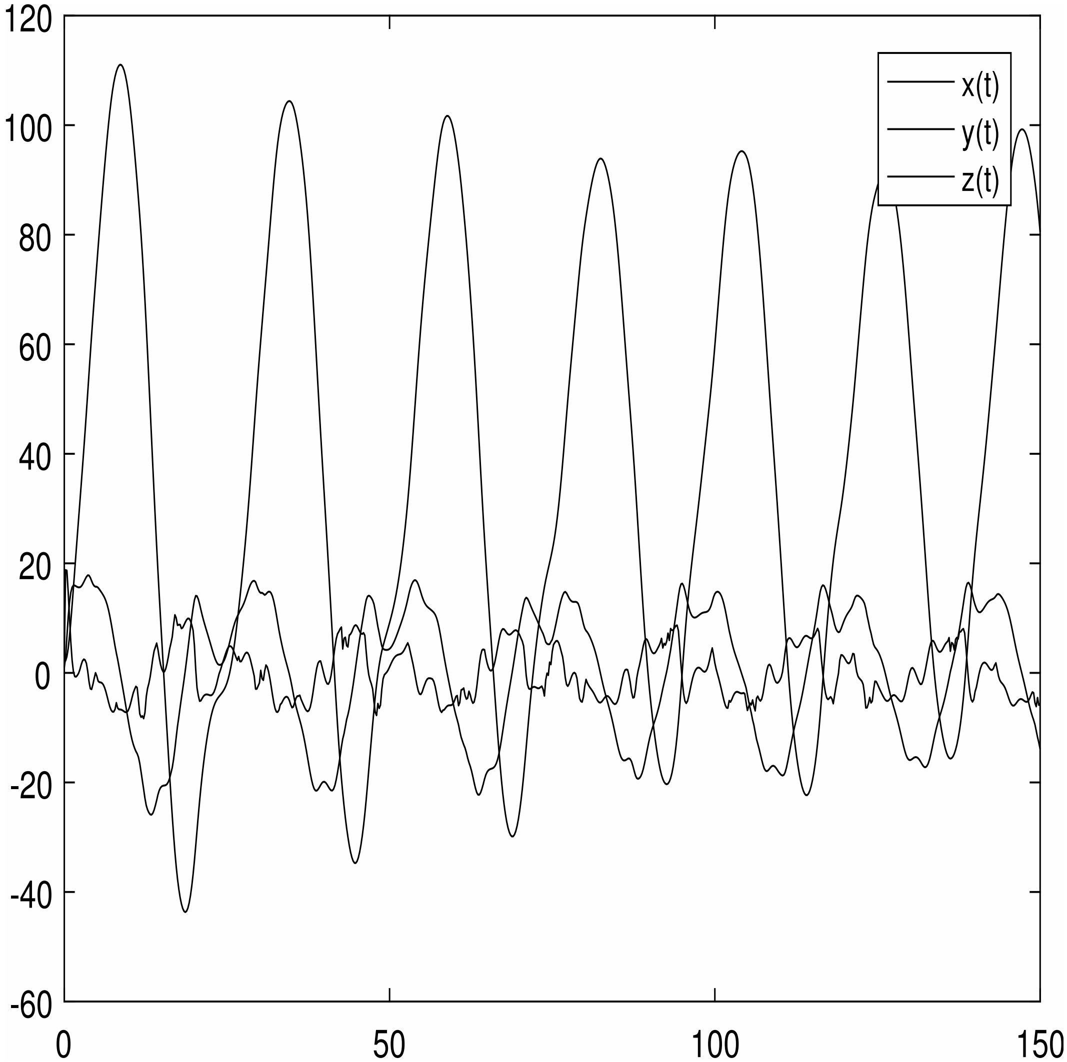

Figure 1. Curve of the pseudo almost automorphic solution of equation (7.1) with multiple delays.

8. Comparaison and discussion

Until now, the studies of nonlinear third-order differential equations mainly focuses on stability and the boundedness of the solutions ([1, 2, 6, 8, 10, 18]). The most effective method in the study the existence of periodic solutions, asymptotic behaviors of stability and boundedness of non-linear differential equations is still by employing an appropriate Lyapunov second method. But in this paper, we study the existence and uniqueness of pseudo almost automorphic solutions by using new conditions and a new approach. However, the methods and techniques used in our work in order to study the uniformly bounded, existence, uniqueness, and the globally exponential stability are totally different from author work since we use a Banach 's fixed point theorem and some properties related to the space pseudo almost automorphic.

In 1974, Hara ([23]) also considered the following third-order nonlinear differential equation:

$$x^{(3)} + a(t)x^{(2)} + b(t)x^{(1)} + c(t)x = e(t)$$

and established conditions under which all solutions of the above equations are uniform bounded and tend to zero as $t \to \infty$.

In([3]), the authors considered the third-order non linear differential equation described as follows:

$$x^{(3)} + a(t)f(x^{(2)}) + b(t)g(x^{(1)}) + c(t)h(x) = e(t).$$

By sufficient conditions, the authors obtained the asymptotic behavior of the solutions which is furthermore bounded.

Also, Zhu ([26]) considered the following third-order non linear delay differential equation:

$$x^{(3)} + ax^{(2)} + bx^{(1)} + f(x(t-r)) = 0,$$

and investigated the asymptotic stability of the trivial solution.

In 1995, Shadman and Mehri ([9]) discussed the existence of periodic solutions to nonlinear third-order differential equation of the form:

$$x^{(3)} + c(t)x^{(2)} + k(t)x^{(1)} + f(t,x) = e(t).$$

In addition, in 2009, Omeike([17]) considered the following nonlinear differential equation of third-order, with a constant deviating argument r,

$$\begin{aligned} x^{(3)}(t) \quad &+ \quad a(t)x^{(2)}(t) + b(t)g(x^{(1)}) + c(t)h(x(t-r)) \\ &= \quad p(t). \end{aligned}$$

discussed the stability and boundedness of solutions of this equation when $p(t) = 0$ and $p(t) \neq 0$. But in our study, we obtain the stability of pseudo almost automorphic solutions without discussing any cases for $p(t)$. Therefore, our study is more generalized than other studies in the literature.

9. Conclusion

Nonlinear differential differential equations of higher order have been extensively studied with high degree of generality. In this paper, the pseudo almost automorphic solutions of the third-order differential equation with multiple deviating arguments are investigated. We have employed mainly the well known Banach contraction principle to prove the existence and uniqueness of the pseudo almost automorphic solution. Some new results are obtained for the stability of the solutions. Finally, an example with its numerical simulations has been provided to demonstrate the feasibility of our results. To the best of our knowledge, this paper is the first one dealing with the pseudo almost automorphic solutions of the third-order differential equation.

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