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INNER AUTOMORPHISMS OF ORDERED TRANSFORMATION SEMIGROUPS

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Abstract. This paper studies ordered subsemigroups of the posemigroup of partial order-preserving transformations on an ordered set whose automorphisms are all inner. As a first step, we determine the automorphism structure of the ordered transformation monoid of all order-preserving self-maps on a nonempty ordered set. We show that every ordered automorphism is inner and arises from conjugation by an order-preserving bijection of the underlying set. Consequently, the automorphism group coincides with the group of order-preserving bijections, providing a complete description of the automorphism structure in the ordered setting.

Keywords. automorphisms; ordered semigroups; constants; constant idempotents
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1. Introduction

Transformation semigroups constitute a fundamental class of algebraic structures in semigroup theory. A classical result states that every abstract semigroup admits a faithful representation as a semigroup of transformations on a suitable set [1]. This representation perspective provides a concrete framework for studying algebraic structures and establishes natural connections with combinatorics, automata theory, and other areas of discrete mathematics.

A central problem in the study of transformation semigroups is the description of their automorphism groups. In the classical unordered setting, Schreier [11] and Mal'cev [7] proved that every automorphism of the full transformation semigroup $\mathcal{T}(A)$ is inner, that is, it arises from conjugation by a bijection of the underlying set A. This remarkable rigidity phenomenon was later extended to several related structures. In particular, similar results were obtained for semigroups of partial transformations and other transformation semigroups by Sutov [17], Liber [5], and Magill [6]. Further investigations by Sullivan [16] and Levi [3], as well as Levi and Wood [4], examined transformation semigroups that are closed under conjugation by permutations. Additionally, Mir and Alali, in [9], investigated the

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automorphisms of a semigroup S of centralizers of idempotent transformations with restricted range. These works collectively demonstrate that in many classical transformation settings the automorphism group is determined entirely by the permutation structure of the underlying set.

Parallel developments in semigroup theory have focused on identifying structural conditions under which all automorphisms are necessarily inner. In this direction, the concept of nearly complete monoids has emerged as an important refinement in the study of automorphism rigidity. Shah and coauthors [8, 14, 12, 13] extended the classical inner automorphism theorem from groups to several classes of monoids and characterized strong semilattices of groups that are nearly complete. Their results highlight the structural role of homomorphisms in preserving internal symmetry and provide conditions under which automorphisms must arise from conjugation.

More recently, Shah and Mir [15] investigated a class of normal transformation semigroups and showed that semigroups of full transformations with restrictions on a fixed subset possess the inner automorphism property. These results further illustrate the persistence of rigidity phenomena in transformation-based algebraic structures.

When an algebraic structure is equipped with a compatible partial order, one obtains an ordered semigroup, where algebraic and order-theoretic properties interact. The presence of order typically imposes additional constraints on homomorphisms and automorphisms and may lead to stronger structural rigidity. Despite the extensive classical theory of transformation semigroups, the automorphism structure of ordered transformation semigroups has not yet been systematically studied.

Let A be a nonempty partially ordered set and let $\mathcal{T}_{\mathcal{M}}(A)$ denote the monoid of all order-preserving self-maps of A, equipped with the natural pointwise order. The primary objective of this paper is to determine the automorphism group of $\mathcal{T}_{\mathcal{M}}(A)$ and to investigate whether the classical Schreier–Mal'cev innerness phenomenon persists in this ordered setting.

We prove that every ordered automorphism of $\mathcal{T}_{\mathcal{M}}(A)$ is inner and is induced by conjugation with an order-preserving bijection of A. More precisely, if $\mathcal{G}_{\mathcal{M}}(A)$ denotes the group of all order-preserving bijections of A, then $\mathrm{Aut}_{\mathcal{M}}(\mathcal{T}_{\mathcal{M}}(A)) \cong \mathcal{G}_{\mathcal{M}}(A)$. Thus the ordered transformation monoid exhibits the same rigidity property as in the classical case, despite the presence of additional order structure.

Furthermore, we study constant-rich ordered subsemigroups of $\mathcal{T}_{\mathcal{M}}(A)$ and characterize those for which every automorphism is inner. This extends Schreier–Mal'cev type results to a broader ordered framework and clarifies the structural role of constant idempotents in determining automorphism behaviour.

2. Preliminaries

We begin by recalling basic notions concerning ordered semigroups and ordered transformation monoids.

Definition 2.1. Let S be a semigroup endowed with a partial order $\ll$. Then $(S, \ll)$ is called an *ordered semigroup* if the semigroup operation is order compatible; that is,

$$u \ll v \text{ and } w \ll z \implies uw \ll vz \quad \text{for all } u, v, w, z \in S.$$

Equivalently, for all $u, v, w \in S$,

$$u \ll v \implies uw \ll vw \text{ and } wu \ll wv.$$

Definition 2.2. An ordered semigroup $(S, \ll)$ is called an *ordered monoid* if S is a monoid and the identity element is compatible with the order. If S is a group under the same compatible order, then $(S, \ll)$ is called an *ordered group*.

Every semigroup S may be viewed as an ordered semigroup by endowing it with the *trivial* (or *discrete*) partial order defined by

$$u \ll v \quad \iff \quad u = v.$$

This relation is clearly reflexive, antisymmetric, and transitive, hence a partial order. Moreover, it is automatically compatible with the semigroup operation: if

$$u \ll v \quad \text{and} \quad w \ll z,$$

then $u = v$ and $w = z$, so $uw = vz$, which implies $uw \ll vz$.

Thus every semigroup admits a canonical ordered structure in which the only comparable elements are identical ones. In this sense, the concept of an ordered semigroup generalizes the classical notion of a semigroup.

Definition 2.3. Let $(S, \ll_S)$ be an ordered semigroup and let $T \subseteq S$. Then T is called an *ordered subsemigroup* of S if T is a subsemigroup of S and is equipped with the induced order

$$\ll_T = \ll_S \cap (T \times T).$$

Definition 2.4. Let $(A, \ll)$ be a partially ordered set. A map $f : A \to A$ is said to be *monotone* (or *order preserving*) if

$$a_1 \ll a_2 \Rightarrow f(a_1) \ll f(a_2) \quad \text{for all } a_1, a_2 \in A.$$

Let $\mathcal{T}_{\mathcal{M}}(A)$ denote the set of all monotone self-maps on A. Since the composition of two monotone maps is again monotone and the identity map on A is monotone, $\mathcal{T}_{\mathcal{M}}(A)$ forms a monoid under composition.

We equip $\mathcal{T}_{\mathcal{M}}(A)$ with the pointwise order defined by

$$f \ll g \quad \iff \quad f(a) \ll g(a) \text{ for all } a \in A.$$

With respect to this order, $\mathcal{T}_{\mathcal{M}}(A)$ becomes an ordered monoid. Indeed, if $f \ll g$ and $\delta \in \mathcal{T}_{\mathcal{M}}(A)$, then for every $a \in A$ we have

$$(f\delta)(a) = f(\delta(a)) \ll g(\delta(a)) = (g\delta)(a),$$

and similarly,

$$(\delta f)(a) = \delta(f(a)) \ll \delta(g(a)) = (\delta g)(a),$$

since δ is monotone. Hence multiplication is compatible with the order.

We refer to $\mathcal{T}_{\mathcal{M}}(A)$ as the *order-preserving transformation monoid* on A. Let $\mathcal{G}_{\mathcal{M}}(A)$ denote the group of all order-preserving bijections on A.

Let S and T be ordered submonoids of $\mathcal{T}_{\mathcal{M}}(A)$. A map $\xi : S \to T$ is called a *monotone monoid homomorphism* if it is a monoid homomorphism and preserves the order. A bijective monotone homomorphism is called an *ordered isomorphism*. In particular, an ordered isomorphism from S onto itself is called an *ordered automorphism*.

We denote by $\mathrm{Aut}_{\mathcal{M}}(S)$ the set of all ordered automorphisms of S. Under composition, $\mathrm{Aut}_{\mathcal{M}}(S)$ forms a group, and it inherits a natural order from the pointwise order on S.

An ordered automorphism $\xi : S \to S$ is called *inner* if there exists an element $f \in \mathcal{G}_{\mathcal{M}}(A)$ such that ξ is implemented by conjugation with f, that is,

$$\xi(g) = fgf^{-1} \quad \text{for all } g \in S.$$

In other words, ξ arises from the natural action of the ordered bijection f on S by conjugation.

It is worth noting that every ordered homomorphism is, in particular, a monoid homomorphism. However, the converse need not hold, as the following example illustrates.

Example 2.5. Let $S = \{1, a, b\}$ be a monoid with multiplication defined by

$*$	1	a	b
1	1	a	b
a	a	a	a
b	b	b	b

Define a partial order on S by

$$\ll_S = \{(a, b)\} \cup \Delta_S,$$

where Δ_S denotes the identity relation on S. Then $(S, \ll_S)$ is an ordered monoid.

Consider the permutation

$$\xi = \begin{pmatrix} 1 & a & b \\ 1 & b & a \end{pmatrix}.$$

It is easy to verify that ξ is a monoid automorphism of S. However, ξ does not preserve the order since $(a, b) \in \ll_S$ but $(\xi(a), \xi(b)) = (b, a) \notin \ll_S$. Thus a monoid automorphism need not be order preserving.

3. Constant Rich Ordered Subsemigroups of $\mathcal{T}_{\mathcal{M}}(A)$

For each $c \in A$, define the constant transformation $\tau_c \in \mathcal{T}_{\mathcal{M}}(A)$ by

$$\tau_c(x) = c \quad \text{for all } x \in A.$$

Clearly, τ_c is idempotent since $\tau_c \circ \tau_c = \tau_c$. We refer to τ_c as the constant idempotent associated with c.

Definition 3.1. An ordered subsemigroup $S \subseteq \mathcal{T}_{\mathcal{M}}(A)$ is said to be *constant rich* if $\tau_c \in S$ for every $c \in A$.

Let $\mathcal{K}_{\mathcal{M}}(S)$ denote the set of all constant elements of S. Then $\mathcal{K}_{\mathcal{M}}(S)$ is an ordered subsemigroup of S.

Lemma 3.2. *Let S be an ordered subsemigroup of $\mathcal{T}_{\mathcal{M}}(A)$ containing at least one constant. For $f \in S$, the following are equivalent:*

$$f \in \mathcal{K}_{\mathcal{M}}(S) \quad \Longleftrightarrow \quad fgf = f \text{ for all } g \in S.$$

Proof. Suppose f is an element of $\mathcal{K}_{\mathcal{M}}(S)$, so we can choose some fixed $c \in A$ such that $f(x) = c$ for all $x \in A$. Then $fgf(x) = fg(f(x)) = f(g(c)) = c = f(x)$. Hence $fgf = f$, as required.

Conversely, assume $fgf = f$ for all $g \in S$. Since this is true for all $g \in S$, so it also holds for $\tau_c \in S$. Taking $g = \tau_c$ gives

$$f\tau_c f = f.$$

Evaluating at $x \in A$ yields

$$f(x) = f(\tau_c(f(x))) = f(c) = \tau_{f(c)},$$

a constant based on the point $f(c)$, as it is independent of x. Thus we have $f \in \mathcal{K}_{\mathcal{M}}(S)$. □

Lemma 3.3. *Let S be a constant rich ordered subsemigroup of $\mathcal{T}_{\mathcal{M}}(A)$ and let $\xi \in \mathrm{Aut}_{\mathcal{M}}(S)$. Then*

$$\xi(\mathcal{K}_{\mathcal{M}}(S)) = \mathcal{K}_{\mathcal{M}}(S).$$

Proof. First we claim that ξ sends constants of S to constants. Let $g \in S$, then for some $f \in S$ with $\xi(f) = g$ as $\xi \in \mathrm{Aut}_{\mathcal{M}}(S)$. Let $c \in A$ and $\tau_c \in S$, so we have

$$\tau_c f \tau_c(x) = \tau_c f(c) = \tau_c(f(c)) = c = \tau_c(x).$$

Hence $\tau_c f \tau_c = \tau_c$. Also as $\xi \in \mathrm{Aut}_{\mathcal{M}}(S)$, we have

$$\xi(\tau_c f \tau_c) = \xi(\tau_c)\xi(f)\xi(\tau_c) \Rightarrow \ \xi(\tau_c) = \xi(\tau_c)g\xi(\tau_c) \quad (\text{ as } \tau_c f \tau_c = \tau_c).$$

Therefore we have $\xi(\tau_c) = \xi(\tau_c)g\xi(\tau_c)$ for all $g \in S$, since Lemma 4.2 is restricted to S. This gives by Lemma 3.2, that $\xi(\tau_c)$ is a constant mapping. Since S is constant rich therefore we have $\xi(\tau_c) = \tau_{c'}$ for some $c' \in A$. We have shown that ξ maps $\mathcal{K}_{\mathcal{M}}(S)$ into $\mathcal{K}_{\mathcal{M}}(S)$.

Now as $\xi \in \mathrm{Aut}_{\mathcal{M}}(S)$, so ξ^{-1} exists and belongs to $\mathrm{Aut}_{\mathcal{M}}(S)$. Therefore we have

$$\begin{aligned} &\xi^{-1}(\tau_c g \tau_c) = \xi^{-1}(\tau_c)\xi^{-1}(g)\xi^{-1}(\tau_c) \\ \Rightarrow \ &\xi^{-1}(\tau_c) = \xi^{-1}(\tau_c) f \xi^{-1}(\tau_c) \quad (\text{ as } \tau_c g \tau_c = \tau_c). \end{aligned}$$

That is, $\xi^{-1}(\tau_c) = \xi^{-1}(\tau_c) f \xi^{-1}(\tau_c) \ \forall \ f \in \mathcal{T}_{\mathcal{M}}(A)$. Thus by Lemma 3.2, it follows that $\xi^{-1}(\tau_c)$ is a constant element of S. Hence $\xi|_{\mathcal{K}_{\mathcal{M}}(S)}$ is surjective. Therefore $\xi|_{\mathcal{K}_{\mathcal{M}}(S)} \in Aut_{\mathcal{M}}(\mathcal{K}_{\mathcal{M}}(S))$, as required. □

Next we show that every automorphisms of $\mathcal{K}_{\mathcal{M}}(S)$ is inner.

Lemma 3.4. *If S is a constant rich ordered subsemigroup of $\mathcal{T}_{\mathcal{M}}(S)$, then every automorphism of $\mathcal{K}_{\mathcal{M}}(S)$ is inner.*

Proof. Let $\xi \in \mathrm{Aut}_{\mathcal{M}}(\mathcal{K}_{\mathcal{M}}(S))$ and $d \in A$. Since S is constant rich, we may choose $\tau_d \in \mathcal{K}_{\mathcal{M}}(S))$. By above Lemma, we have $\xi(\tau_d) = \tau_{d'}$ for some $\tau_{d'} \in \mathcal{K}_{\mathcal{M}}(S)$. Now define a map $f : A \to A$ by $f(d) = d'$ if and only if $\xi(\tau_d) = \tau_{d'}$.
We claim that $f \in \mathcal{G}_{\mathcal{M}}(A)$.

Let $d, d' \in A$ be such that

$$d = d' \Leftrightarrow \tau_d = \tau_{d'} \Leftrightarrow \xi(\tau_d) = \xi(\tau_{d'}) \Leftrightarrow \tau_{f(d)} = \tau_{f(d')} \Leftrightarrow f(d) = f(d').$$

Thus f is one-one. Since ξ is an automorphism, so ξ^{-1} exists and belongs to $\mathrm{Aut}_{\mathcal{M}}(S)$. Therefore there exists $d \in A$ such that $\xi^{-1}(\tau_{d'}) = \tau_d$, implies $\xi(\tau_d) = \tau_{d'}$, that is, $f(d) = d'$. Therefore f is surjective.

Let $d, d' \in A$, with

$$d \ll d' \Rightarrow \tau_d \ll \tau_{d'} \Rightarrow \xi(\tau_d) \ll \xi(\tau_{d'}) \Rightarrow \tau_{f(d)} \ll \tau_{f(d')} \Rightarrow f(d) \ll f(d').$$

Thus f is monotone and hence the claim is true, that is to say $f \in \mathcal{G}_{\mathcal{M}}(A)$.
If $g \in \mathcal{K}_M(S)$, then $g = \tau_a$ for some $a \in A$. Then $fgf^{-1} = f\tau_a f^{-1}$, which is exactly the constant map $\tau_{f(a)}$. Since $\xi(\tau_a) = \tau_{f(a)}$ by the definition of f, it immediately follows that $\xi(g) = fgf^{-1}$. Thus the automorphism ξ of $\mathcal{K}_{\mathcal{M}}(S)$ is inner. □

In next theorem we prove that every automorphism of a constant rich ordered subsemigroups of $\mathcal{T}_{\mathcal{M}}(A)$ is always inner.

Theorem 3.5. *If S is a constant rich ordered subsemigroup of $\mathcal{T}_{\mathcal{M}}(A)$, then every automorphism of S is inner.*

Proof. Suppose $\xi \in \mathrm{Aut}_{\mathcal{M}}(S)$ then by Lemma 3.3 and Lemma 3.4, there exists $f \in \mathcal{G}_{\mathcal{M}}(A)$ such that $\xi(g) = fgf^{-1}$ for all $g \in \mathcal{K}_{\mathcal{M}}(S)$.

Now suppose $\delta \in S$. Since S is constant rich, then for each $c \in A$, we have $\tau_{f\delta f^{-1}(c)} \in \mathcal{T}_{\mathcal{M}}(A)$ and $\tau_{f\delta f^{-1}(c)} = \tau_{f(\delta f^{-1}(c))}$, and by definition of f in Lemma 3.4, this implies $\tau_{f\delta f^{-1}(c)} = \xi(\tau_{\delta f^{-1}(c)})$.
Also for any $a \in A$ we have

$$\tau_{\delta f^{-1}(c)}(a) = \delta f^{-1}(c) = \delta(f^{-1}(c)) = \delta(\tau_{f^{-1}(c)})(a).$$

Thus $\tau_{\delta f^{-1}(c)}(a) = \delta(\tau_{f^{-1}(c)})(a)$.
Therefore we have

$$\tau_{f\delta f^{-1}(c)} = \xi(\tau_{\delta f^{-1}(c)}) = \xi(\delta(\tau_{f^{-1}(c)})) = \xi(\delta)\xi(\tau_{f^{-1}(c)}) = \xi(\delta)\tau_{ff^{-1}(c)} = \xi(\delta)\tau_c.$$

Hence we get $\tau_{f\delta f^{-1}(c)} = \xi(\delta)\tau_c$ and so $\tau_{f\delta f^{-1}(c)}(a) = \xi(\delta)\tau_c(a)$ for all $a \in A$. This implies, $f\delta f^{-1}(c) = \xi(\delta)(c)$, that is, $\xi(\delta) = f\delta f^{-1}$ for all $\delta \in S$. Thus the automorphism ξ of S is inner. □

Next is an example of an ordered semigroup S which illustrates above theorem.

Example 3.6. Define a partial order on a set $A = \{a_1, a_2, a_3, a_4,\}$ by $\ll_A = \{(a_1, a_4)\} \cup \Delta_A$, where $\Delta_A = \{(a, a) : a \in A\}$. Then A is a poset. Let $S = \{\tau_{a_1}, \tau_{a_2}, \tau_{a_3}, \tau_{a_4}\}$, where τ_{a_i} is the map with domain A and range $\{a_i\}$. Clearly S

is an ordered semigroup, which is constant rich. Also $\mathcal{G}_{\mathcal{M}}(A) = \{id, f = (a_2\ a_3)\}$. Now consider a map $\xi = (\tau_{a_2}\ \tau_{a_3})$, then $\xi \in \mathrm{Aut}_{\mathcal{M}}(S)$. Thus $\mathrm{Aut}_{\mathcal{M}}(S) = \{id_S, \xi\}$. Clearly, $\xi(s)$ inner. For $f \in \mathcal{G}_{\mathcal{M}}(A)$ we have $\xi(s) = fsf^{-1}$ for all $s \in S$. If $s = \tau_{a_i}$ for $i = 1, 4$, then there is nothing to prove. Now if $s = \tau_{a_2}$, then we have

$$\begin{aligned}\xi(\tau_{a_2})(a) = \tau_{a_3}(a) = a_3 &= (a_2\ a_3)(a_3) = (a_2\ a_3)\tau_{a_2}(a)\\ &= (a_2\ a_3)\tau_{a_2}(a_2\ a_3)(a)\\ &= f\tau_{a_2}f^{-1}(a),\ (\text{where } f = (a_2\ a_3)).\end{aligned}$$

Also if $s = \tau_{a_3}$, then we have

$$\begin{aligned}\xi(\tau_{a_3})(a) = \tau_{a_2}(a) = a_2 &= (a_2\ a_3)(a_3) = (a_2\ a_3)\tau_{a_3}(a)\\ &= (a_2\ a_3)\tau_{a_2}(a_2\ a_3)(a)\\ &= f\tau_{a_3}f^{-1}(a),\ (\text{where } f = (a_2\ a_3)).\end{aligned}$$

Thus in all cases we have $\xi(s) = fsf^{-1}$ for all $s \in S$. Therefore ξ is inner.

Definition 3.7. A subsemigroup $S \subseteq \mathcal{T}_{\mathcal{M}}(A)$ is called $\mathcal{G}_{\mathcal{M}}(A)$-normal if $gfg^{-1} \in S \quad$ for all $f \in S,\ g \in \mathcal{G}_{\mathcal{M}}(A)$.

Next we show that the automorphism of $\mathcal{G}_{\mathcal{M}}(A)$-normal constant rich ordered subsemigroup of $\mathcal{T}_{\mathcal{M}}(A)$ are precisely the ordered bijections on A.

Theorem 3.8. *If S is constant rich and $\mathcal{G}_{\mathcal{M}}(A)$-normal, then* $\mathrm{Aut}_{\mathcal{M}}(S) \cong \mathcal{G}_{\mathcal{M}}(A)$.

Proof. Let $f \in \mathcal{G}_{\mathcal{M}}(A)$ define a map $\xi^f : S \to S$ by $\xi^f(\theta) = f\theta f^{-1}$ for all $\theta \in S$. Suppose $\xi^f(\theta_1) = \xi^f(\theta_2)$ for some $\theta_1, \theta_2 \in S$. Therefore we have $f\theta_1 f^{-1} = f\theta_2 f^{-1}$, implies that, $\theta_1 = \theta_2$, that is, ξ^f is injective. Let $\theta \in S$ then $f\theta f^{-1} \in S$, as S is $\mathcal{G}_{\mathcal{M}}(A)$-normal and $\xi^f(f\theta f^{-1}) = \theta$ showing that ξ^f is surjective. Also let $\theta_1, \theta_2 \in S$, then

$$\xi^f(\theta_1\theta_2) = f(\theta_1\theta_2)f^{-1} = f\theta_1 f^{-1} f\theta_2 f^{-1} = \xi^f(\theta_1)\xi^f(\theta_2).$$

Thus ξ^f satisfies the morphism property. Finally, we show that ξ^f preserves the order. For this let $\theta_1, \theta_2 \in S$, then

$$\theta_1(x) \ll \theta_2(x) \Rightarrow f\theta_1 f^{-1}(x) \ll f\theta_2 f^{-1}(x) \Rightarrow \xi^f(\theta_1)(x) \ll \xi^f(\theta_2)(x).$$

Hence ξ^f is monotone so forms an automorphism of S. Therefore by Theorem 3.5 it follows that

$$\mathrm{Aut}_{\mathcal{M}}(S) = \{\xi^f : f \in \mathcal{G}_{\mathcal{M}}(A)\}.$$

Now define a map $\Psi : \mathrm{Aut}_{\mathcal{M}}(S) \to \mathcal{G}_{\mathcal{M}}(A)$ by $\psi(\xi^f) = f$ for all $f \in \mathcal{G}_{\mathcal{M}}(A)$. So for any $\xi^f, \xi^{f'} \in \mathrm{Aut}(S)$, assume $\xi^f = \xi^{f'}$. Then $f = f'$. Thus, $\Psi(\xi^f) = \Psi(\xi^{f'})$. That is, Ψ is one-one map. Clearly Ψ is also surjective. Therefore we have Ψ is a bijection. Let $\xi^f, \xi^{f'} \in \mathrm{Aut}_{\mathcal{M}}(S)$, then we have

$$\Psi(\xi^f\xi^{f'}) = \Psi(\xi^{ff'})\ (\text{as } \xi^f = ff' = \Psi(\xi^f)\Psi(\xi^{f'})).$$

That is, Ψ is a morphism. Now we show that Ψ preserves the order, for this we choose $b = \tau_y \in \mathcal{K}_M(S)$. Then $(f\tau_y f^{-1})(x) = f(y) \ll f'(y) = (f'\tau_y f'^{-1})(x)$ for all $y \in A$, which properly yields $f \ll f'$. Therefore Ψ is as required. Hence $\mathrm{Aut}_{\mathcal{M}}(S) \cong \mathcal{G}_{\mathcal{M}}(A)$. ∎

Corollary 3.9. *Every ordered automorphism of $\mathcal{T}_{\mathcal{M}}(A)$ is inner and $\mathrm{Aut}_{\mathcal{M}}(\mathcal{T}_{\mathcal{M}}(A)) \cong \mathcal{G}_{\mathcal{M}}(A)$.*

Proof. The monoid $\mathcal{T}_{\mathcal{M}}(A)$ is constant rich and $\mathcal{G}_{\mathcal{M}}(A)$-normal. Therefore it follows by Theorem 3.5, that every element of $\mathrm{Aut}_{\mathcal{M}}(\mathcal{T}_{\mathcal{M}}(A))$ is inner and by Theorem 3.8 that automorphism group of $\mathcal{T}_{\mathcal{M}}(A)$ is $\mathcal{G}_{\mathcal{M}}(A)$. $\square$

4. Conclusions

We have established that the ordered transformation monoid $\mathcal{T}_{\mathcal{M}}(A)$ exhibits the same rigidity phenomenon as in the classical unordered case: every ordered automorphism is implemented by conjugation with an order-preserving bijection of the underlying set. Consequently, $\mathrm{Aut}_{\mathcal{M}}(\mathcal{T}_{\mathcal{M}}(A)) \cong \mathcal{G}_{\mathcal{M}}(A)$. This provides a structural description of the automorphism group and extends the classical innerness results to the ordered framework.

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