

A VISCOSITY IMPLICIT MIDPOINT RULE FOR QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we establish the viscosity implicit midpoint rule under appropriate assumptions regarding the sequence of parameters for the fixed point of quasi-nonexpansive mappings within the context of a Hilbert space. We then demonstrate strong convergence based on specific assumptions placed on the sequence of parameters. Additionally, we present a unique solution to the variational inequality problem.

Keywords. Viscosity implicit midpoint rule, quasi-nonexpansive mapping, iterative scheme, fixed point, variational inequality.

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1. INTRODUCTION

In this paper, we assume that $\mathcal{H}$ is a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$. Let $\mathcal{M}$ denote a nonempty subset of $\mathcal{H}$. We then have the nearest point projection from $\mathcal{H}$ onto $\mathcal{M}$, $P_{\mathcal{M}}$, defined by

$$P_{\mathcal{M}}\varkappa := \arg \min_{u \in \mathcal{M}} \|\varkappa - u\|^2, \quad \varkappa \in \mathcal{H}. \quad (1.1)$$

The objective $\|\varkappa - u\|^2$ over $u \in \mathcal{M}$ is minimized at only point $P_{\mathcal{M}}\varkappa$.

Observe that $P_{\mathcal{M}}\varkappa$ is defined as follows:

$$P_{\mathcal{M}}\varkappa \quad \text{and} \quad \langle \varkappa - P_{\mathcal{M}}\varkappa, u - P_{\mathcal{M}}\varkappa \rangle \leq 0, \quad \forall u \in \mathcal{M}. \quad (1.2)$$

The viscosity approximation method for nonexpansive mappings in Hilbert spaces was introduced by Moudafi [20], building on the concepts developed by Attouch [4]. This method generates the sequence $\{\varkappa_k\}$ through an explicit viscosity approach:

$$\varkappa_{k+1} = \alpha_k f(\varkappa_k) + (1 - \alpha_k) \Psi \varkappa_k, \quad k \geq 0. \quad (1.3)$$

In this context, f is defined as a contraction mapping, while Ψ represents a nonexpansive mapping on the Hilbert space $\mathcal{H}$. Additionally, $\{\alpha_k\}$ denotes a sequence

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of values in the interval $(0, 1)$. The results of Browder [9] and Halpern [15] are expanded upon by the iterative scheme (1.3). Many authors have studied the convergence of this explicit iterative scheme; under appropriate conditions, these iterations strongly converge to the unique solution $q \in \text{Fix}(\Psi)$ of the variational inequality (VI).

$$((I - f)q, \varkappa - q) \geq 0, \quad \varkappa \in \text{Fix}(\Psi), \quad (1.4)$$

where, $\text{Fix}(\Psi)$ represents the set of fixed points of Ψ . This fact enables the application of this method to areas such as convex optimization, linear programming, and monotone inclusions. In 2004, Xu [27] expanded upon Moudafi's [20] results, applying them to uniformly smooth Banach spaces and demonstrating a strong convergence theorem.

The implicit midpoint rule is an effective method for solving ordinary differential equations; refer to [5, 6, 13, 19, 22, 23, 26] and the associated references for further details. For example, examine the initial value problem associated with the differential equation

$$\frac{d\varkappa}{dt} = \Psi(\varkappa(t)), \quad (1.5)$$

with the initial condition $(\varkappa(0) = \varkappa_0)$.

The accretive and monotone mappings are associated with the evolution model (1.5), which enhances their utility and relevance. According to [8], the model (1.5) can be solved when Ψ is locally Lipschitzian and accretive to $\mathcal{H}$. Estimating the zero of a nonlinear mapping Ψ , specifically $0 \in \Psi\varkappa$, is a significant and powerful tool in approximation theory. This approach allows for the derivation of solutions to various problems, including elliptic differential equations, optimization problems, inclusion problems, and fixed point problems. These solutions can be framed within the context of the inclusion problem $0 \in \Psi\varkappa$, which corresponds to the equilibrium state characterized by the equations $\frac{d\varkappa}{dt} = 0$ and $\Psi(\varkappa) = 0$; refer to [8, 12]. The implicit midpoint rule generates a sequence $\{\varkappa_k\}$ by the following recursion:

$$\frac{1}{h}(\varkappa_{k+1} - \varkappa_k) = \Psi\left(\frac{\varkappa_k + \varkappa_{k+1}}{2}\right), \quad k \in \mathbb{N}, \quad (1.6)$$

where $h > 0$ denotes a step size and $\mathbb{N}$ represents the set of positive integers for a continuous function map $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$.

The implicit midpoint rule (IMR) has been extended in [3] for nonexpansive mappings, which results in the generates of a sequence $\{\varkappa_k\}$ through an implicit scheme:

$$\varkappa_{k+1} = (1 - \varrho_k)\varkappa_k + \varrho_k \Psi\left(\frac{\varkappa_{k+1} + \varkappa_k}{2}\right), \quad k \in \mathbb{N}, \quad (1.7)$$

where $\{\varrho_k\}$ in $(0, 1)$ and $\Psi : \mathcal{H} \rightarrow \mathcal{H}$ is nonexpansive. In 2017, Hu [16] proposed and analyzed a more general viscosity implicit midpoint rule (MGVIMR for short)

$$\varkappa_{k+1} = \alpha_k \rho U(\varkappa_k) + \beta_k \varkappa_k + ((1 - \beta_k)I - \alpha_k \mu F) \Psi\left(\frac{\varkappa_k + \varkappa_{k+1}}{2}\right), \quad \forall k \geq 0,$$

where Ψ is a nonexpansive mapping, F is a k -Lipschitzian, and U a τ -Lipschitzian mapping. The authors introduced the more general viscosity implicit midpoint

rule for fixed points of nonexpansive mappings in Hilbert space and proved a strong convergence theorem for these mappings under specific assumptions regarding the sequence of parameters. The fixed point theory has emerged as a rapidly developing field in recent years, largely due to the process and diversity of the implicit midpoint rules employed to approximate fixed points across different classes of mappings (see [2, 3, 11, 17, 21, 24, 29, 30]).

Motivated by the aforementioned works, in the present manuscript, we establish the viscosity implicit midpoint rule under specific condition on the sequence of parameters for quasi-nonexpansive mappings in the Hilbert space. Let $\mathcal{F}$ be a ϕ -Lipschitzian mapping and σ -strongly monotone mapping. Additionally, let $\mathcal{K}$ be a t -Lipschitzian mapping, consider the following viscosity implicit midpoint rule.

$$\begin{aligned} \varkappa_{k+1} = & \varrho_k \rho \mathcal{K}(\varkappa_k) + \eta_k \varkappa_k + \sigma_k \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \\ & + ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F}) \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \quad k \geq 0, \end{aligned} \quad (1.8)$$

where $\{\varrho_k\}$, $\{\sigma_k\}$ and $\{\eta_k\} \in (0, 1)$ for all k .

Furthermore, we present a strong convergence theorem for quasi-nonexpansive mappings, given certain assumptions on the sequence of parameters. The novelty of this paper lies in proposing a viscosity implicit midpoint rule for quasi-nonexpansive mappings involving an additional Lipschitz perturbation. Unlike existing schemes that require nonexpansiveness, our results apply to the class of quasi-nonexpansive mappings and extend the work of Hu [16].

2. PRELIMINARIES

In this section, we will review some concepts and results that will be used in the main results.

Definition 2.1. A mapping $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ is said to be L -Lipschitzian if there exists a constant $L > 0$ satisfying

$$\|\Psi \varkappa - \Psi u\| \leq L \|\varkappa - u\|, \quad \forall \varkappa, u \in \mathcal{H},$$

when $L = \alpha \in (0, 1)$, is called α -contractive mapping. If $L = 1$, then Ψ is known as a nonexpansive mapping. We denote the set of fixed points of Ψ by $Fix(\Psi)$.

It is clear that the Lipschitz operator is more general than the contractive operator.

Definition 2.2. A mapping $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ is said to be quasi-nonexpansive if

$$\|\Psi(\varkappa) - u\| \leq \|\varkappa - u\|$$

for all $\varkappa \in \mathcal{M}$, and $u \in Fix(\Psi)$.

In 1916, Tricomi [25] introduced the concept of quasi-nonexpansiveness for real functions. This notion was subsequently studied by Diaz and Metcalf in the context of mappings in Banach spaces [14].

Definition 2.3. A bounded linear operator $\mathcal{B} : \mathcal{H} \rightarrow \mathcal{H}$ is called strongly positive if there exists a constant $\gamma > 0$ such that

$$\langle \mathcal{B}\mathfrak{x}, \mathfrak{x} \rangle \geq \gamma \|\mathfrak{x}\|^2, \quad \forall \mathfrak{x} \in \mathcal{H}.$$

Definition 2.4. The metric (or nearest point) projection from $\mathcal{H}$ onto $\mathcal{M}$ is defined as the mapping $P_{\mathcal{M}} : \mathcal{H} \rightarrow \mathcal{M}$. This mapping assigns to each point $\mathfrak{x} \in \mathcal{H}$ a unique point $P_{\mathcal{M}}\mathfrak{x} \in \mathcal{M}$ that satisfies the property.

$$\|\mathfrak{x} - P_{\mathcal{M}}\mathfrak{x}\| = \inf_{y \in \mathcal{M}} \|\mathfrak{x} - y\| := d(\mathfrak{x}, \mathcal{M}).$$

Definition 2.5. $\Psi : \mathcal{M} \rightarrow \mathcal{H}$ is a mapping that is known as

(i) monotone if

$$(\Psi\mathfrak{x} - \Psi u, \mathfrak{x} - u) \geq 0, \quad \forall \mathfrak{x}, u \in \mathcal{M};$$

(ii) η -strongly monotone if there exists a constant $\eta > 0$ such that

$$(\Psi\mathfrak{x} - \Psi u, \mathfrak{x} - u) \geq \eta \|\mathfrak{x} - u\|^2, \quad \forall \mathfrak{x}, u \in \mathcal{M};$$

(iii) ζ -inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$(\Psi\mathfrak{x} - \Psi u, \mathfrak{x} - u) \geq \zeta \|\Psi\mathfrak{x} - \Psi u\|^2, \quad \forall \mathfrak{x}, u \in \mathcal{M}.$$

It is obvious that if Ψ is ζ -inverse-strongly monotone, then it is monotone and $\frac{1}{\zeta}$ -Lipschitz continuous. Furthermore, we have that, for any $u, v \in \mathcal{M}$ and $\lambda > 0$

$$\|(I - \lambda\Psi)u - (I - \lambda\Psi)v\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\zeta)\|\Psi u - \Psi v\|^2.$$

So, if $\lambda < 2\zeta$, then $I - \lambda\Psi$ is nonexpansive mapping.

The following proposition presents some important properties of projection.

Proposition 2.6. For given $\mathfrak{x} \in \mathcal{H}$ and $w \in \mathcal{M}$

- (i.) $w = P_{\mathcal{M}}\mathfrak{x} \Leftrightarrow (\mathfrak{x} - z, u - z) \leq 0, \quad \forall u \in \mathcal{M}$
- (ii.) $w = P_{\mathcal{M}}\mathfrak{x} \Leftrightarrow \|\mathfrak{x} - z\|^2 \leq \|\mathfrak{x} - u\|^2 - \|u - z\|^2, \quad \forall u \in \mathcal{M}$
- (iii.) $(P_{\mathcal{M}}\mathfrak{x} - P_{\mathcal{M}}u, \mathfrak{x} - u) \geq \|P_{\mathcal{M}}\mathfrak{x} - P_{\mathcal{M}}u\|^2, \quad \forall u \in \mathcal{H}.$

Consequently, $P_{\mathcal{M}}$ is nonexpansive and monotone.

Lemma 2.7. ([7, 18] (The demiclosedness principle)). Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a closed convex subset of $\mathcal{H}$. Let $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ be a nonexpansive mapping with $\text{Fix}(\Psi) \neq \emptyset$. If a sequence $\{\mathfrak{x}_k\} \subset \mathcal{M}$ satisfying

- (i) $\{\mathfrak{x}_k\}$ weakly converges to $\mathfrak{x}$
- and (ii) $\{(I - \Psi)\mathfrak{x}_k\}$ converges strongly to 0. Then, it follows that $\mathfrak{x} = \Psi\mathfrak{x}$.

Remark 2.8. Although Lemma 2.7 is stated for nonexpansive mappings, the conclusion remains valid for quasi-nonexpansive mappings provided Ψ is Lipschitz continuous, since the proof relies only on the weak sequential closedness of $I - \Psi$.

Lemma 2.9. [7, 18] In real Hilbert space $\mathcal{H}$, the following hold:

- (i) $\|u + w\|^2 \leq \|u\|^2 + 2\langle w, u + w \rangle, \quad \forall u, w \in \mathcal{H};$
- (ii) $\|\lambda u + (1 - \lambda)w\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|w\|^2 - \lambda(1 - \lambda)\|u - w\|^2, \text{ for all } u, w \in \mathcal{H}$
and $\lambda \in (0, 1)$.

Lemma 2.10. [28] *Let $\{a_k\}$ be a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \gamma_k)a_k + \delta_k \quad k \geq 0,$$

where $\{\gamma_k\} \subset (0, 1)$ and $\{\delta_k\} \subset \mathbb{R}$ such that

- (i) $\sum_{k=1}^{\infty} \gamma_k = \infty$;
- (ii) $\limsup_{k \rightarrow \infty} \frac{\delta_k}{\gamma_k} \leq 0$ or $\sum_{k=1}^{\infty} |\delta_k| < \infty$.

Then the sequence $\{a_k\}$ converges to zero, that is $\lim_{k \rightarrow \infty} a_k = 0$.

Lemma 2.11. [10] *Let $\mathcal{H}$ be a real Hilbert space. Let $V: \mathcal{H} \rightarrow \mathcal{H}$ be a l -Lipschitzian mapping with a constant $l \geq 0$, and let $G: \mathcal{H} \rightarrow \mathcal{H}$ be a κ -Lipschitzian and η -strongly monotone mapping with constants $\kappa, \eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$((\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y) \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Lemma 2.12. *Let $\mathcal{H}$ be a real Hilbert space and let $\Psi: \mathcal{M} \rightarrow \mathcal{M}$ be a quasi-nonexpansive mapping with $\text{Fix}(\Psi) \neq \emptyset$, where $\mathcal{M} \subset \mathcal{H}$ is nonempty, closed, and convex. Let $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{H}$ be Lipschitz continuous and strongly monotone. Then the variational inequality*

$$\langle \mathcal{F}z, x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(\Psi),$$

admits a unique solution $z \in \text{Fix}(\Psi)$, which is equivalently characterized by

$$z = P_{\text{Fix}(\Psi)}(I - \lambda \mathcal{F})z,$$

can be derived by combining the projection characterization in Hilbert spaces and the viscosity approximation framework (see Xu [27], Marino and Xu [19]).

Remark 2.13. Although the above lemma is well known for nonexpansive mappings, the same conclusion remains valid for quasi-nonexpansive mappings since the proof relies only on the closedness and convexity of the fixed point set.

3. THE VISCOSITY TECHNIQUE FOR IMPLICIT MIDPOINT RULE

In this section, we demonstrate a strong convergence theorem derived from a viscosity implicit midpoint rule for fixed points of quasi-nonexpansive mapping.

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed convex subset of $\mathcal{H}$, and $\Psi : \mathcal{M} \rightarrow \mathcal{M}$ a quasi-nonexpansive mapping with $\text{Fix}(\Psi) \neq \emptyset$. Let $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}$ be a ϕ -Lipschitz continuous and σ -strongly monotone mapping, where $\phi, \sigma > 0$, and $\mathcal{K} : \mathcal{M} \rightarrow \mathcal{M}$ a t -Lipschitzian mapping with constant $t > 0$. Assume that Ψ is uniformly Lipschitz continuous on bounded subsets; that is, there exists $L > 0$ such that $\|\Psi\kappa - \Psi u\| \leq L\|\kappa - u\|$ for all κ, u in bounded subset of $\mathcal{M}$. Let $\nu = 1 - \sqrt{1 - \lambda(2\sigma - \lambda\phi^2)}$. Suppose the parameters satisfy $0 < \lambda < \frac{2\sigma}{\phi^2}$, $0 < \rho t < \nu$. Let $\{\kappa_k\}$ be generated by the viscosity implicit midpoint rule (1.8). Then $\{\kappa_k\}$ converges in norm to a fixed point p of Ψ , which is also the unique solution of the variational inequality.*

We additionally assume the following conditions:

- (A1) $\lim_{k \rightarrow \infty} \varrho_k = 0$, $\sum_{k=0}^{\infty} \varrho_k = \infty$,
- (A2) $\sum_{k=0}^{\infty} |\varrho_{k+1} - \varrho_k| < \infty$,
- (A3) $\sum_{k=0}^{\infty} |\eta_{k+1} - \eta_k| < \infty$ and $\sum_{k=0}^{\infty} |\sigma_{k+1} - \sigma_k| < \infty$
- (A4) $0 < \eta_k + \sigma_k < 1$, $\forall k$
- (A5) Ψ is quasi-nonexpansive and uniformly Lipschitz continuous on bounded subsets of $\mathcal{H}$.

Then the sequence $\{\kappa_k\}$ converges strongly to $z \in \text{Fix}(\Psi)$, where $z = P_{\text{Fix}(\Psi)}(I - \lambda\mathcal{F} + \rho\mathcal{K})(z)$. In other words, which is also the unique solution of the variational inequality

$$\langle (\rho\mathcal{K} - \lambda\mathcal{F})(z), \kappa - z \rangle \leq 0, \quad \forall \kappa \in \text{Fix}(\Psi). \quad (3.1)$$

Remark 3.2. Assumption (A5) is imposed to guarantee the demiclosedness of the operator $I - \Psi$ at zero and to control the implicit midpoint term $\Psi\left(\frac{\kappa_k + \kappa_{k+1}}{2}\right)$. In the absence of Lipschitz continuity, quasi-nonexpansiveness alone is insufficient to ensure weak sequential closedness of $\text{Fix}(\Psi)$, which is required in Step 2 and Step 3 of the proof. Similar assumptions are standard in viscosity-type approximation methods (see Xu [27], Marino and Xu [19]).

Proof. Since $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}$ be a ϕ -Lipschitzian continuous and σ -strongly monotone mapping, and $\mathcal{K} : \mathcal{M} \rightarrow \mathcal{M}$ a t -Lipschitzian continuous mapping, we have

$$\begin{aligned} \|(I - \lambda\mathcal{F})\kappa - (I - \lambda\mathcal{F})u\|^2 &= \|\kappa - u\|^2 - 2\lambda\langle \kappa - u, \mathcal{F}\kappa - \mathcal{F}u \rangle \\ &\quad + \lambda^2\|\mathcal{F}\kappa - \mathcal{F}u\|^2 \\ &\leq (1 - 2\lambda\sigma + \lambda^2\phi^2)\|\kappa - u\|^2 \\ &= (1 - \nu)^2\|\kappa - u\|^2, \end{aligned} \quad (3.2)$$

where $\nu = 1 - \sqrt{1 - \lambda(2\sigma - \lambda\phi^2)}$, and hence

$$\begin{aligned}
\|P_{\text{Fix}(\Psi)}(I - \lambda\mathcal{F} + \rho\mathcal{K})\varkappa - P_{\text{Fix}(\Psi)}(I - \lambda\mathcal{F} + \rho\mathcal{K})u\| &\leq \|(I - \lambda\mathcal{F} + \rho\mathcal{K})\varkappa \\
&\quad - (I - \lambda\mathcal{F} + \rho\mathcal{K})u\| \\
&\leq \|(I - \lambda\mathcal{F})\varkappa - (I - \lambda\mathcal{F})u\| \\
&\quad + \rho\|\mathcal{K}\varkappa - \mathcal{K}u\| \\
&\leq (1 - \nu)\|\varkappa - u\| + \rho t\|\varkappa - u\| \\
&\leq (1 - (\nu - \rho t))\|\varkappa - u\|,
\end{aligned} \tag{3.3}$$

for all $\varkappa, u \in \mathcal{M}$. Therefore, $P_{\text{Fix}(\Psi)}(I - \lambda\mathcal{F} + \rho\mathcal{K})$ is a contraction of $\mathcal{M}$ into itself, which implies that there exists a unique $z \in \mathcal{M}$ such that

$$z = P_{\text{Fix}(\Psi)}(I - \lambda\mathcal{F} + \rho\mathcal{K})z.$$

We divide the proof into several steps:

Step I. Since $\text{Fix}(\Psi) \neq \emptyset$, let $p \in \text{Fix}(\Psi)$. Using the quasi-nonexpansive mapping of Ψ , we obtain

$$\begin{aligned}
\|\varkappa_{k+1} - p\| &= \left\| \left(\varrho_k \rho \mathcal{K}(\varkappa_k) + \eta_k \varkappa_k + \sigma_k \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right. \right. \\
&\quad \left. \left. + ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F}) \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right) - p \right\|
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\leq \|\varrho_k(\rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F}(p))\| + \eta_k \|\varkappa_k - p\| + \sigma_k \left\| \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) - p \right\| \\
&\quad + \left\| ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F}) \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right. \\
&\quad \left. - ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F})(p) \right\| \\
&\leq \varrho_k \|\rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F}(p)\| + \eta_k \|\varkappa_k - p\| + \sigma_k \left\| \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) - p \right\| \\
&\quad + (1 - \eta_k - \sigma_k) \left\| \left(1 - \frac{\varrho_k \lambda}{1 - \eta_k - \sigma_k} \mathcal{F} \right) \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right. \\
&\quad \left. - \left(1 - \frac{\varrho_k \lambda}{1 - \eta_k - \sigma_k} \mathcal{F} \right) p \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \varrho_k \|\rho\mathcal{K}(\mathfrak{x}_k) - \rho\mathcal{K}(p)\| + \varrho_k \|\rho\mathcal{K}(p) - \lambda\mathcal{F}(p)\| + \sigma_k \left\| \Psi\left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2}\right) - p \right\| \\
&\quad + \eta_k \|\mathfrak{x}_k - p\| + (1 - \eta_k - \sigma_k) \left(1 - \frac{\varrho_k \nu}{1 - \eta_k - \sigma_k}\right) \left\| \Psi\left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2}\right) - p \right\| \\
&= \frac{1 + \eta_k - \varrho_k \nu + 2\varrho_k \rho t}{2} \|\mathfrak{x}_k - p\| + \frac{1 - \eta_k - \varrho_k \nu}{2} \|\mathfrak{x}_{k+1} - p\| \\
&\quad + \varrho_k \|\rho\mathcal{K}(p) - \lambda\mathcal{F}(p)\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&= \left(1 - \frac{2(\varrho_k(\nu - \rho t))}{1 + \eta_k + \varrho_k \nu}\right) \|\mathfrak{x}_k - p\| \\
&\quad + \frac{2\varrho_k(\nu - \rho t)}{1 + \eta_k + \varrho_k \nu} \left\{ \frac{1}{(\nu - \rho t)} \|\rho\mathcal{K}(p) - \lambda\mathcal{F}(p)\| \right\} \\
&\leq \max \left\{ \|\mathfrak{x}_0 - p\|, \frac{1}{(\nu - \rho t)} \|(\rho\mathcal{K} - \lambda\mathcal{F})p\| \right\}.
\end{aligned} \tag{3.5}$$

By induction, we readily obtain

$$\|\mathfrak{x}_{k+1} - p\| \leq \max \left\{ \|\mathfrak{x}_0 - p\|, \frac{1}{(\nu - \rho t)} \|(\rho\mathcal{K} - \lambda\mathcal{F})p\| \right\},$$

for all k and $\mathfrak{x}_0 \in \mathcal{M}$. It turns out that $\{\mathfrak{x}_k\}$ is bounded.

Step 2. $\lim_{k \rightarrow \infty} \|\mathfrak{x}_{k+1} - \mathfrak{x}_k\| = 0$. To see this, we apply (1.8) to get

$$\begin{aligned}
\|\mathfrak{x}_{k+1} - \mathfrak{x}_k\| &= \left\| \varrho_k \rho \mathcal{K}(\mathfrak{x}_k) + \eta_k \mathfrak{x}_k + \sigma_k \Psi\left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2}\right) \right. \\
&\quad \left. - \left[\varrho_{k-1} \rho \mathcal{K}(\mathfrak{x}_{k-1}) + \eta_{k-1} \mathfrak{x}_{k-1} + \sigma_{k-1} \Psi\left(\frac{\mathfrak{x}_{k-1} + \mathfrak{x}_k}{2}\right) \right] \right. \\
&\quad \left. + ((1 - \eta_{k-1} - \sigma_{k-1})I - \varrho_{k-1} \lambda \mathcal{F}) \Psi\left(\frac{\mathfrak{x}_{k-1} + \mathfrak{x}_k}{2}\right) \right\| \\
&\leq \left\| \varrho_k (\rho \mathcal{K}(\mathfrak{x}_k) - \rho \mathcal{K}(\mathfrak{x}_{k-1})) + (\varrho_k - \varrho_{k-1}) (\rho \mathcal{K}(\mathfrak{x}_{k-1})) \right. \\
&\quad \left. + \eta_k (\mathfrak{x}_k - \mathfrak{x}_{k+1}) + (\eta_k - \eta_{k-1}) (\mathfrak{x}_{k-1}) + \sigma_k \left(\Psi\left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2}\right) - \Psi\left(\frac{\mathfrak{x}_{k-1} + \mathfrak{x}_k}{2}\right) \right) \right\|
\end{aligned}$$

$$\begin{aligned}
& + (\sigma_k - \sigma_{k-1}) \left(\Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right) \\
& + ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F}) \left(\Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) - \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right) \\
& + ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F} - ((1 - \eta_{k-1} - \sigma_{k-1})I - \varrho_{k-1} \lambda \mathcal{F})) \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \Big\| \\
& \leq \varrho_k \rho t \|\varkappa_k - \varkappa_{k-1}\| + |\varrho_k - \varrho_{k-1}| \|\rho \mathcal{K} \varkappa_{k-1}\| + \eta_k \|\varkappa_k - \varkappa_{k+1}\| + |\eta_k - \eta_{k-1}| \|\varkappa_{k-1}\| \\
& + \frac{\sigma_k L}{2} (\|\varkappa_k - \varkappa_{k-1}\| + \|\varkappa_{k+1} - \varkappa_k\|) + |\sigma_k - \sigma_{k-1}| \left\| \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right\| \\
& + \frac{(1 - \eta_k - \sigma_k - \varrho_k \nu) L}{2} (\|\varkappa_k - \varkappa_{k-1}\| + \|\varkappa_{k+1} - \varkappa_k\|) \\
& + |(1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F} - ((1 - \eta_{k-1} - \sigma_{k-1})I - \varrho_{k-1} \lambda \mathcal{F})| \left\| \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right\| \\
& \leq \left(\varrho_k \rho t + \eta_k + \frac{(1 - \eta_k - \sigma_k - \varrho_k \nu) L}{2} + \frac{\sigma_k L}{2} \right) \|\varkappa_k - \varkappa_{k-1}\| + |\varrho_k - \varrho_{k-1}| \|\rho \mathcal{K}(\varkappa_{k-1})\| \\
& + |\eta_k - \eta_{k-1}| \|\varkappa_{k-1}\| + |\eta_k - \eta_{k-1}| \left\| \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right\| \\
& + |\sigma_k - \sigma_{k-1}| \left\| \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right\| + |\varrho_k - \varrho_{k-1}| \left\| \lambda \mathcal{F} \Psi \left(\frac{\varkappa_{k-1} + \varkappa_k}{2} \right) \right\| \\
& + \frac{(1 - \eta_k - \varrho_k \nu) L}{2} \|\varkappa_{k+1} - \varkappa_k\| \\
& = \left(\frac{(1 + \eta_k - \varrho_k \nu) L + 2\eta_k + 2\varrho_k \rho t}{2 - (1 - \eta_k - \varrho_k \nu) L} \right) \|\varkappa_k - \varkappa_{k-1}\| + \left(\frac{2|\sigma_k - \sigma_{k-1}|}{2 - (1 - \eta_k - \varrho_k \nu) L} \right. \\
& \quad \left. + \frac{2|\varrho_k - \varrho_{k-1}|}{2 - (1 - \eta_k - \varrho_k \nu) L} + \frac{2|\eta_k - \eta_{k-1}|}{2 - (1 - \eta_k - \varrho_k \nu) L} \right) S
\end{aligned} \tag{3.6}$$

Here $S > 0$ is a constant such that

$$\begin{aligned}
S = \max \Big\{ & \sup_{k \geq 0} \left(\left\| \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \right), \sup_{k \geq 0} \left(\|\varkappa_k\| + \left\| \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \right), \\
& \sup_{k \geq 0} \left\| \rho \mathcal{K}(\varkappa_k) + \lambda \mathcal{F} \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \Big\}.
\end{aligned}$$

It turns out that

$$\begin{aligned}
& = \left(1 - \frac{2((1 + L)\eta_k - (1 - L) - \varrho_k(\nu L - \rho t))}{2 - (1 - \eta_k - \varrho_k \nu) L} \right) \|\varkappa_k - \varkappa_{k-1}\| \\
& + \frac{2S}{2 - (1 - \eta_k - \varrho_k \nu) L} (|\varrho_k - \varrho_{k-1}| + |\eta_k - \eta_{k-1}| + |\sigma_k - \sigma_{k-1}|).
\end{aligned} \tag{3.7}$$

Let

$$a_k = \frac{2((1 + L)\eta_k - (1 - L) - \varrho_k(\nu L - \rho t))}{2 - (1 - \eta_k - \varrho_k \nu) L},$$

and

$$\epsilon_k = \frac{2S}{2 - (1 - \eta_k - \varrho_k \nu)L} (|\varrho_k - \varrho_{k-1}| + |\eta_k - \eta_{k-1}| + |\sigma_k - \sigma_{k-1}|).$$

Then inequality (3.7) can be written as

$$\|\varkappa_{k+1} - \varkappa_k\| = (1 - a_k)\|\varkappa_k - \varkappa_{k-1}\| + \epsilon_k. \quad (3.8)$$

Using the boundedness of $\{\varkappa_k\}$ and the Lipschitz continuity of $\mathcal{F}$, $\mathcal{K}$ and Ψ standard estimates yield inequality (3.6). From assumptions (A1)–(A5), we have $a_k > 0$ and $\epsilon_k \rightarrow 0$. Hence, by Lemma 2.10, $\lim_{k \rightarrow \infty} \|\varkappa_{k+1} - \varkappa_k\| = 0$ as $k \rightarrow \infty$.

Step 3. $\lim_{k \rightarrow \infty} \|\varkappa_k - \Psi \varkappa_k\| = 0$. This follows the argument below:

$$\begin{aligned} \|\varkappa_k - \Psi \varkappa_k\| &\leq \|\varkappa_k - \varkappa_{k+1}\| + \|\varkappa_{k+1} - \Psi \varkappa_k\| \\ &\leq \|\varkappa_k - \varkappa_{k+1}\| + \left\| \varkappa_{k+1} - \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) + \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) - \Psi \varkappa_k \right\|. \end{aligned}$$

Then apply triangle inequality

$$\leq \|\varkappa_k - \varkappa_{k+1}\| + \left\| \varkappa_{k+1} - \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| + \left\| \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) - \Psi \varkappa_k \right\|.$$

Now apply Lipschitz property

$$\begin{aligned} &\leq \|\varkappa_k - \varkappa_{k+1}\| + \left\| \varkappa_{k+1} - \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| + \frac{L}{2} \|\varkappa_{k+1} - \varkappa_k\| \\ &\leq \left(\frac{2+L}{2} \right) \|\varkappa_k - \varkappa_{k+1}\| + \varrho_k \left\| \rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F} \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \\ &\quad + \eta_k \left\| \varkappa_k - \Psi \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \\ &\leq \left(\frac{2+L}{2} \right) \|\varkappa_k - \varkappa_{k+1}\| + \varrho_k \left\| \rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F} \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \\ &\quad + \frac{\eta_k}{2} \|\varkappa_k - \Psi(\varkappa_k)\| + \frac{\eta_k}{2} \|\varkappa_k - \Psi(\varkappa_{k+1})\| \\ &\leq \left(\frac{2+\eta_k+L}{2-\eta_k} \right) \|\varkappa_k - \varkappa_{k+1}\| + \frac{2\varrho_k}{2-\eta_k} \left\| \rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F} \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\|, \end{aligned}$$

and therefore

$$\begin{aligned} \|\varkappa_k - \Psi(\varkappa_k)\| &\leq \left(\frac{2+\eta_k+L}{2-\eta_k} \right) \|\varkappa_k - \varkappa_{k+1}\| + \frac{2\varrho_k}{2-\eta_k} \left\| \rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F} \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\| \\ &\leq C \|\varkappa_k - \varkappa_{k+1}\| + \frac{2\varrho_k}{2-\eta_k} \left\| \rho \mathcal{K}(\varkappa_k) - \lambda \mathcal{F} \left(\frac{\varkappa_k + \varkappa_{k+1}}{2} \right) \right\|. \end{aligned}$$

Where $C > 0$ is a constant depending on the Lipschitz constant L and the parameters of the iteration. Since $\lim_{k \rightarrow \infty} \|\varkappa_k - \varkappa_{k+1}\| = 0$ and $\varrho_k \rightarrow 0$ as $k \rightarrow \infty$, we obtain

$$\lim_{k \rightarrow \infty} \|\varkappa_k - \Psi \varkappa_k\| = 0.$$

Step 4. We prove that $\mathfrak{x}_u(\mathfrak{x}_k) \subset \text{Fix}(\Psi)$. Here $\mathfrak{x}_u(\mathfrak{x}_k) = \{\mathfrak{x}_k \in \mathcal{M} : \text{there exists a subsequence of } \{\mathfrak{x}_k\} \text{ weakly converging to } z\}$ is the weak u -limit set of $\{\mathfrak{x}_k\}$. This follows directly from Step 3 and Lemma 2.7.

Step 5. We claim

$$\limsup_{k \rightarrow \infty} \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_k - z \rangle \leq 0, \quad (3.9)$$

where $z \in \text{Fix}(\Psi)$ is the unique solution of variational inequality (3.1)

Indeed, we can find a subsequence $\mathfrak{x}_{k_i}$ from the sequence $\mathfrak{x}_k$, which converges weakly to a point u . Furthermore,

$$\limsup_{k \rightarrow \infty} \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_k - z \rangle = \limsup_{i \rightarrow \infty} \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_{k_i} - z \rangle. \quad (3.10)$$

We can use Proposition 2.6(i) to combine (3.9) and (3.10) and come to the conclusion that $z \in \text{Fix}(\Psi)$ by Step 4.

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_k - z \rangle &= \limsup_{i \rightarrow \infty} \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_{k_i} - z \rangle \\ &= \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, u - z \rangle \leq 0. \end{aligned} \quad (3.11)$$

Step 6. We finally prove that $\mathfrak{x}_k \rightarrow z$ in norm. Here again $z \in \text{Fix}(\Psi)$ is the unique solution of variational inequality (3.1). We present the details as follows:

$$\begin{aligned} \|\mathfrak{x}_{k+1} - z\|^2 &= \left\langle \left(\varrho_k \rho \mathcal{K}(\mathfrak{x}_k) + \eta_k(\mathfrak{x}_k) + \sigma_k \Psi \left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2} \right) \right. \right. \\ &\quad \left. \left. + ((1 - \eta_k - \sigma_k)I - \varrho_k \lambda \mathcal{F}) \Psi \left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2} \right) \right) - z, \mathfrak{x}_{k+1} - z \right\rangle \\ &\leq \varrho_k \rho \langle (\mathcal{K}(\mathfrak{x}_k) - \mathcal{K}(z)), \mathfrak{x}_{k+1} - z \rangle + \varrho_k \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle \\ &\quad + \eta_k \langle \mathfrak{x}_k - z, \mathfrak{x}_{k+1} - z \rangle + \sigma_k \left\langle \Psi \left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2} \right) - z, \mathfrak{x}_{k+1} - z \right\rangle \\ &\quad + (1 - \eta_k - \sigma_k) \left(1 - \frac{\varrho_k \lambda}{1 - \eta_k - \sigma_k} \mathcal{F} \right) \left\langle \Psi \left(\frac{\mathfrak{x}_k + \mathfrak{x}_{k+1}}{2} \right) - z, \mathfrak{x}_{k+1} - z \right\rangle \\ &\leq \varrho_k \rho t \|\mathfrak{x}_k - z\| \|\mathfrak{x}_{k+1} - z\| + \varrho_k \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle \\ &\quad + \eta_k \|\mathfrak{x}_k - z\| \|\mathfrak{x}_{k+1} - z\| + \frac{\sigma_k}{2} \|\mathfrak{x}_k - z\| \|\mathfrak{x}_{k+1} - z\| \\ &\quad + \frac{\sigma_k}{2} \|\mathfrak{x}_{k+1} - z\| \|\mathfrak{x}_{k+1} - z\| + \frac{1 - \eta_k - \sigma_k - \varrho_k \nu}{2} \|\mathfrak{x}_k - z\| \|\mathfrak{x}_{k+1} - z\| \\ &\quad + \frac{1 - \eta_k - \sigma_k - \varrho_k \nu}{2} \|\mathfrak{x}_{k+1} - z\| \|\mathfrak{x}_{k+1} - z\| \\ &= \left(\frac{1 + \eta_k + 2\varrho_k \rho t - \varrho_k \nu}{2} \right) \|\mathfrak{x}_k - z\| \|\mathfrak{x}_{k+1} - z\| \\ &\quad + \left(\frac{1 - \eta_k - \varrho_k \nu}{2} \right) \|\mathfrak{x}_k - z\|^2 + \varrho_k \langle (\rho\mathcal{K} - \lambda\mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle \end{aligned} \quad (3.12)$$

It turns out that

$$\begin{aligned}
\left(\frac{1 + \eta_k + \varrho_k \nu}{2}\right) \|\mathfrak{x}_{k+1} - z\|^2 &\leq \left(\frac{1 + \eta_k + 2\varrho_k \rho t - \varrho_k \nu}{2}\right) \|\mathfrak{x}_k - z\| \|\mathfrak{x}_{k+1} - z\| \\
&\quad + \varrho_k \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle. \\
&\leq \left(\frac{1 + \eta_k + 2\varrho_k \rho t - \varrho_k \nu}{1 + \eta_k + \varrho_k \nu}\right) \|\mathfrak{x}_k - z\|^2 \\
&\quad + \left(\frac{1 + \eta_k + 2\varrho_k \rho t - \varrho_k \nu}{1 + \eta_k + \varrho_k \nu}\right) \|\mathfrak{x}_{k+1} - z\|^2 \\
&\quad + \frac{2\varrho_k}{1 + \eta_k + \varrho_k \nu} \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle.
\end{aligned} \tag{3.13}$$

Hence, we get

$$\begin{aligned}
\left(1 - \left(\frac{1 + \eta_k + 2\varrho_k \rho t - \varrho_k \nu}{1 + \eta_k + \varrho_k \nu}\right)\right) \|\mathfrak{x}_{k+1} - z\|^2 &\leq \left(\frac{1 + \eta_k + 2\varrho_k \rho t - \varrho_k \nu}{1 + \eta_k + \varrho_k \nu}\right) \|\mathfrak{x}_k - z\|^2 \\
&\quad + \frac{\varrho_k}{1 + \eta_k + \varrho_k \nu} \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle. \\
\frac{1 + \eta_k + 3\varrho_k \nu - 2\varrho_k \rho t}{2(1 + \eta_k + \varrho_k \nu)} \|\mathfrak{x}_{k+1} - z\|^2 &\leq \frac{1 + \eta_k + \varrho_k (2\rho t - \nu)}{2(1 + \eta_k + \varrho_k \nu)} \|\mathfrak{x}_{k+1} - z\|^2 \\
&\quad + \frac{4\varrho_k}{1 + \eta_k + \varrho_k \nu} \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle.
\end{aligned} \tag{3.14}$$

Which implies that

$$\begin{aligned}
\|\mathfrak{x}_{k+1} - z\|^2 &\leq \frac{1 + \eta_k + \varrho_k (2\rho t - \nu)}{1 + \eta_k + 3\varrho_k \nu - 2\varrho_k \rho t} \|\mathfrak{x}_k - z\|^2 \\
&\quad + \frac{4\varrho_k}{1 + \eta_k + 3\varrho_k \nu - 2\varrho_k \rho t} \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle. \\
&= \left(1 - \frac{4\varrho_k (\nu - \rho t)}{1 + \eta_k + \varrho_k (3\nu - 2\rho t)}\right) \|\mathfrak{x}_k - z\|^2 \\
&\quad + \frac{4\varrho_k (\nu - \rho t)}{1 + \eta_k + \varrho_k (3\nu - 2\rho t)} \times \left\{ \frac{1}{(\nu - \rho t)} \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle \right\} \\
&= (1 - \theta_k) \|\mathfrak{x}_k - z\|^2 + \varpi_k,
\end{aligned} \tag{3.15}$$

where $\gamma_k = \frac{4\varrho_k (\nu - \rho t)}{1 + \eta_k + \varrho_k (3\nu - 2\rho t)}$ and $\delta_k = \frac{1}{(\nu - \rho t)} \langle (\rho \mathcal{K} - \lambda \mathcal{F})z, \mathfrak{x}_{k+1} - z \rangle$. Since $\lim_{k \rightarrow \infty} \varrho_k = 0$ and $\sum_{k=0}^{\infty} \varrho_k = \infty$, it is easy to see that $\lim_{k \rightarrow \infty} \gamma_k = 0$, $\sum_{k=0}^{\infty} \gamma_k = \infty$, and $\limsup_{k \rightarrow \infty} \delta_k \leq 0$. It follows from (3.14), (3.15), and Lemma 2.10 that $\mathfrak{x}_k \rightarrow z$. This completes the proof. $\square$

Remark 3.3. If Ψ is nonexpansive, then it is quasi-nonexpansive. Therefore, our result recovers Hu's [16] finding as a special case. Since the converse is not true, our result is more general. Hence, the applicability of Theorem 3.1 extends the results of Hu [16].

4. CONCLUSION

In this paper, we demonstrate a strong convergence theorem derived from a proposed viscosity implicit midpoint for fixed points of quasi-nonexpansive mapping. Additionally, we present a unique solution to the variational inequality problem. Our result is more general and extends the results of Hu [16].

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